Name:	**Teacher's**	Class	Reg No.
Date:		4	

Chapter 4: Polynomials and Partial Fractions
Reference Book: Additional Maths 360 Textbook, Marshall Cavendish

4.1 Polynomials and Identities

You will learn how to,

- Add, subtract, multiply and divide polynomials,
- Find the unknown constants in a polynomial identity,
- Divide one polynomial by another using the division algorithm.

What are Polynomials?

"Polynomial" is a Greek word which literally means 'many terms'.

(a) A **polynomial** is a sum of terms, each of the form ax^n, where a is a constant and the power n is a **non-negative integer**.

Examples of polynomials include $x + 1$, $3x^2 - 4x + 5$, $x^3 - \frac{1}{4}x^2 + \frac{1}{2}$, and $-x^3$.

(b) In the term ax^n, a is called the **coefficient** of x^n.

(c) The **degree** or **order** of a polynomial in x is the **highest** power of x.

Example 1

Determine whether each of the following is a polynomial. Give a brief reason for your answer. If it is a polynomial, state its degree.

Polynomial	Degree/ Order
$x + 1$	1
$3x^2 - 4x + 5$	2
$x^3 - \frac{1}{4}x^2 + \frac{1}{2}$	3
$-x^3$	3
4	0
$x^2 + \frac{1}{2}x$	2
$(x - 2)(x + 1)$	2
$x^2 + 2\sqrt{x} + 3$	Not a polynomial as it has a term with a fractional power.

(d) A polynomial is often denoted by P(x), Q(x), f(x) etc. If $Q(x) = x^2 + 2x + 3$, then the **value** of this polynomial at $x = 2$ is denoted by **Q(2)** $= 2^2 + 2(2) + 3 = 11$.
→ More generally, the value of the polynomial, Q(x) at $x = a$, is denoted by **Q(a)**.

Example 2
Evaluate P(3) and P(−1) if P(x) = $x^3 - 2x + 1$.

$$P(3) = (3)^3 - 2(3) + 1$$
$$= 22$$
$$P(-1) = (-1)^3 - 2(-1) + 1$$
$$= 4$$

Addition and Subtraction and Multiplication of Polynomials
When two polynomials are added, subtracted or multiplied, the result is still a **polynomial**. Addition and subtracting polynomials can be done by combining **like terms** while multiplication of two polynomials can be done through either using the distributive law.

Example 3
Consider the polynomials $P(x) = x^2 + x + 1$ and $Q(x) = 2x^2 - 3x + 2.$ Find
(i) $Q(x) - P(x)$,
(ii) $P(x) + 2Q(x)$.

(i) $Q(x) - P(x) = 2x^2 - 3x + 2 - (x^2 + x + 1)$
$$= 2x^2 - 3x + 2 - x^2 - x - 1$$
$$= x^2 - 4x + 1$$

(ii) $P(x) + 2Q(x) = x^2 + x + 1 + 2(2x^2 - 3x + 2)$
$$= x^2 + x + 1 + 4x^2 - 6x + 4$$
$$= 5x^2 - 5x + 5$$

Example 4
By observation, find the coefficient of x^3 and x^2 in the expansion of
(i) $(3x^3 - 2x^2 + 5x - 1)(2x + 1)$,
(ii) $(2x^2 - 3)(x^2 + 1)$.

(i) Coefficient of $x^3 = 3(1) - 2(2)$
$$= -1$$

Coefficient of $x^2 = -2(1) + 5(2)$
$$= 8$$

(ii) Coefficient of $x^3 = 0$

Coefficient of $x^2 = 2(1) - 3(1)$
$$= -1$$

Finding Unknowns in Identities

Consider the polynomials $x + 3$ and $x^2 - x$. Now, $x + 3 = x^2 - x$ holds only for $x = -1$ and $x = 3$. We call $x + 3 = x^2 - x$ **<u>an equation</u>** with solutions $x = -1$ and $x = 3$.

Now, consider the polynomials $x^2 - 9$ and $(x - 3)(x + 3)$.
These two polynomials are **<u>identical</u>**, i.e. they are the same except that they are written differently.

Since $x^2 - 9 = (x - 3)(x + 3)$ is true **<u>for all real values of x</u>**, we call such an equation an **identity**. An identity in x is an equation that is true for all values of the variable x. Identities can be written using the symbol " $\equiv$ ". Hence, the equation can be rewritten as $x^2 - 9 \equiv (x - 3)(x + 3)$. Mathematical formulae are actually identities. For example, $(a+b)^2 = a^2 + 2ab + b^2$ is an identity.
One <u>non-example</u> is $x^2 + 2x = 4x + 3$.

For solving for unknown(s) in an identity, you can
1. Substitute suitable values of x,
2. Equate the coefficients of the corresponding terms and/or constants on both sides of the identity.

<u>Example 5</u>
Given that $x^3 - 2x^2 + 5 \equiv ax(x-1)^2 + b(x-1) + c$, find the values of a, b and c.

<u>Method 1: Substituting suitable values of x</u>

How do you know what values to substitute into the polynomial?

When $x = 1$, $\quad 1^3 - 2(1)^2 + 5 = c$
$\therefore \quad c = 4$
When $x = 0$, $\quad 5 = b(-1) + 4$
$\therefore \quad b = -1$
When $x = -1$,
$(-1)^3 - 2(-1)^2 + 5 = a(-1)(-1-1)^2 + (-1)(-1-1) + 4$
$-1 - 2 + 5 = -4a + 2 + 4$
$\therefore \quad a = 1$

$\therefore \quad a = 1, \ b = -1, \ c = 4$

<u>Method 2: Comparing the coefficients of like powers of x</u>

$$\left.\begin{aligned} x^3 - 2x^2 + 5 &\equiv ax(x^2 - 2x + 1) + bx - b + c \\ &\equiv ax^3 - 2ax^2 + ax + bx - b + c \\ &\equiv ax^3 - 2ax^2 + (a+b)x - b + c \end{aligned}\right\}$$

it is not necessary to expand;
we can also do by observation

By comparing the coefficients of like-terms,
$\therefore a = 1, \ b = -1, \ c = 4$

<u>Example 6</u>

Given that $4x^3 + cx + 6 \equiv (2x-1)(x+2)(ax+b)$, find the values of a, b and c.

Compare the coefficients of x^3 – term and constant term,

x^3 : $\quad 4 = 2(1)a$

$\qquad\quad a = 2$

x^0 : $\quad 6 = -1(2)b$

$\qquad\quad b = -3$

When $x = 1$, $\quad 4 + c + 6 = (1)(3)(-1)$

$\qquad\qquad\qquad c = -13$

$\therefore \quad a = 2,\ b = -3,\ c = -13$

You can also use a combination of substitution and comparing the coefficients to solve a question!

Division of Polynomials

Recall long division for integers:

$$\begin{array}{rcrcl} & & 24 & \longleftarrow & \textbf{quotient} \\ \textbf{divisor} & \longrightarrow & 5\overline{)123} & \longleftarrow & \textbf{dividend} \\ & & 10\ \ & & \\ \hline & & 23 & & \\ & & 20 & & \\ \hline & & 3 & \longleftarrow & \textbf{remainder} \end{array}$$

We can express 123 = 5 × 24 + 3. Note that in any division,

dividend = **<u>divisor</u>** × **<u>quotient</u>** + **<u>remainder</u>**

A similar process works for division of polynomials.

When a polynomial P(x) is divided by another polynomial D(x),
we can express the division algorithm as
P(x) = D(x) × Q(x) + R(x),
where the quotient Q(x) and the remainder R(x) are polynomials of x.

In addition, if the degree of D(x) is not zero, then the
degree of the remainder R(x) is always less than the degree of the divisor D(x).

Example 7

Complete the following long division.

$$
\begin{array}{r|l}
 & 2x^2 - x + 1 \\
\hline
x-2 & 2x^3 - 5x^2 + 3x + 1 \\
 & -\left(2x^3 - 4x^2\right) \\
\hline
 & -x^2 + 3x + 1 \\
 & -\left(-x^2 + 2x\right) \\
\hline
 & x + 1 \\
 & -\left(x - 2\right) \\
\hline
 & 3
\end{array}
$$

$2x^3 - 5x^2 + 3x + 1 \equiv (x-2)(2x^2 - x + 1) + 3$

$$
\begin{array}{r|l}
 & 6x^2 - 3 \\
\hline
x^2+1 & 6x^4 + 3x^2 - 4x + 7 \\
 & -\left(6x^4 + 6x^2\right) \\
\hline
 & -3x^2 - 4x + 7 \\
 & -\left(-3x^2 - 3\right) \\
\hline
 & -4x + 10
\end{array}
$$

$6x^4 + 3x^2 - 4x + 7 \equiv (x^2+1)(6x^2 - 3) + 10 - 4x$

Class Practice 1

1 Use long division to find the remainder when $3x^4 - 5x^3 + 2x^2 + 10$ is divided by

(a) $2 - x$, **(b)** $x^2 - 1$.

Write the expression in the form **"dividend = divisor × quotient + remainder"** for each part.

(a)

$$
\begin{array}{r|l}
 & -3x^3 - x^2 - 4x - 8 \\
\hline
-x+2 & 3x^4 - 5x^3 + 2x^2 + 0x + 10 \\
 & -\left(3x^4 - 6x^3\right) \\
\hline
 & x^3 + 2x^2 \\
 & -\left(x^3 - 2x^2\right) \\
\hline
 & 4x^2 + 0x \\
 & -\left(4x^2 - 8x\right) \\
\hline
 & 8x + 10 \\
 & -\left(8x - 16\right) \\
\hline
 & 26
\end{array}
$$

Remainder $= 26$

$3x^4 - 5x^3 + 2x^2 + 10 = (2-x)\left(-3x^3 - x^2 - 4x - 8\right) + 26$

(b)

$$
\begin{array}{r|l}
 & 3x^2 - 5x + 5 \\
\hline
x^2-1 & 3x^4 - 5x^3 + 2x^2 + 0x + 10 \\
 & -\left(3x^4 - 3x^2\right) \\
\hline
 & -5x^3 + 5x^2 \\
 & -\left(-5x^3 + 5x\right) \\
\hline
 & 5x^2 - 5x + 10 \\
 & -\left(5x^2 - 5\right) \\
\hline
 & -5x + 15
\end{array}
$$

Remainder $= 15 - 5x$

$3x^4 - 5x^3 + 2x^2 + 10 = \left(x^2 - 1\right)\left(3x^2 - 5x + 5\right) + 15 - 5x$

2 The function P is defined by $P(x)=3x^3-5x^2+4x-9$ and the function Q is defined by $Q(x)=2x^4+4x^3-6x-5$.

(a) Find the remainders when $P(x)$ and $Q(x)$ are each divided by $(x-2)$.

(b) Deduce the remainder when $P(x)+Q(x)$ and $2P(x)-Q(x)$ are each divided by $(x-2)$.

(a)

$$
\begin{array}{r}
3x^2+x+6 \\
x-2\,\big)\,\overline{3x^3-5x^2+4x-9} \\
-(3x^3-6x^2) \\
\hline
x^2+4x \\
-(x^2-2x) \\
\hline
6x-9 \\
-(6x-12) \\
\hline
3
\end{array}
$$

The remainder when $P(x)$ is divided by $x-2$ is 3.

$$
\begin{array}{r}
2x^3+8x^2+16x+26 \\
x-2\,\big)\,\overline{2x^4+4x^3+0x^2-6x-5} \\
-(2x^4-4x^3) \\
\hline
8x^3+0x^2 \\
-(8x^3-16x^2) \\
\hline
16x^2-6x \\
-(16x^2-32x) \\
\hline
26x-5 \\
-(26x-52) \\
\hline
47
\end{array}
$$

The remainder when $Q(x)$ is divided by $x-2$ is 47.

(b)

The remainder of $P(x)+Q(x)=3+47$
$=50$

The remainder of $2P(x)-Q(x)=2(3)-47$
$=-41$

What do you notice about the degree of the remainder?
The degree of the remainder is 1 less than that of the divisor.

What is the degree of the remainder if

(a) $2x^6-9x^4+5x-9$ is divided by x^3-1? Ans: 2

(b) $3x^4-5x^3+2x^2+10$ is divided by $(x-2)(x-3)$? Ans: 1

3 Find the coefficient of x^2 in each of the following expansions.

(a) $\left(2x^2 - x + 1\right)\left(3x - 2\right)$ **(b)** $\left(x^2 + 3x + 2\right)\left(8x^2 - 5x - 4\right)$

(a) $$\text{Coefficient of } x^2 = 2\left(-2\right) + \left(-1\right)\left(3\right)$$
$$= -7$$

(b) $$\text{Coefficient of } x^2 = \left(1\right)\left(-4\right) + \left(3\right)\left(-5\right) + \left(2\right)\left(8\right)$$
$$= -3$$

4 Find the values of the unknowns in each of the following identities.

(a) $a(x-1) + b(x+3) = 3x + 1$

(b) $ax^3 - x + 3 = 4x^3 + bx^2 + c(x-1) + d$

(a) $$a(x-1) + b(x+3) = 3x + 1$$

When $x = 1$, $a(1-1) + b(1+3) = 3\left(1\right) + 1$
$$4b = 4$$
$$b = 1$$

When $x = -3$, $a(-3-1) + b(-3+3) = 3\left(-3\right) + 1$
$$-4a = -8$$
$$a = 2$$

$\therefore \quad a = 2, \; b = 1$

(b) $$ax^3 - x + 3 = 4x^3 + bx^2 + c(x-1) + d$$

By equating the coefficients of:

x^3: $a = 4$

x^2: $b = 0$

x: $c = -1$

x^0: $3 = -c + d$
$$3 = -\left(-1\right) + d$$
$$d = 2$$

$\therefore \quad a = 4, \; b = 0, \; c = -1, \; d = 2$

5 It is given that $\mathrm{P}(x)=x+1$ and $\mathrm{Q}(x)=x^2-x+1$.

(i) Find the coefficient of x^2 in the expansion of

(a) $\mathrm{P}(x)\bullet\mathrm{Q}(x)$, **(b)** $2\mathrm{P}(x)-\left[\mathrm{Q}(x)\right]^2$, **(c)** $\mathrm{Q}(x)\left[3x^2+\mathrm{P}(x)\right]$.

(ii) State the degree of each polynomial in part **(i)**.

(i)(a) $\mathrm{P}(x)\bullet\mathrm{Q}(x)=(x+1)(x^2-x+1)$

$$\begin{aligned}\text{Coefficient of } x^2 &= (1)(-1)+(1)(1)\\ &= 0\end{aligned}$$

(i)(b)
$$\begin{aligned}2\mathrm{P}(x)-\left[\mathrm{Q}(x)\right]^2 &= 2(x+1)-(x^2-x+1)^2\\ &= 2x+2-(x^2-x+1)(x^2-x+1)\end{aligned}$$

$$\begin{aligned}\text{Coefficient of } x^2 &= -\left[(1)(1)+(-1)(-1)+(1)(1)\right]\\ &= -3\end{aligned}$$

(i)(c) $\mathrm{Q}(x)\left[3x^2+\mathrm{P}(x)\right]=(x^2-x+1)(3x^2+x+1)$

$$\begin{aligned}\text{Coefficient of } x^2 &= (1)(1)+(-1)(1)+(1)(3)\\ &= 3\end{aligned}$$

(ii) Degree of polynomial for (a) is 3.
Degree of polynomial for (b) is 4.
Degree of polynomial for (c) is 4.

6 Find the values of the unknowns in
$$x^3+3x^2-2x+16-c(x+2)=ax^2(x-1)+b(x-2)^2(x-1).$$

$$x^3+3x^2-2x+16-c(x+2)=ax^2(x-1)+b(x-2)^2(x-1)$$
$$x^3+3x^2-2x+16-c(x+2)=ax^2(x-1)+b(x^2-4x+4)(x-1)$$

When $x=1$,
$$\begin{aligned}(1)^3+3(1)^2-2(1)+16-c(1+2)&=0\\ 18-3c&=0\\ c&=6\end{aligned}$$

By equating the coefficients of:

x^3: $a+b=1$ $\cdots$ (1)

x^2: $-a-b-4b=3$

x: $-a-5b=3$ $\cdots$ (2)

$(1)+(2)$: $-4b=4$
$b=-1$

Sub. $b=-1$ into (1): $a-1=1$
$a=2$

$\therefore\ a=2,\ b=-1$

4.2 The Remainder and Factor Theorems

You will learn how to,

- Use the Remainder Theorem and the Factor theorem.

The Remainder Theorem

Is there a simpler way to find the remainder?

Let us consider the polynomial $\mathrm{f}(x) = 2x^4 - 3x^3 + x^2 + 1$.

Given that $2x^4 - 3x^3 + x^2 + 1 \equiv (x-2)(2x^3 + x^2 + 3x + 6) + 13$, state the remainder when polynomial is divided by $x - 2$. Ans: 13

Write down the value of $\mathrm{f}(2)$. What did you notice? $f(2) = 2(2)^4 - 3(2)^3 + (2)^2 + 1 = 13$

Hence, the Remainder Theorem states that:

> **If a polynomial f(x) is divided by a linear divisor $ax - b$, the remainder is $\mathrm{f}\left(\frac{b}{a}\right)$.**
>
> **In particular, if the divisor is $x - a$, the remainder is f(a).**

Example 8

Verify your answers in Class Practice 1, Q1(a) and 1(b) on page 5 using Remainder Theorem.

(a) Let $\mathrm{f}(x) = 3x^4 - 5x^3 + 2x^2 + 10$

Divisor $= 2 - x$

$\mathrm{f}(2) = 3(2)^4 - 5(2)^3 + 2(2)^2 + 10$

$\mathrm{f}(2) = 26$

(b) Divisor $= x^2 - 1$

Let $3x^4 - 5x^3 + 2x^2 + 10 = (x^2 - 1)\mathrm{Q}(x) + (ax + b)$

$3x^4 - 5x^3 + 2x^2 + 10 = (x-1)(x+1)\mathrm{Q}(x) + (ax + b)$

* Remainder is a polynomial 1 degree less than the divisor

When $x = 1$, $a + b = 10$ $\cdots$ (1)

When $x = -1$, $-a + b = 20$ $\cdots$ (2)

$(1) + (2)$: $b = 15$

Sub $b = 15$ into (1): $a = -5$

$\therefore$ Remainder $= 15 - 5x$

<u>Example 9</u>

By applying the remainder theorem, find the value of p, if $x^2(x+3)-p(x+4)$ leaves a remainder of $2p$ when divided by $x-2$.

Let $f(x)=x^2(x+3)-p(x+4)$.

Given that $f(2)=2p$,

$2^2(2+3)\ -p(2+4)=2p$

$20-6p=2p$

$8p=20$

$p=2.5$

The Factor Theorem

Suppose that $f(x)=(x-1)(x-2)(x-3)$. Since $f(1)=0$, the remainder when f(x) is divided by $(x-1)$ is 0. We say that f(x) is **exactly divisible by** $(x-1)$ or $(x-1)$ is a **factor** of $f(x)$. (Note that this factor is **<u>linear</u>**).

Write down the other linear **factors** of $f(x)$, and give a quadratic factor too.

Linear factors are $(x-2)$ or $(x-3)$.

Quadratic factors are x^2-3x+2, x^2-4x+3 or x^2-5x+6.

Note that 1, 2, and 3, are also called the **solutions/roots** of the equation

$$(x-1)(x-2)(x-3)=0.$$

Recall the Remainder Theorem:

If a polynomial f(x) is divided by a <u>linear divisor</u> (x – a), the remainder is f(a).

If $(x-a)$ is a factor of $f(x)$, $f(x)$ is **exactly divisible by** $(x-a)$, and the remainder is 0. Thus, $f(a)=0$.

Hence, the **Factor Theorem** states that:

If $(x-a)$ is a factor of a polynomial $f(x)$, then $f(a)=0$.

Conversely, if $f(a)=0$, then $(x-a)$ is a factor of f(x).

This also means that $f(x)$ is divisible by $(x-a)$.

Example 10

Determine whether $(x+1)$ is a factor of the following polynomials:

(a) $x^2+\frac{1}{2}x$ **(b)** $3x^3+6x^2-3$

(a) Let $f(x) = x^2+\frac{1}{2}x$.

$f(-1)=(-1)^2+\frac{1}{2}(-1)$

$f(-1)=\frac{1}{2}\neq 0$

By Factor Theorem, $(x+1)$ is not a factor of $f(x)$.

(b) Let $q(x) = 3x^3+6x^2-3$.

$q(-1)=3(-1)^3+6(-1)^2-3$

$q(-1)=0$

By Factor Theorem, $(x+1)$ is a factor of $q(x)$.

*[Students to take note of presentation.]

Example 11

(a) Find the value of p for which $x-2$ is a factor of $2x^3+x^2+px-4$.

(b) Hence factorise $2x^3+x^2+px-4$ completely.

(a) Let $f(x)=2x^3+x^2+px-4$.

By Factor Theorem, $f(2)=0$.

$2(2)^3+2^2+2p-4=0$

$p=-8$

(b)
$$\begin{aligned}2x^3+x^2+px-4&=2x^3+x^2-8x-4\\&=(x-2)(2x^2+5x+2)\\&=(x-2)(2x+1)(x+2)\end{aligned}$$

$$\begin{array}{r|l} & 2x^2+5x+2 \\ x-2 & 2x^3+x^2-8x-4 \\ & -(2x^3-4x^2) \\ \hline & 5x^2-8x \\ & -(5x^2-10x) \\ \hline & 2x-4 \\ & -(2x-4) \\ \hline & 0 \end{array}$$

Example 12

The polynomial $x^3 + ax^2 + 2bx - 5$ is exactly divisible by $x+1$ but leaves a remainder of 51 when divided by $x-2$. Find $2a-b$.

Let $\mathrm{f}(x) = x^3 + ax^2 + 2bx - 5$.

By Factor Theorem, $\mathrm{f}(-1) = 0$

$(-1)^3 + a(-1)^2 + 2b(-1) - 5 = 0$

$a - 2b = 6 \quad \cdots \text{ (1)}$

By Remainder Theorem, $\mathrm{f}(2) = 51$.

$(2)^3 + a(2)^2 + 2b(2) - 5 = 51$

$4a + 4b = 48$

$a + b = 12 \quad \cdots \text{ (2)}$

$(2)-(1):$ $\quad 3b = 6$

$b = 2$

Sub. $b = 2$ into (2): $\quad a + 2 = 12$

$a = 10$

$\therefore 2a - b = 2(10) - 2$

$= 18$

Example 13

Given that $\mathrm{f}(a) = a^3 - ba^2 - 4b^2a + 4b^3$,

(a) show that $a-2b$ is a factor of $\mathrm{f}(a)$.

(b) Find, in terms of b, the remainder when $\mathrm{f}(a)$ is divided by $a+b$.

(a) $\mathrm{f}(2b) = (2b)^3 - b(2b)^2 - 4b^2(2b) + 4b^3$

$= 8b^3 - 4b^3 - 8b^3 + 4b^3$

$= 0$

By Factor Theorem, $(a-2b)$ is a factor of f(a).

(b) Remainder when $\mathrm{f}(a)$ is divided by $(a+b)$

$= \mathrm{f}(-b)$

$= (-b)^3 - b(-b)^2 - 4b^2(-b) + 4b^3$

$= -b^3 - b^3 + 4b^3 + 4b^3$

$= 6b^3$

Class Practice 2:

1 Use the Remainder Theorem to find the remainder in each of the following divisions.

(a) $\left(x^4-4x^3+3x^2+x+3\right)\div\left(x+1\right)$

(b) $\left[x\left(x-1\right)\left(1-2x^2\right)^2+x^2-3\right]\div\left(2-x\right)$

(c) $\left[3\left(x+4\right)^2-\left(1-x\right)^3\right]\div x$

(a) Let $\text{f}(x)=x^4-4x^3+3x^2+x+3.$

Remainder when f(x) is divided by $(x+1)$

$=\text{f}(-1)$

$=(-1)^4-4(-1)^3+3(-1)^2-1+3$

$=10$

(b) Let $\text{f}(x)=x\left(x-1\right)\left(1-2x\right)^2+x^2-3.$

Remainder when f(x) is divided by $(2-x)$

$=\text{f}(2)$

$=2\left(2-1\right)\left[1-2\left(2\right)\right]^2+\left(2\right)^2-3$

$=19$

(c) Let $\text{f}(x)=3\left(x+4\right)^2-\left(1-x\right)^3.$

Remainder when f(x) is divided by x

$=\text{f}(0)$

$=3\left(4\right)^2-\left(1\right)^3$

$=47$

2 The expression x^3+3x^2-kx+4 has a remainder k when divided by $x-2$. Find the value of k.

Let $\text{f}(x)=x^3+3x^2-kx+4.$

By Remainder Theorem, remainder $=k$.

$\text{f}(2)=k$

$(2)^3+3(2)^2-k(2)+4=k$

$24-2k=k$

$3k=24$

$k=8$

3 The expression $3x^2 - 4ax - 4a^2$ is exactly divisible by $x + 2$. Find the value of a.

Let $\mathrm{f}(x) = 3x^2 - 4ax - 4a^2$.

By Factor Theorem, $\mathrm{f}(-2) = 0$.

$3(-2)^2 - 4a(-2) - 4a^2 = 0$

$12 + 8a - 4a^2 = 0$

$4a^2 - 8a - 12 = 0$

$a^2 - 2a - 3 = 0$

$(a-3)(a+1) = 0$

$a - 3 = 0$ or $a + 1 = 0$

$a = 3$ $\quad$ $a = -1$

4 The function f is defined by $\mathrm{f}(x) = ax^2 + bx - 6$, where a and b are constants.

(a) Given that $\mathrm{f}(x)$ has a remainder of 9 when divided by $x + 3$, find b in terms of a.

(b) Hence find, in terms of a, the remainder when $2x^3 - bx^2 + 2ax - 4$ is divided by $x - 2$.

(a) Let $\mathrm{f}(x) = ax^2 + bx - 6$.

By Remainder Theorem, $\mathrm{f}(-3) = 9$.

$a(-3)^2 + b(-3) - 6 = 9$

$9a - 3b = 15$

$3a - b = 15$

$b = 3a - 5$

(b) Let $\mathrm{g}(x) = 2x^3 - bx^2 + 2ax - 4$.

Remainder $= \mathrm{g}(2)$

$= 2(2)^3 - b(2)^2 + 2a(2) - 4$

$= 16 - 4b + 4a - 4$

$= 12 - 4(3a - 5) + 4a$ [from part (a)]

$= 12 - 12a + 20 + 4a$

$= 32 - 8a$

5 The expression $8x^3 + ax^2 + bx - 9$, where a and b are constants, has a remainder of -95 when divided by $x+2$ and has a remainder of 3 when divided by $2x-3$.
Find the value of a and b.

Let $f(x) = 8x^3 + ax^2 + bx - 9$.

By Remainder Theorem, $f(-2) = -95$.

$8(-2)^3 + a(-2)^2 + b(-2) - 9 = -95$

$-64 + 4a - 2b - 9 = -95$

$4a - 2b = -22 \quad \cdots \ (1)$

By Remainder Theorem, $f\left(\frac{3}{2}\right) = 3$.

$8\left(\frac{3}{2}\right)^3 + a\left(\frac{3}{2}\right)^2 + b\left(\frac{3}{2}\right) - 9 = 3$

$27 + \frac{9}{4}a + \frac{3}{2}b - 9 = 3$

$\frac{9}{4}a + \frac{3}{2}b = -15$

$9a + 6b = -60$

$3a + 2b = -20 \quad \cdots \ (2)$

$(1)+(2)$: $\quad 7a = -42$

$a = -6$

Sub. $a = -6$ into (2): $\quad 3(-6) + 2b = -20$

$2b = -2$

$b = -1$

$\therefore \quad a = -6, \quad b = -1$

6 The polynomial $\mathrm{f}(x)$ leaves a remainder of -19 when divided by $x+2$ and a remainder of 2 when divided by $x-1$.

(i) What will the degree of the remainder be if $\mathrm{f}(x)$ is divided by $(x-1)(x+2)$? Explain.

(ii) Find the remainder when $\mathrm{f}(x)$ is divided by $(x-1)(x+2)$.

(i) The degree of the divisor $(x-1)(x+2)$ is 2.
As the degree of the remainder is always less than the degree of the divisor, the degree of the remainder can be 0 or 1 only. Given that both $(x+2)$ and $(x-1)$ are not a factor of f(x), so the degree of the remainder cannot be 0.
Hence the degree of the remainder is 1.

(ii) Let $\mathrm{f}(x)=(x-1)(x+2)\mathrm{Q}(x)+(ax+b)$,
where Q(x) is a polynomial in x, and a and b are constants.

By Remainder Theorem, $\mathrm{f}(-2)=-19$.
$(-2-1)(-2+2)\mathrm{Q}(x)+(-2a+b)=-19$
$-2a+b=-19 \quad \cdots (1)$

By Remainder Theorem, $\mathrm{f}(1)=2$.
$(1-1)(1+2)\mathrm{Q}(x)+(a+b)=2$
$a+b=2 \quad \cdots (2)$

$(2)-(1):$ $\quad 3a=21$
$a=7$

Sub. $a=7$ into (2): $\quad 7+b=2$
$b=-5$

$\therefore$ The remainder when $\mathrm{f}(x)$ is divided by $(x-1)(x+2)$ is $7x-5$.

7 The remainder when the expression ax^3+bx^2+2x+c is divided by $x-1$ is twice of that when it is divided by $x+1$. Show that $c=3a-b+6$.

Let $\mathrm{f}(x)=ax^3+bx^2+2x+c$.

$\mathrm{f}(1)=a(1)^3+b(1)^2+2(1)+c$
$\mathrm{f}(1)=a+b+2+c$

$\mathrm{f}(-1)=a(-1)^3+b(-1)^2+2(-1)+c$
$\mathrm{f}(-1)=-a+b-2+c$

$\mathrm{f}(1)=2\mathrm{f}(-1)$
$a+b+2+c=2(-a+b-2+c)$
$a+b+2+c=-2a+2b-4+2c$
$2c-c=a+2a+b-2b+2+4$
$c=3a-b+6$ (shown)

8 The expression $2x^2+6x+3$ has the same remainder when divided by $x+p$ and by $x-2q$, where $p\neq-2q$. Find the value of $p-2q$.

Let $\mathrm{f}(x)=2x^2+6x+3$.

$\mathrm{f}(-p)=2(-p)^2+6(-p)+3$
$\mathrm{f}(-p)=2p^2-6p+3$

$\mathrm{f}(2q)=2(2q)^2+6(2q)+3$
$\mathrm{f}(2q)=8q^2+12q+3$

$\mathrm{f}(-p)=\mathrm{f}(2q)$
$2p^2-6p+3=8q^2+12q+3$
$2p^2-6p=8q^2+12q$
$p^2-3p=4q^2+6q$
$p^2-4q^2-3p-6q=0$
$(p+2q)(p-2q)-3(p+2q)=0$
$(p+2q)(p-2q-3)=0$
$p+2q=0$ or $p-2q-3=0$
$p=-2q$ $\quad$ $p-2q=3$
(NA, $p\neq-2q$)

9 The function f is defined by $\mathrm{f}(x) = ax^3 + bx^2 - 5x + 2a$, where a and b are constants.

(i) Given that $x^2 - 3x - 4$ is a factor of $\mathrm{f}(x)$, find the value of a and of b.

(ii) Hence, factorise $\mathrm{f}(x)$ completely.

(i) Let $\mathrm{f}(x) = ax^3 + bx^2 - 5x + 2a$.

$x^2 - 3x - 4 = (x+1)(x-4)$

Both $(x+1)$ and $(x-4)$ are factors of $\mathrm{f}(x)$.

By Factor Theorem, $\mathrm{f}(-1) = 0$.

$a(-1)^3 + b(-1)^2 - 5(-1) + 2a = 0$

$-a + b + 5 + 2a = 0$

$a + b = -5 \quad \cdots (1)$

By Factor Theorem, $\mathrm{f}(4) = 0$.

$a(4)^3 + b(4)^2 - 5(4) + 2a = 0$

$64a + 16b - 20 + 2a = 0$

$66a + 16b = 20$

$33a + 8a = 10 \quad \cdots (2)$

$(1) \times 8: \quad 8a + 8b = -40 \quad \cdots (3)$

$(2) - (3): \quad 25a = 50$

$a = 2$

Sub. $a = 2$ into (1): $2 + b = -5$

$b = -7$

$\therefore \quad a = 2, \quad b = -7$

(ii) $\mathrm{f}(x) = 2x^3 - 7x^2 - 5x + 4$.

$$
\begin{array}{r}
2x - 1 \\
x^2 - 3x - 4 \;\big)\; 2x^3 - 7x^2 - 5x + 4 \\
-\left(2x^3 - 6x^2 - 8x\right) \\
\hline
-x^2 + 3x + 4 \\
-\left(-x^2 + 3x + 4\right) \\
\hline
0
\end{array}
$$

$\therefore \quad \mathrm{f}(x) = \left(x^2 - 3x - 4\right)(2x - 1)$

$\mathrm{f}(x) = (x-4)(x+1)(2x-1)$

10 The function f is defined by $f(x)=(2p+1)x^2+px+2p^2$, where p is a constant.
Find the value(s) of p when $f(x)$ has a factor of

(i) $x-1$, **(ii)** $x+2$, **(iii)** $x+2$ but not $x-1$.

(i) $f(x)=(2p+1)x^2+px+2p^2$.

By Factor Theorem, $f(1)=0$.

$(2p+1)(1)^2+p(1)+2p^2=0$

$2p+1+p+2p^2=0$

$2p^2+3p+1=0$

$(p+1)(2p+1)=0$

$p+1=0$ or $2p+1=0$

$p=-1$ $\quad p=-\frac{1}{2}$

(ii) $f(x)=(2p+1)x^2+px+2p^2$

By Factor Theorem, $f(-2)=0$.

$(2p+1)(-2)^2+p(-2)+2p^2=0$

$8p+4-2p+2p^2=0$

$2p^2+6p+4=0$

$p^2+3p+2=0$

$(p+1)(p+2)=0$

$p+1=0$ or $p+2=0$

$p=-1$ $\quad p=-2$

(iii) When $(x+2)$ is a factor of $f(x)$, $p=-1$ or $p=-2$.

As $(x+1)$ is not a factor of $f(x)$, from part (i), we can deduce that $p\neq-1$.

Hence, $p=-2$.

11 Let $f(x) = 2x^2 + 3px - 2q$ and $g(x) = x^2 + q$, where p and q are non-zero constants. Given that $x - a$, where $a \neq 0$, is a common factor of $f(x)$ and $g(x)$, show that $9p^2 + 16q = 0$.

$f(x) = 2x^2 + 3px - 2q$

By Factor Theorem, $f(a) = 0$.

$2(a)^2 + 3p(a) - 2q = 0$

$2a^2 - 2q = -3pa \quad \cdots (1)$

$g(x) = x^2 + q$

By Factor Theorem, $g(a) = 0$.

$a^2 + q = 0$

$a^2 = -q \quad \cdots (2)$

Sub. (2) into (1):

$2(-q) - 2q = -3pa$

$-2q - 2q = -3pa$

$-4q = -3pa$

$a = \dfrac{4q}{3p}$

Sub. $a = \dfrac{4q}{3p}$ into (2):

$\left(\dfrac{4q}{3p}\right)^2 = -q$

$\dfrac{16q^2}{9p^2} = -q$

$16q^2 = -9p^2q$

$16q^2 + 9p^2q = 0$

$q\left(16q + 9p^2\right) = 0$

$q = 0$ or $16q + 9p^2 = 0$

$(\text{NA since } q \neq 0)$ $\quad 9p^2 + 16q = 0$ (shown)

4.3 Cubic Polynomials and Equations

You will learn how to,

- Use $a^3+b^3=(a+b)(a^2-ab+b^2)$ and $a^3-b^3=(a-b)(a^2+ab+b^2)$.
- Factorise and solve cubic polynomials.

Sum and Difference of Cubes (Definition)

A polynomial in the form a^3+b^3 is called the sum of cubes.
A polynomial in the form a^3-b^3 is called the difference of cubes.

Let us go through an example to learn about the factorization of a^3+b^3 and a^3-b^3.

Example 14

Show that $a+b$ is a factor of $\mathrm{f}(a)=a^3+b^3$ and factorise a^3+b^3.

Hence factorise a^3-b^3 and x^3+8.

$$\begin{aligned}\mathrm{f}(-b)&=(-b)^3+b^3\\&=0\end{aligned}$$

By Factor Theorem, $(a+b)$ is a factor of $\mathrm{f}(a)$.

$$\begin{array}{r|l}
 & a^2 \quad -ab \quad +b^2 \\
a+b & a^3 \qquad\qquad\quad +b^3 \\
 & -(a^3+a^2b) \\
 & \quad -a^2b \qquad +b^3 \\
 & \quad -(-a^2b-ab^2) \\
 & \qquad\quad ab^2+b^3 \\
 & \qquad\quad -(ab^2+b^3) \\
 & \qquad\qquad\quad 0
\end{array}$$

$\therefore a^3+b^3=(a+b)(a^2-ab+b^2)$

or

$\therefore a^3+b^3=(a+b)(a^2+ka+b^2)$

By comparing coefficients of a,

$b^2+kb=0$

$kb=-b^2$

$k=-b$

$\therefore a^3+b^3=(a+b)(a^2-ab+b^2)$

Let $\mathrm{g}(x)=a^3-b^3$

$$\begin{aligned}\mathrm{g}(b)&=(b)^3-b^3\\&=0\end{aligned}$$

$\therefore$ By Factor Theorem, $(a-b)$ is a factor of $\mathrm{g}(x)$.

Replacing "b" in $a^3+b^3=(a+b)(a^2-ab+b^2)$ with "$-b$",

we will get $a^3-b^3=(a-b)(a^2+ab+b^2)$.

$$\begin{aligned}\therefore\quad x^3+8&=x^3+2^3\\&=(x+2)(x^2-2x+4)\end{aligned}$$

Through observation, we can conclude that the sum and difference of cubes are

$$\boxed{\begin{aligned}a^3+b^3&=(a+b)(a^2-ab+b^2)\\a^3-b^3&=(a-b)(a^2+ab+b^2)\end{aligned}}$$

Factorising Cubic Polynomials
Imagine you need to make a cuboid which has a volume of 120 cm^3. Suppose the cuboid has the dimensions x cm by $(x-1)$ cm by $(x-2)$ cm, how can you determine its dimension?

$x(x-1)(x-2) = 120$
$x^3 - 3x^2 + 2x - 120 = 0$

Let $f(x) = x^3 - 3x^2 + 2x - 120$.

$f(6) = 6^3 - 3(6)^2 + 2(6) - 120$
$= 0$
(How do you know which values to try?)
Therefore, $(x - 6)$ is a factor of $f(x)$.
$f(x) = (x - 6)(x^2 + 3x + 20)$
$f(x) = 0$
$(x - 6)(x^2 + 3x + 20) = 0$
$x - 6 = 0$ or $x^2 + 3x + 20 = 0$
$x = 6$ or $x = \dfrac{-3\pm\sqrt{-72}}{2}$ (NA)
Therefore, the dimensions of the cuboid are 6 cm by 5 cm by 4 cm.

General steps for solving cubic equations of the form $px^3 + qx^2 + rx + s = 0$

<u>Step 1</u>: Let $f(x) = px^3 + qx^2 + rx + s$. Then, by trial and error, substitute possible integer values of α until we obtain $f(\alpha) = 0$ (Factor Theorem). This would tell us that $(x-\alpha)$ is a linear factor of $f(x)$.

<u>Step 2</u>: Rewrite $f(x)$ in the form $(x-\alpha)(ax^2 + bx + c)$.

Obtain the remaining quadratic equation $ax^2 + bx + c$ by long division, synthetic division or observation.

<u>Step 3</u>: Solve the quadratic equation $ax^2 + bx + c = 0$ to find the remaining roots.

<u>Example 15</u>
Factorise the polynomial $x^3 - 2x^2 - 4x + 8$ completely.

Let $f(x) = x^3 - 2x^2 - 4x + 8$.
$f(2) = (2)^3 - 2(2)^2 - 4(2) + 8$
$= 0$
By Factor Theorem, $(x-2)$ is a factor of $f(x)$.

$x^3 - 2x^2 - 4x + 8 = (x-2)(x^2 + px - 4)$
By comparing the coefficient of x, $-4 = -4 - 2p$
$-2p = 0$
$p = 0$

$\therefore x^3 - 2x^2 - 4x + 8 = (x-2)(x^2 - 4)$
$= (x-2)(x-2)(x+2)$
$= (x-2)^2(x+2)$

Solving Cubic Equations
After we have factorised a cubic polynomial $f(x)$, we can proceed to find the roots of the corresponding cubic equation $f(x) = 0$.

For any cubic polynomial $f(x)$,
- If $x - k$, where k is a real number, is a factor of $f(x)$, then $x = k$ is a root of $f(x) = 0$.
- If the graph of $y = f(x)$ intersects the x-axis at $x = k$, then $x = k$ is a linear factor of $f(x)$.
- The number of x-intercepts of the graph is the number of distinct real roots of $f(x) = 0$.
- The graph of $y = f(x)$ can intersect the x-axis at up to three points. Thus, $f(x) = 0$ has at most three distinct real roots.

Example 16

Solve $3x^3+5x^2=3x+2$, giving your answers in surd form, where necessary.

$3x^3+5x^2=3x+2$

$3x^3+5x^2-3x-2=0$

Let $\mathrm{f}(x)=3x^3+5x^2-3x-2$.

$\mathrm{f}(-2)=3(-2)^3+5(-2)^2-3(-2)-2$

$\quad\quad=0$

By Factor Theorem, $(x+2)$ is a factor of $\mathrm{f}(x)$.

$3x^3+5x^2-3x-2=(x+2)(3x^2+px-1)$

By comparing the coefficient of x, $-3=-1+2p$

$2p=-2$

$p=-1$

$(x+2)(3x^2-x-1)=0$

$x+2=0$ or $3x^2-x-1=0$

$x=-2$

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(3)(-1)}}{2(3)}$$

$$x=\frac{1\pm\sqrt{13}}{6}$$

Example 17

Given that kx^3+2x^2+2x+3 and kx^3-2x+9 have a common linear factor, determine the possible values of k.

Let the factor be $(x-a)$.

$ka^3+2a^2+2a+3=0 \quad \cdots \ (1)$

$ka^3-2a+9=0 \quad \cdots \ (2)$

$(1)-(2):\quad 2a^2+4a-6=0$

$a^2+2a-3=0$

$(a+3)(a-1)=0$

$a+3=0$ or $a-1=0$

$a=-3$ or $a=1$

If $(x+3)$ is a factor, $kx^3-2x+9=(x+3)(kx^2+px+3)$ comparing coefficients of x, $-2=3+3p$ $p=-\frac{5}{3}$ comparing coefficients of x^2, $0=3k+p$ $3k=\frac{5}{3}$ $k=\frac{5}{9}$	If $(x-1)$ is a factor, $kx^3-2x+9=(x-1)(kx^2+qx-9)$ comparing coefficients of x, $-2=-9-q$ $q=-7$ comparing coefficients of x^2, $0=-k+q$ $k=q$ $k=-7$ $\therefore$ Possible values of k are -7 or $\frac{5}{9}$.

[Can use the other equation .... we will have same k values.]

If $(x+3)$ is a factor, $kx^3+2x^2+2x+3=(x+3)(kx^2+mx+1)$

comparing coefficients of x, $2=1+3m$

$m=\frac{1}{3}$

comparing coefficients of x^2, $2=3k+m$

$3k=2-\frac{1}{3}$

$k=\frac{5}{9}$

If $(x-1)$ is a factor, $kx^3+2x^2+2x+3=(x-1)(kx^2+nx-3)$

comparing coefficients of x, $2=-3-n$

$n=-5$

comparing coefficients of x^2, $2=-k+n$

$k=-5-2$

$k=-7$

Class Practice 3:

1 Factorise each of the following polynomials.

(a) x^3-64 **(b)** $27a^3+125x^3$ **(c)** $8+27x^3$ **(d)** $432x^3-2y^3$

(a) $x^3-64=x^3-4^3$

$$=(x-4)(x^2+4x+16)$$

(b) $27a^3+125x^3=(3a)^3+(5x)^3$

$$=(3a+5x)(9a^2-15ax+25x^2)$$

(c) $8+27x^3=2^3+(3x)^3$

$$=(2+3x)(4-6x+9x^2)$$

(d) $432x^3-2y^3=2(216x^3-y^3)$

$$=2\left[(6x)^3-y^3\right]$$

$$=2(6x-y)(36x^2+6xy+y^2)$$

2 Solve each of the following equations.

(a) $4x^3+18=7x^2+21x$ **(b)** $x^3+4=x(x+4)$

(a)
$$4x^3+18=7x^2+21x$$
$$4x^3-7x^2-21x+18=0$$

Let $\mathrm{f}(x)=4x^3-7x^2-21x+18$.

$$\mathrm{f}(3)=4(3)^3-7(3)^2-21(3)+18$$
$$=0$$

By Factor Theorem, $(x-3)$ is a factor of $\mathrm{f}(x)$.

$$4x^3-7x^2-21x+18=(x-3)(4x^2+px-6)$$

By comparing the coefficient of x^2, $-7=p-12$

$p=5$

$$(x-3)(4x^2+5x-6)=0$$
$$(x-3)(x+2)(4x-3)=0$$

$x-3=0$ or $x+2=0$ or $4x-3=0$

$x=3$ $\quad x=-2$ $\quad x=\dfrac{3}{4}$

(b)
$$x^3+4=x(x+4)$$
$$x^3+4=x^2+4x$$
$$x^3-x^2-4x+4=0$$

Let $\mathrm{f}(x)=x^3-x^2-4x+4$.

$$\mathrm{f}(1)=(1)^3-(1)^2-4(1)+4$$
$$=0$$

By Factor Theorem, $(x-1)$ is a factor of $\mathrm{f}(x)$.

$$x^3-x^2-4x+4=(x-1)(x^2+px-4)$$

By comparing the coefficient of x^2, $-1=p-1$

$p=0$

$$(x-1)(x^2-4)=0$$
$$(x-1)(x+2)(x-2)=0$$

$x-1=0$ or $x+2=0$ or $x-2=0$

$x=1$ $\quad x=-2$ $\quad x=2$

3 Solve $2x^3+6x-6=(13x-6)(x-1)$. Leave your answers in surd form, where necessary.

$2x^3+6x-6=(13x-6)(x-1)$
$2x^3+6x-6=13x^2-13x-6x+6$
$2x^3-13x^2+25x-12=0$

Let f $(x)=2x^3-13x^2+25x-12$.
$\text{f}(3)=2(3)^3-13(3)^2+25(3)-12$
$\quad =0$
By Factor Theorem, $(x-3)$ is a factor of f (x).

$2x^3-13x^2+25x-12=(x-3)(2x^2+px+4)$
By comparing the coefficient of x, $25=4-3p$
$3p=-21$
$p=-7$

$(x-3)(2x^2-7x+4)=0$

$x-3=0$ or $2x^2-7x+4=0$

$x=3$ $\quad x=\dfrac{-(-7)\pm\sqrt{(-7)^2-4(2)(4)}}{2(2)}$

$x=\dfrac{7\pm\sqrt{17}}{4}$

4 **(i)** Show that $2x+1$ is a factor of $2x^3-x^2+3x+2$.

(ii) Show that the equation $2x^3+4=(x-1)(x-2)$ has only 1 real root.

(i) Let f $(x)=2x^3-x^2+3x+2$.

$$\text{f}\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^3-\left(-\frac{1}{2}\right)^2+3\left(-\frac{1}{2}\right)+2$$
$$=0$$

By Factor Theorem, $(2x+1)$ is a factor of f (x). (shown)

(ii) $2x^3+4=(x-1)(x-2)$

$2x^3+4=x^2-3x+2$

$2x^3-x^2+3x+2=0$

Let $2x^3-x^2+3x+2=(2x+1)(x^2+px+2)$.

By comparing the coefficient of x, $3=4+p$

$p=-1$

$(2x+1)(x^2-x+2)=0$

$2x+1=0$ or $x^2-x+2=0$

$x=-\frac{1}{2}$ $\qquad x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(2)}}{2(1)}$

$x=\frac{1\pm\sqrt{-7}}{2}$ (NA)

Alternative working:
For $x^2-x+2=0$,
discriminant $=(-1)^2-4(1)(2)$
$=-7 < 0$
Thus $x^2-x+2=0$ has no real roots.

Since $x=-\frac{1}{2}$ is the only real solution, the equation $2x^3+4=(x-1)(x-2)$ has only 1 real root.

5 **(a)** The coefficient of x^3 in the cubic polynomial $\text{f}(x)$ is 5. The roots of the equation $\text{f}(x)=0$ are -4, -1 and 3. Find an expression for $\text{f}(x)$.

(b) The graph of a cubic polynomial g(x) meets the x-axis at -2, 1 and 2, and passes through the point $(0, 12)$. Find an expression for $\text{g}(x)$.

(a) f $(x)=5(x-3)(x+4)(x+1)$

(b) Let g $(x)=k(x-1)(x-2)(x+2)$.

At $(0, 12)$, $12=k(0-1)(0-2)(0+2)$

$k=3$

Hence, g $(x)=k(x-1)(x-2)(x+2)$.

6 It is given that $\mathrm{f}(x)=2x^3+9x^2+7x+3$. $\mathrm{f}(x)$ has a remainder of 9 when divided by $x-k$, where k is a constant.

(i) Show that $2k^3+9k^2+7k-6=0$.

(ii) Hence find the values of k.

(i) $\mathrm{f}(x)=2x^3+9x^2+7x+3$

By Remainder Theorem, $\mathrm{f}(k)=9$

$2k^3+9k^2+7k+3=9$

$2k^3+9k^2+7k-6=0$ (shown)

(ii) Let $\mathrm{g}(k)=2k^3+9k^2+7k-6$.

$$\mathrm{g}(-2)=2(-2)^3+9(-2)^2+7(-2)-6$$
$$=0$$

By Factor Theorem, $(k+2)$ is a factor of $\mathrm{g}(k)$.

$$2k^3+9k^2+7k-6=(k+2)(2k^2+pk-3)$$

By comparing the coefficient of k, $7=-3+2p$

$2p=10$

$p=5$

$$(k+2)(2k^2+5k-3)=0$$
$$(k+2)(2k-1)(k+3)=0$$

$k+2=0$ or $2k-1=0$ or $k+3=0$

$k=-2$ $\quad k=\frac{1}{2}$ $\quad k=-3$

7 Find the coordinates of the points where the curves $y = 2x^3$ and $y = (2-x)(5x+6)$ intersect.

$y = 2x^3 \quad \cdots (1)$

$y = (2-x)(5x+6) \quad \cdots (2)$

Sub. (1) into (2):
$$2x^3 = (2-x)(5x+6)$$
$$2x^3 = 10x + 12 - 5x^2 - 6x$$
$$2x^3 + 5x^2 - 4x - 12 = 0$$

Let f $(x) = 2x^3 + 5x^2 - 4x - 12$.

$$\begin{aligned} \text{f}(-2) &= 2(-2)^3 + 5(-2)^2 - 4(-2) - 12 \\ &= 0 \end{aligned}$$

By Factor Theorem, $(x+2)$ is a factor of f (x).

$$2x^3 + 5x^2 - 4x - 12 = (x+2)(2x^2 + px - 6)$$

By comparing the coefficient of x^2, $\quad 5 = p + 4$

$$p = 1$$

$$(x+2)(2x^2 + x - 6) = 0$$
$$(x+2)(x+2)(2x-3) = 0$$

$x + 2 = 0$ or $x + 2 = 0$ or $2x - 3 = 0$

$x = -2 \qquad x = -2 \qquad x = \frac{3}{2}$

When $x = -2$, $y = 2(-2)^3$

$y = -16$

When $x = \frac{3}{2}$, $y = 2\left(\frac{3}{2}\right)^3$

$y = \frac{27}{4}$

Points of intersection are at $(-2, -16)$ and $\left(\frac{3}{2}, \frac{27}{4}\right)$.

8 It is given that $f(x) = x^4 + ax^3 - x^2 + bx - 12$ is exactly divisible by $(x-2)(x+1)$, where a and b are constants.

(i) Find the value of a and of b.

(ii) Hence solve the equation $f(x) = 0$.

(i) $f(x) = x^4 + ax^3 - x^2 + bx - 12.$

By Factor Theorem, $f(2) = 0$.

$(2)^4 + a(2)^3 - (2)^2 + b(2) - 12 = 0$

$16 + 8a - 4 + 2b - 12 = 0$

$8a + 2b = 0$

$4a + b = 0 \quad \cdots \text{ (1)}$

By Factor Theorem, $f(-1) = 0$.

$(-1)^4 + a(-1)^3 - (-1)^2 + b(-1) - 12 = 0$

$1 - a - 1 - b - 12 = 0$

$a + b = -12 \quad \cdots \text{ (2)}$

$(1) - (2)$: $\quad 3a = 12$

$a = 4$

Sub. $a = 4$ into (2): $\quad 4 + b = -12$

$b = -16$

$\therefore \quad a = 4, \quad b = -16$

(ii) $f(x) = x^4 + 4x^3 - x^2 - 16x - 12$

$(x-2)(x+1) = x^2 - x - 2$

$$
\begin{array}{r l}
 & x^2 + 5x + 6 \\
x^2 - x - 2 \,\big) & x^4 + 4x^3 - x^2 - 16x - 12 \\
 & -\left(x^4 - x^3 - 2x^2\right) \\
\hline
 & 5x^3 + x^2 - 16x \\
 & -\left(5x^3 - 5x^2 - 10x\right) \\
\hline
 & 6x^2 - 6x - 12 \\
 & -\left(6x^2 - 6x - 12\right) \\
\hline
 & 0
\end{array}
$$

$(x^2 - x - 2)(x^2 + 5x + 6) = 0$

$(x-2)(x+1)(x+2)(x+3) = 0$

$x - 2 = 0$ or $x + 1 = 0$ or $x + 2 = 0$ or $x + 3 = 0$

$x = 2 \qquad x = -1 \qquad x = -2 \qquad x = -3$

9 Given that x^2+2x-3 is a factor of $\mathrm{f}(x)$, where $\mathrm{f}(x)=x^4+6x^3+2ax^2+bx-3a$, find

(i) the values of the constants a and b,

(ii) the other quadratic factor of $\mathrm{f}(x)$. Explain why the equation $\mathrm{f}(x)=0$ has only two distinct real roots.

(i) $\mathrm{f}(x)=x^4+6x^3+2ax^2+bx-3a$

$x^2+2x-3=(x-1)(x+3)$

By Factor Theorem, $\mathrm{f}(1)=0$.

$(1)^4+6(1)^3+2a(1)^2+b(1)-3a=0$

$-a+b=-7 \quad \cdots (1)$

By Factor Theorem, $\mathrm{f}(-3)=0$.

$(-3)^4+6(-3)^3+2a(-3)^2+b(-3)-3a=0$

$81-162+18a-3b-3a=0$

$15a-3b=81$

$5a-b=27 \quad \cdots (2)$

$(1)+(2): \quad 4a=20$

$a=5$

Sub. $a=5$ into (1): $-5+b=-7$

$b=-2$

$\therefore \quad a=5, \quad b=-2$

(ii) $\mathrm{f}(x)=x^4+6x^3+10x^2-2x-15$

$x^4+6x^3+10x^2-2x-15=(x^2+2x-3)(x^2+px+5)$

By comparing the coefficient of x^2,

$10=5+2p-3$

$2p=8$

$p=4$

Hence the other quadratic factor of $\mathrm{f}(x)$ is x^2+4x+5.

For $x^2+2x-3=0$, discriminant $=2^2-4(1)(-3)$

$=16 > 0$

So, $x^2+2x-3=0$ has two distinct real roots.

For $x^2+4x+5=0$, discriminant $=4^2-4(1)(5)$

$=-4<0$

So, $x^2+4x+5=0$ has no real roots.

Thus, $\mathrm{f}(x)=0$ has only two distinct real roots.

10 The function f is defined by $\mathrm{f}(x) = 4x^3 + 2ax^2 - 2x^2 + ax - a$, where a is a constant.

(i) Show that $2x - 1$ is a factor of $\mathrm{f}(x)$.

(ii) Factorise $\mathrm{f}(x)$ completely.

(iii) Find the range of values of a for which the equation $\mathrm{f}(x) = 0$ has only one real root.

(i)
$$\mathrm{f}\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 + 2a\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^2 + a\left(\frac{1}{2}\right) - a$$
$$= 0$$

By Factor Theorem, $(2x-1)$ is a factor of f (x). (shown)

(ii) $\mathrm{f}(x) = 4x^3 + 2ax^2 - 2x^2 + ax - a$

$$4x^3 + 2ax^2 - 2x^2 + ax - a = (2x-1)(2x^2 + px + a)$$

By comparing the coefficient of x^2, $2a - 2 = 2p - 2$
$$2p = 2a$$
$$p = a$$

$$\mathrm{f}(x) = 4x^3 + 2ax^2 - 2x^2 + ax - a$$
$$\mathrm{f}(x) = (2x-1)(2x^2 + ax + a)$$

(iii) For $\mathrm{f}(x) = 0$ to have only one real root, the equation $2x^2 + ax + a = 0$ cannot have any real roots.

So, for $2x^2 + ax + a = 0$, the discriminant < 0.
$$a^2 - 4(2)(a) < 0$$
$$a^2 - 8a < 0$$
$$a(a-8) < 0$$

$0 < a < 8$

4.4 Partial Fractions

You will learn how to,
- Express a proper algebraic fraction in partial fractions,
- Identify whether an algebraic fraction in a proper or an improper fraction,
- Express an improper algebraic fraction in partial fractions.

Proper Fractions and Improper Fractions (Definition)

A ***rational*** expression is an expression of the form $\frac{\mathrm{p}(x)}{\mathrm{q}(x)}$, where p($x$) and q($x$) are polynomials, and $\mathrm{q}(x) \neq 0$.

If the degree of the numerator p(x) is ***less than*** that of the denominator q(x), then we say that the fraction $\frac{\mathrm{p}(x)}{\mathrm{q}(x)}$ is a *proper fraction.* Otherwise, we say that it is an *improper fraction.*

QUESTION: What if the degree of the numerator is equal to the degree of the denominator? Is the fraction proper or improper?

Improper fraction

Example 18

State whether the fraction is a proper or improper fraction, and if they are rational or irrational.

$\frac{x+2}{x^2+4}$	Proper, rational	$\frac{(x-3)(2-3x)}{x^2+9}$	Improper, rational
$\frac{x^2+x-7}{(x+2)(x-3)}$	Improper, rational	$\frac{4x^3}{(x^2-1)(x^2+1)}$	Proper, rational
$\frac{x^3+1}{x^2-5}$	Improper, rational	$\frac{2x+\sqrt{x}}{x-1}$	Irrational (note that proper or improper describe only rational expression)

Partial Fraction Decomposition

We have learnt to add or subtract two proper fractions,

e.g. $\frac{2}{x+1}+\frac{3}{x+2}=\frac{2(x+2)+3(x+1)}{(x+1)(x+2)}=\frac{5x+7}{(x+1)(x+2)}$

Can the *reverse* be done? Can we find a way to express $\frac{5x+7}{(x+1)(x+2)}$ as the sum of 2 (or more) simpler fractions $\left(\text{i.e. } \frac{2}{x+1}+\frac{3}{x+2}\right)$?

This process is known as ***decomposing*** $\frac{5x+7}{(x+1)(x+2)}$ into its *partial fractions.*

Combining algebraic fractions into a single fraction and expressing the single fraction in partial fractions are **reverse processes** of each other.
One purpose of writing an algebraic fraction into partial fractions is for applying integration rules; you need to rewrite some algebraic fractions into simpler fractions in order to apply the integration rules that you have learnt or will be learning.

We have a standard procedure for writing a proper fraction in partial fractions.
In the event that the fraction is improper, we need to write the improper fraction as a sum of a polynomial (which can be a constant) and a proper fraction; similar to writing $\frac{10}{3}=3+\frac{1}{3}$,

we write $\frac{x^3+1}{x^2-5}=\frac{x\left(x^2-5\right)+5x+1}{x^2-5}=x+\frac{5x+1}{x^2-5}$.

Then we apply the standard procedure for writing the proper fraction part into partial fractions where needed. Note that the quotient (x) and remainder $(5x+1)$ can be obtained by observing (as above) or by using the long division algorithm.

<u>Procedure for Writing in Partial Fractions</u>

Check whether the given algebraic fraction is <u>**proper**</u> or <u>**improper**</u>.

- If the given algebraic fraction is <u>proper</u>, apply the three rules in **Section 3.6.1** to express it in partial fractions.

- If the given algebraic fraction is <u>improper</u>, we will apply the extra step of writing the improper fraction as a sum of polynomials and a proper algebraic fraction, then use the three rules in **Section 3.6.1**.

<u>Decomposing Proper Fractions</u>

<u>Rule #1:</u> Distinct Linear Factors

To each linear factor $(ax+b)$ in the denominator, there corresponds a partial fraction of the form $\frac{A}{ax+b}$.

Example 19

Express the following rational expressions into its partial fractions.

(a) $\dfrac{3x+4}{(x+1)(x+2)}$ **{numerator is of degree 1 while denominator is of degree 2}**

Let $\dfrac{3x+4}{(x+1)(x+2)} = \dfrac{A}{x+1}+\dfrac{B}{x+2}$, where A and B are constants.

$$\frac{3x+4}{(x+1)(x+2)} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$\therefore \quad 3x+4 = A(x+2)+B(x+1)$

When $x=-1$, $3(-1)+4 = A(-1+2)+B(-1+1)$
$A=1$

When $x=-2$, $3(-2)+4 = A(-2+2)+B(-2+1)$
$-B=-2$
$B=2$

$$\therefore \frac{3x+4}{(x+1)(x+2)} = \frac{1}{x+1}+\frac{2}{x+2}$$

(b) $\dfrac{x^2+3}{(x+1)(x+2)(x-3)}$ **{numerator is of degree 2 while denominator is of degree 3}**

Let $\dfrac{x^2+3}{(x+1)(x+2)(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x-3}$, where A, B and C are constants.

$$\frac{x^2+3}{(x+1)(x+2)(x-3)} = \frac{A(x+2)(x-3)+B(x+1)(x-3)+C(x+1)(x+2)}{(x+1)(x+2)(x-3)}$$

$\therefore \quad x^2+3 = A(x+2)(x-3)+B(x+1)(x-3)+C(x+1)(x+2)$

When $x=-1$, $(-1)^2+3 = A(-1+2)(-1-3)+B(-1+1)(-1-3)+C(-1+1)(-1+2)$
$-4A=4$
$A=-1$

When $x=-2$, $(-2)^2+3 = A(-2+2)(-2-3)+B(-2+1)(-2-3)+C(-2+1)(-2+2)$
$5B=7$
$B=\dfrac{7}{5}$

When $x=3$, $(3)^2+3 = A(3+2)(3-3)+B(3+1)(3-3)+C(3+1)(3+2)$
$20C=12$
$C=\dfrac{3}{5}$

$\therefore \dfrac{x^2+3}{(x+1)(x+2)(x+2)} = -\dfrac{1}{x+1} + \dfrac{7}{5(x+2)} + \dfrac{3}{5(x+2)}$

<u>Rule #2:</u> Irreducible Quadratic Factors
To each quadratic factor (ax^2+bx+c) (which <u>cannot</u> be factorised) in the denominator, there corresponds a partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$.

Note that if the quadratic factor in the denominator is factorisable, we will factorise it and apply Rule #1 instead.

Example 20

Express the following rational expressions into its partial fractions.

(a) $\dfrac{3x+1}{(x+1)(x^2+1)}$ **{numerator is of degree 1 while denominator is of degree 3}**

Let $\dfrac{3x+1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$, where A, B and C are constants.

$$\frac{3x+1}{(x+1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$\therefore \quad 3x+1 = A(x^2+1)+(Bx+C)(x+1)$

By comparing the coefficients of

x^2: $\quad A+B=0 \quad \cdots \text{ (1)}$

x: $\quad B+C=3 \quad \cdots \text{ (2)}$

x^0: $\quad A+C=1 \quad \cdots \text{ (3)}$

$(1)-(3)$: $\quad B-C=-1 \quad \cdots \text{ (4)}$

$(2)+(4)$: $\quad 2B=2$

$B=1$

Sub. $B=1$ into (1): $\quad A+1=0$

$A=-1$

Sub. $B=1$ into (2): $\quad 1+C=3$

$C=2$

$$\therefore \quad \frac{3x+1}{(x+1)(x^2+1)} = -\frac{1}{x+1} + \frac{x+2}{x^2+1}$$

(b) $\dfrac{4x^3}{(x^2-1)(x^2+1)}$ **{numerator is of degree 3 while denominator is of degree 4}**

$\dfrac{4x^3}{(x^2-1)(x^2+1)} = \dfrac{4x^3}{(x-1)(x+1)(x^2+1)}$ **{factorise the denominator whenever possible}**

Let $\dfrac{4x^3}{(x^2-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$, where A, B, C and D are constants.

$$\frac{4x^3}{(x^2-1)(x^2+1)} = \frac{A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)(x+1)}{(x-1)(x+1)(x^2+1)}$$

$\therefore \quad 4x^3 = A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x-1)(x+1)$

When $x=-1$, $4(-1)^3 = A(-1+1)\left[(-1)^2+1\right]+B(-1-1)\left[(-1)^2+1\right]+\left[C(-1)+D\right](-1-1)(-1+1)$
$-4B=-4$
$B=1$

When $x=1$, $4(1)^3 = A(1+1)(1^2+1)+B(1-1)(1^2+1)+\left[C(1)+D\right](1-1)(1+1)$
$4A=4$
$A=1$

When $x=0$, $A=1$ and $B=1$, $0=(1)(1)(1)+(1)(-1)(1)+D(-1)(1)$
$D=0$

When $x=2$, $A=1$, $B=1$ and $D=0$, $4(2)^3=(1)(2+1)(2^2+1)+(1)(2-1)(2^2+1)+2C(2-1)(2+1)$
$32=15+5+6C$
$6C=12$
$C=2$

$$\therefore \quad \frac{4x^3}{(x^2-1)(x^2+1)} = \frac{1}{x-1}+\frac{1}{x+1}+\frac{2x}{x^2+1}$$

<u>Rule #3:</u> Repeated Linear Factors
To each linear factor $(ax+b)$ repeated k times in the denominator, there corresponds k partial fraction of the form $\dfrac{A_1}{ax+b}, \dfrac{A_2}{(ax+b)^2}, \dfrac{A_3}{(ax+b)^3}, \ldots \dfrac{A_k}{(ax+b)^k}$.
In particular, $(ax+b)^2$ in the denominator corresponds to partial fractions of the form $\dfrac{A_1}{ax+b}, \dfrac{A_2}{(ax+b)^2}$.

Example 21

Express the following rational expressions into its partial fractions.

(a) $\dfrac{2x^2+1}{(x+1)^3}$ **{numerator is of degree 2 while denominator is of degree 3}**

Let $\dfrac{2x^2+1}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}$, where A, B and C are constants.

$$\frac{2x^2+1}{(x+1)^3} = \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3}$$

$$\therefore \quad 2x^2+1 = A(x+1)^2 + B(x+1) + C$$

By comparing the coefficients of

x^2: $\quad A = 2$

x: $\quad 2A + B = 0 \quad \cdots \quad (1)$

x^0: $\quad A + B + C = 1 \quad \cdots \quad (2)$

Sub. $A = 2$ into (1): $\quad 2(2) + B = 0$

$B = -4$

Sub. $A = 2$ and $B = -4$ into (2): $\quad 2 - 4 + C = 1$

$C = 3$

$$\therefore \quad \frac{2x^2+1}{(x+1)^3} = \frac{2}{x+1} - \frac{4}{(x+1)^2} + \frac{3}{(x+1)^3}$$

(b) $\dfrac{x+7}{(x-1)(x+1)^2}$ **{numerator is of degree 1 while denominator is of degree 3}**

Let $\dfrac{x+7}{(x-1)(x+1)^2}=\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$, where A, B and C are constants

$$\frac{x+7}{(x-1)(x+1)^2}=\frac{A(x+1)^2+B(x-1)(x+1)+C(x-1)}{(x-1)(x+1)^2}$$

$\therefore \quad x+7=A(x+1)^2+B(x-1)(x+1)+C(x-1)$

When $x=1$, $\quad 1+7=A(1+1)^2+B(1-1)(1+1)+C(1-1)$

$4A=8$

$A=2$

By comparing the coefficients of

x^2 : $\quad A+B=0 \quad \cdots \quad (1)$

x^0 : $\quad A-B-C=7 \quad \cdots \quad (2)$

Sub. $A=2$ into (1): $\quad 2+B=0$

$B=-2$

Sub. $A=2$ and $B=-2$ into (2): $\quad 2-(-2)-C=7$

$C=-3$

$$\therefore \quad \frac{x+7}{(x-1)(x+1)^2}=\frac{2}{x-1}-\frac{2}{x+1}-\frac{3}{(x+1)^2}$$

Decomposing Improper Fractions

Step 1: Use long division to express the improper fraction, $\frac{p(x)}{q(x)}$ as $f(x)+\frac{g(x)}{q(x)}$, where $\frac{g(x)}{q(x)}$ is a **proper** fraction.

Dividend = Divisor × Quotient + remainder
OR
$p(x) = f(x) \times q(x) + g(x)$

Step 2: Decompose **proper** fraction $\frac{g(x)}{q(x)}$ into its partial fractions using the three rules that you have learnt in **Section 3.6.1**.

*REMINDER: Even when the degree of the numerator is **equal** to the degree of the denominator, the fraction is still considered **improper**.*

Example 22

Express $\frac{x^2+x-7}{(x+2)(x-3)}$ into its partial fractions.

$$\frac{x^2+x-7}{(x+2)(x-3)} = \frac{x^2+x-7}{x^2-x-6} = \frac{(x^2-x-6)+2x-1}{x^2-x-6}$$ (alternative method to long division)

$$\therefore \quad \frac{x^2+x-7}{(x+2)(x-3)} = 1+\frac{2x-1}{(x+2)(x-3)}$$ **{the quotient in this case is the constant 1}**

Let $\frac{2x-1}{(x+2)(x-3)} = \frac{A}{x+2}+\frac{B}{x-3}$, where A and B are constants

$$\frac{2x-1}{(x+2)(x-3)} = \frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$$

$$\therefore \quad 2x-1 = A(x-3)+B(x+2)$$

When $x=-2$, $2(-2)-1 = A(-2-3)+B(-2+2)$

$$-5A=-5$$
$$A=1$$

When $x=3$, $2(3)-1 = A(3-3)+B(3+2)$

$$5B=5$$
$$B=1$$

$$\therefore \quad \frac{2x-1}{(x+2)(x-3)} = \frac{1}{x+2}+\frac{1}{x-3}$$

Thus, $\frac{x^2+x-7}{(x+2)(x-3)} = 1+\frac{1}{x+2}+\frac{1}{x-3}$.

Class Practice 4:

1 Express $\dfrac{(x+3)}{(x-2)(x^2-4)}$ in partial fractions.

$$\frac{x+3}{(x-2)(x^2-4)} = \frac{x+3}{(x-2)(x+2)(x-2)}$$
$$= \frac{x+3}{(x+2)(x-2)^2}$$

Let $\dfrac{x+3}{(x+2)(x-2)^2} = \dfrac{A}{x+2} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$, where A, B and C are constants.

$$\frac{x+3}{(x+2)(x-2)^2} = \frac{A(x-2)^2 + B(x+2)(x-2) + C(x+2)}{(x+2)(x-2)^2}$$

$\therefore \quad x+3 = A(x-2)^2 + B(x+2)(x-2) + C(x+2)$

When $x=2$, $2+3 = A(2-2)^2 + B(2+2)(2-2) + C(2+2)$
$4C = 5$
$C = \dfrac{5}{4}$

When $x=-2$, $-2+3 = A(-2-2)^2 + B(-2+2)(-2-2) + C(-2+2)$
$16A = 1$
$A = \dfrac{1}{16}$

When $x=0$, $A = \dfrac{1}{16}$ and $C = \dfrac{5}{4}$, $\quad 3 = \dfrac{1}{16}(-2)^2 + B(2)(-2) + \dfrac{5}{4}(2)$
$4B = -\dfrac{1}{4}$
$B = -\dfrac{1}{16}$

$$\therefore \quad \frac{x+3}{(x+2)(x-2)^2} = \frac{1}{16(x+2)} - \frac{1}{16(x-2)} + \frac{5}{4(x-2)^2}$$

2 Express $\dfrac{x+3}{(x-2)(x^2+4)}$ in partial fractions.

Let $\dfrac{x+3}{(x-2)(x^2+4)}=\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+4}$, where A, B and C are constants.

$$\frac{x+3}{(x-2)(x^2+4)}=\frac{A(x^2+4)+(Bx+C)(x-2)}{(x-2)(x^2+4)}$$

$\therefore\quad x+3=A(x^2+4)+(Bx+C)(x-2)$

When $x=2$, $\quad 2+3=A(2^2+4)+(2B+C)(2-2)$

$$8A=5$$

$$A=\frac{5}{8}$$

When $x=0$ and $A=\dfrac{5}{8}$, $\quad 3=\dfrac{5}{8}(4)+(C)(-2)$

$$2C=-\frac{1}{2}$$

$$C=-\frac{1}{4}$$

When $x=1$, $A=\dfrac{5}{8}$ and $C=-\dfrac{1}{4}$, $\quad 1+3=\dfrac{5}{8}(1^2+4)+\left(B-\dfrac{1}{4}\right)(1-2)$

$$4=\frac{25}{8}-B+\frac{1}{4}$$

$$B=-\frac{5}{8}$$

$$\therefore\quad \frac{x+3}{(x-2)(x^2+4)}=\frac{5}{8(x-2)}+\left(-\frac{5x}{8}-\frac{1}{4}\right)\left(\frac{1}{x^2+4}\right)$$

$$\frac{x+3}{(x-2)(x^2+4)}=\frac{5}{8(x-2)}+\left(-\frac{5x+2}{8}\right)\left(\frac{1}{x^2+4}\right)$$

$$\frac{x+3}{(x-2)(x^2+4)}=\frac{5}{8(x-2)}-\frac{5x+2}{8(x^2+4)}$$

3 Express $\dfrac{6x^3-5x^2-19x+28}{(2x+3)(x-1)^2}$ in partial fractions.

$$\begin{aligned}(2x+3)(x-1)^2 &= (2x+3)(x^2-2x+1)\\ &= 2x^3-4x^2+2x+3x^2-6x+3\\ &= 2x^3-x^2-4x+3\end{aligned}$$

$$\begin{array}{r|l} & \quad 3 \\ \hline 2x^3-x^2-4x+3 & \; 6x^3-5x^2-19x+28 \\ & -(6x^3-3x^2-12x+9) \\ \hline & \qquad -2x^2-7x+19 \end{array}$$

$$\frac{6x^3-5x^2-19x+28}{(2x+3)(x-1)^2} = 3+\frac{-2x^2-7x+19}{(2x+3)(x-1)^2}$$

Let $\dfrac{-2x^2-7x+19}{(2x+3)(x-1)^2} = \dfrac{A}{2x+3}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$, where A, B and C are constants.

$$\frac{-2x^2-7x+19}{(2x+3)(x-1)^2} = \frac{A(x-1)^2+B(2x+3)(x-1)+C(2x+3)}{(2x+3)(x-1)^2}$$

$\therefore \quad -2x^2-7x+19 = A(x-1)^2+B(2x+3)(x-1)+C(2x+3)$

When $x=1$, $\quad -2(1)^2-7(1)+19 = A(1-1)^2+B[2(1)+3](1-1)+C[2(1)+3]$

$5C=10$

$C=2$

When $x=-\dfrac{3}{2}$ and $C=2$,

$$-2\left(-\frac{3}{2}\right)^2-7\left(-\frac{3}{2}\right)+19 = A\left(-\frac{3}{2}-1\right)^2+B\left[2\left(-\frac{3}{2}\right)+3\right]\left(-\frac{3}{2}-1\right)+2\left[2\left(-\frac{3}{2}\right)+3\right]$$

$\dfrac{25}{4}A=25$

$A=4$

When $x=0$, $A=4$ and $C=2$, $\quad 19=4(-1)^2+B(3)(-1)+2(3)$

$3B=-9$

$B=-3$

$$\frac{6x^3-5x^2-19x+28}{(2x+3)(x-1)^2} = 3+\frac{4}{2x+3}-\frac{3}{x-1}+\frac{2}{(x-1)^2}$$

4 Given that $\dfrac{4x^3+12x^2-x-4}{4x^2-1}=Ax+B+\dfrac{C}{2x-1}+\dfrac{D}{2x+1}$, where A, B, C and D are constants, find the value of A, of B, of C and of D.

$$\begin{array}{r@{\,}l} & \quad x+3 \\ 4x^2-1 \,\big) & \; 4x^3+12x^2-x-4 \\ & -\left(4x^3-0x^2-x+0\right) \\ \hline & \qquad 12x^2+0x-4 \\ & \quad -\left(12x^2+0x-3\right) \\ \hline & \qquad\qquad -1 \end{array}$$

$$\frac{4x^3+12x^2-x-4}{4x^2-1}=x+3+\frac{-1}{4x^2-1}$$

Let $\dfrac{-1}{4x^2-1}=\dfrac{C}{2x-1}+\dfrac{D}{2x+1}$.

$$\frac{-1}{4x^2-1}=\frac{C(2x+1)+D(2x-1)}{4x^2-1}$$

$$\therefore \quad -1=C(2x+1)+D(2x-1)$$

When $x=\dfrac{1}{2}$, $\quad -1=C\left[2\left(\dfrac{1}{2}\right)+1\right]+D\left[2\left(\dfrac{1}{2}\right)-1\right]$

$$2C=-1$$

$$C=-\frac{1}{2}$$

When $x=-\dfrac{1}{2}$, $\quad -1=C\left[2\left(-\dfrac{1}{2}\right)+1\right]+D\left[2\left(-\dfrac{1}{2}\right)-1\right]$

$$-2D=-1$$

$$D=\frac{1}{2}$$

$$\therefore \quad A=1,\ B=3,\ C=-\frac{1}{2},\ D=\frac{1}{2}$$

5 **(i)** Factorise $x^3 - 3x^2 + 4$.

Let f $(x) = x^3 - 3x^2 + 4$.

$$f(2) = (2)^3 - 3(2)^2 + 4$$
$$= 0$$

By Factor Theorem, $(x-2)$ is a factor of f (x).

$$x^3 - 3x^2 + 4 = (x-2)(x^2 + px - 2)$$

By comparing the coefficient of x,
$$0 = -2 - 2p$$
$$2p = -2$$
$$p = -1$$

$$x^3 - 3x^2 + 4 = (x-2)(x^2 - x - 2)$$
$$x^3 - 3x^2 + 4 = (x-2)(x+1)(x-2)$$
$$x^3 - 3x^2 + 4 = (x+1)(x-2)^2$$

5 **(ii)** Hence express $\dfrac{x^5-36x^2+53x+18}{x^3-3x^2+4}$ in partial fractions.

$$\begin{array}{r}
x^2+3x+9 \\
x^3-3x^2+4\;\overline{)\;x^5+0x^4+0x^3-36x^2+53x+18} \\
-\left(x^5-3x^4+0x^3+4x^2\right) \\
\hline
3x^4+0x^3-40x^2+53x+18 \\
-\left(3x^4-9x^3+0x^2+12x\right) \\
\hline
9x^3-40x^2+41x+18 \\
-\left(9x^3-27x^2+0x+36\right) \\
\hline
-13x^2+41x-18
\end{array}$$

$$\frac{x^5-36x^2+53x+18}{x^3-3x^2+4}=x^2+3x+9+\frac{-13x^2+41x-18}{x^3-3x^2+4}$$

$$\frac{x^5-36x^2+53x+18}{x^3-3x^2+4}=x^2+3x+9+\frac{-13x^2+41x-18}{(x+1)(x-2)^2}$$

Let $\dfrac{-13x^2+41x-18}{(x+1)(x-2)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$, where A, B and C are constants.

$$\frac{-13x^2+41x-18}{(x+1)(x-2)^2}=\frac{A(x-2)^2+B(x+1)(x-2)+C(x+1)}{(x+1)(x-2)^2}$$

$\therefore \quad -13x^2+41x-18=A(x-2)^2+B(x+1)(x-2)+C(x+1)$

When $x=2$, $\quad -13(2)^2+41(2)-18=A(2-2)^2+B(2+1)(2-2)+C(2+1)$

$3C=12$

$C=4$

When $x=-1$ and $C=4$,

$-13(-1)^2+41(-1)-18=A(-1-2)^2+B(-1+1)(-1-2)+4(-1+1)$

$9A=-72$

$A=-8$

When $x=0$, $A=-8$ and $C=4$, $\quad -18=-8(-2)^2+B(1)(-2)+4(1)$

$2B=-10$

$B=-5$

$$\therefore \quad \frac{x^5-36x^2+53x+18}{x^3-3x^2+4}=x^2+3x+9-\frac{8}{x+1}-\frac{5}{x-2}+\frac{4}{(x-2)^2}$$

6 **(i)** Factorise $x^3 - 8$.

(ii) Hence express $\dfrac{x^2+4}{x^3-8}$ in partial fractions.

(i) $x^3 - 8 = x^3 - 2^3$

$x^3 - 8 = (x-2)(x^2+2x+4)$

(ii) $\dfrac{x^2+4}{x^3-8} = \dfrac{x^2+4}{(x-2)(x^2+2x+4)}$

Let $\dfrac{x^2+4}{(x-2)(x^2+2x+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+2x+4}$, where A, B and C are constants.

$$\frac{x^2+4}{(x-2)(x^2+2x+4)} = \frac{A(x^2+2x+4)+(Bx+C)(x-2)}{(x-2)(x^2+2x+4)}$$

$\therefore \quad x^2+4 = A(x^2+2x+4)+(Bx+C)(x-2)$

When $x=2$, $\quad 2^2+4 = A\left[2^2+2(2)+4\right]+(2B+C)(2-2)$

$12A = 8$

$A = \dfrac{2}{3}$

When $x=0$ and $A=\dfrac{2}{3}$, $\quad 4 = \dfrac{2}{3}(4) - 2C$

$2C = -\dfrac{4}{3}$

$C = -\dfrac{2}{3}$

When $x=1$, $A=\dfrac{2}{3}$ and $C=-\dfrac{2}{3}$, $\quad 1^2+4 = \dfrac{2}{3}(1^2+2+4)+\left(B-\dfrac{2}{3}\right)(1-2)$

$5 = \dfrac{14}{3} - B + \dfrac{2}{3}$

$B = \dfrac{1}{3}$

$\therefore \quad \dfrac{x^2+4}{x^3-8} = \dfrac{2}{3(x-2)} + \dfrac{x-2}{3(x^2+2x+4)}$

Summary (Partial Fractions)

To express the proper fraction $\frac{f(x)}{g(x)}$ in partial fractions, factorise g(x) completely and write down the forms of the partial fractions as follows:

g(x) has	**Corresponding partial fraction(s)**	**Example**
Linear factor $ax+b$	$\frac{A}{ax+b}$	$\frac{7x+4}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$
Repeated linear factor $(ax+b)^2$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$	$\frac{7x+4}{(2x+1)(x-2)^2}=\frac{A}{2x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}$
Quadratic factor x^2+c^2 (which cannot be factorised)	$\frac{Ax+b}{x^2+c^2}$	$\frac{4x+7}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$