Note: H3 Inequalities is very different in nature from the Inequalities content in H2 Math.

Example 1
Show that if a and b are real numbers such that $0 < a \le 1 \le b$, then $a+b \ge ab+1$.

Solution

The Arithmetic Mean-Geometric Mean Inequality

For any nonnegative real numbers $x_1, x_2, \dots, x_n$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 x_2 \dots x_n} \ ,$$

where the equality holds if and only if $x_1 = x_2 = \dots = x_n$

Proof
In addendum.

Example 2 (Do not use AM-GM)
Prove that for $a, b > 0$, $\dfrac{a+b}{2} \ge \sqrt{ab}$. Prove that for $a, b, c, d > 0$, $\dfrac{a+b+c+d}{4} \ge \sqrt[4]{abcd}$.

Solution

Example 3

Find the minimum value of $\dfrac{x^2+361}{x}$ for $x>0$.

Solution

Example 4

Show that $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{d}+\dfrac{d}{a}\geq 4$ where a, b, c, d are positive real numbers.

Solution

Example 5

Show that if x is a positive real number, $8x^5+10x^{-4}\geq 18$.

Solution

Example 6

Show that $x^2 + y^2 + z^2 \geq xy + yz + zx$ where *x*, *y* and *z* are non-negative real numbers.

Thought Process

AM-GM tells us that $\frac{x^2 + y^2 + z^2}{3} \geq \sqrt[3]{x^2 y^2 z^2}$. Does this help us?

Solution

Example 7

Show that $x^3 + y^3 \geq x^2 y + xy^2$ where *x* and *y* are non-negative real numbers.

Show further that $x^n + y^n \geq x^{n-1} y + xy^{n-1}$ for any positive integer *n*.

Solution

The Cauchy-Schwarz Inequality

For any real numbers $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$,

$$\left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right) \geq \left(\sum_{i=1}^{n} u_i v_i\right)^2 ,$$

where the equality holds if there exists a nonzero constant *k* such that $u_i = kv_i$ for all $i = 1, 2, \ldots, n$

<u>Proof</u>
In addendum.

Example 8

Show that, if a, b and c are non-zero real numbers, $(a+b+c)\left(\frac{25}{a}+\frac{36}{b}+\frac{81}{c}\right)\geq 400$.

Solution
By Cauchy-Schwarz,

Example 9
Find the maximum value of $x+2y+3z$, given that $x^2+y^2+z^2=1$.

Thought Process
Cauchy-Schwarz involves <u>a product of two sums</u> on the "large" side, and <u>the square of a sum (of square-root terms)</u> on the "small" side. It takes familiarity to be able to "guess" what goes where.

Solution

Example 10
Find the minimum value of a^2+b^2, given that $3a+4b=15$.

Solution

Example 11

Let $x, y, z \in \mathbb{R}^+$. Prove that $\sqrt{x(3x+y)}+\sqrt{y(3y+z)}+\sqrt{z(3z+x)} \le 2(x+y+z)$.

Solution

Example 12

Let $a,b,c,d \in \mathbb{R}^+$ such that $a+b+c+d=1$. Find the minimum value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$.

Solution

Example 13

Show that $2\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right) \ge x+y+z$ for $x, y, z > 0$.

Thought Process

How did Example 12 "get rid of" denominators?

Solution

Example 14
Heron's formula states that the area of a triangle with sides a, b and c is equal to $\sqrt{s(s-a)(s-b)(s-c)}$ where $s=\dfrac{a+b+c}{2}$ is the semiperimeter.

A triangle with sides a, b and c has perimeter P and area T.

(i) Show that $T \le \dfrac{P^2}{12\sqrt{3}}$.

(ii) Hence, show that $a^2+b^2+c^2 \ge \left(4\sqrt{3}\right)T$.

Solution

(i)

(ii)

The Triangle Inequality

Suppose the lengths of the sides of a triangle are a, b, c. Then the sum of the lengths of any two sides must be larger than the length of the third side, i.e. the following must all be true:

$$a+b>c\,,\ b+c>a\,,\ c+a>b\,.$$

This can be shown algebraically, but a powerful intuitive argument goes: the third side is the straight line segment joining its end points, and so must be the shortest path between its endpoints. In particular, the path that follows the other two sides is a longer path.

A less common but equivalent way of stating this fact is that the length of any side must be longer than the difference of the lengths of the other two sides:

$$a>|b-c|\,,\ b>|a-c|\,,\ c>|a-b|\,.$$

The inequality signs above are strict for non-degenerate triangles; when the triangle is degenerate in the sense that the three points lie on a straight line, the signs must be replaced with $\geq$.

Instead of considering points lying on some plane, we can consider points on a number line. Since these points form degenerate triangles, we can conclude that for any real numbers x and y, we must have $|x+y|\leq|x|+|y|$. Using induction, we can extend this to:

> For any real numbers $x_1, x_2, \ldots, x_n$,
>
> $$|x_1+x_2+\cdots+x_n|\leq|x_1|+|x_2|+\cdots+|x_n|\ ,$$
>
> where the equality holds if and only if $x_1, x_2, \ldots, x_n$ are all non-negative

<u>Proof</u>

In addendum.

Example 15

Show that for all $x, y\in\mathbb{R}$, $|x-y|\geq|x|-|y|$.

Hence, show that $|x-y|\geq\big||x|-|y|\big|$.

Solution

By the triangle equality, for all $x_1, x_2\in\mathbb{R}$, $|x_1+x_2|\leq|x_1|+|x_2|$

For all $x, y\in\mathbb{R}$, we define $x_1=x-y$ and $x_2=y$

Then $|x|\leq|x-y|+|y|$

$|x-y|\geq|x|-|y|\ldots(*)$ (shown)

Exchanging x and y in $(*)$, $|y-x|\geq|y|-|x|$

$$|x-y|\geq-\left(|x|-|y|\right)\ldots(**)$$

By (*) and (**), $|x-y|\geq\max\left\{|x|-|y|, -\left(|x|-|y|\right)\right\}=\big||x|-|y|\big|$

Example 16

Let a, b, c be the sides of a triangle. Prove that $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2$.

Solution

Since a, b, c are the sides of a triangle,
we have $a+b>c$, $a+c>b$ and $b+c>a$ (why are they strict inequalities?)

Method 1	**Method 2** Without loss of generality, let

Keep in mind when attempting these tutorial questions:
Our primary objective is to **gain familiarity and mastery** in applying AM-GM and Cauchy-Schwarz inequalities, as well as general inequality-solving and problem-solving techniques.
Thus, **resist to urge to look up solutions.** At least 5 elegant proofs of Nesbitt's Inequality (Q8) are just a few mouse clicks away, but looking them up works against our primary objective.
The longer you spend thinking about a question, the more you gain out of it. Crucially, effort is seldom wasted: the experience gained from trying approaches that do not work is as important as finding what finally works – you build a sense of what "going astray" looks like!
It is not unusual to spend days thinking about the harder problems here.

Tutorial Questions

1 Find the minimum value of $\frac{4x^2+9}{x}$ where $x>0$.

2 Find the minimum value of $\frac{x^2}{x-9}$ where $x>9$.

3 Let a, b, c be positive real numbers such that $a+b+c=2019$. State the maximum possible value of $\sqrt{3a+12}+\sqrt{3b+12}+\sqrt{3c+12}$, and prove that this is a maximum.

4 Prove that $x^3+y^3+z^3 \geq x^2y+y^2z+z^2x$ for positive numbers x, y, z.

5 Given that $p+q+r=1$, find the minimum value of $p^2+2q^2+r^2$.

6 Harmonic Means.

By considering the AM-GM inequality for positive real numbers $b_1, b_2, \cdots, b_n$ and the substitution $b_r = \frac{1}{a_r}$, show that $\sqrt[n]{a_1 a_2 \cdots a_n} \geq \dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$.

7 Let $a, b, c \in \mathbb{R}^+$ such that $abc = 1$. Find the minimum value of $\dfrac{1+ab}{1+a} + \dfrac{1+bc}{1+b} + \dfrac{1+ca}{1+c}$.

8 Nesbitt's Inequality.

Show that $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b} \geq \dfrac{3}{2}$ for positive real numbers *a*, *b*, *c*.

9 Titu's Lemma.

Let $a_1, a_2, \ldots, a_n$ be real numbers and $b_1, b_2, \ldots, b_n$ be positive numbers. Show that

$$\frac{{a_1}^2}{b_1} + \frac{{a_2}^2}{b_2} + \cdots + \frac{{a_n}^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n} .$$

10 Let $a, b, c > 0$. Prove that $\dfrac{a^3}{bc} + \dfrac{b^3}{ca} + \dfrac{c^3}{ab} \geq a + b + c$.

11 $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 + a_2 + \ldots + a_n = s$. Show that

$$\sum_{i=1}^{n} \frac{a_i}{s - a_i} \geq \frac{n}{n-1} \text{ and } \sum_{i=1}^{n} \frac{s - a_i}{a_i} \geq n(n-1) .$$

12 Find the minimum value of $36^{x^2+y} + 36^{y^2+x}$ where $x, y \in \mathbb{R}$.

13 x, y, z are positive real numbers. Show that $\dfrac{2x}{y} + \dfrac{y}{z} \geq \dfrac{3x}{\sqrt[3]{xyz}}$, and hence show that

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}} .$$

14 Let *a*, *b*, *c* be nonnegative real numbers such that $\dfrac{1}{a^2+1} + \dfrac{1}{b^2+1} + \dfrac{1}{c^2+1} = 2$.

State the value of $\dfrac{a^2}{a^2+1} + \dfrac{b^2}{b^2+1} + \dfrac{c^2}{c^2+1}$ and hence show that $ab + bc + ca \leq \dfrac{3}{2}$.

15 Let *a*, *b*, *c* be positive numbers such that $a^2 + b^2 + c^2 = 3$.

(i) Show that $a^4 + b^4 + c^4 \geq 3$.

(ii) Deduce from (i) that $a^2b^2 + b^2c^2 + c^2a^2 \leq 3$.

(iii) Hence, using your result in part (ii), show that $a^2b + b^2c + c^2a \leq 3$.

16 (i) Let $p,q,r \in \mathbb{R}^+$ such that $pqr = 1$. Show that $\dfrac{p^2}{q+r}+\dfrac{q^2}{r+p}+\dfrac{r^2}{p+q} \geq \dfrac{3}{2}$.

(ii) Hence, by performing a suitable substitution, deduce that if *a*, *b*, *c* are positive real numbers such that $abc = 1$, then $\dfrac{1}{a^3(b+c)}+\dfrac{1}{b^3(c+a)}+\dfrac{1}{c^3(a+b)} \geq \dfrac{3}{2}$

17 HCI 2020 Prelim Q7 modified

(ii) Given that $x \geq 1,\ y \geq 1,\ z \geq 1$, and $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=2$, use the Cauchy-Schwarz Inequality to prove that $\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$ [3]

(iii) Given that $p+q+r+s=3$ and $p^2+2q^2+3r^2+6s^2=5$, show that $1 \leq p \leq 2$. [3]

18 (Modified from SMO 2013 Senior Section 2nd Round)
Let $c_1, c_2, \cdots$ be a sequence of positive real numbers such that for each $n \geq 1$,

$$c_{n+1}^2 \geq \frac{c_1^2}{1^3}+\frac{c_2^2}{2^3}+\cdots+\frac{c_n^2}{n^3} .$$

Prove that $\displaystyle\sum_{n=1}^{199} \frac{c_{n+1}}{c_1+c_2+\cdots+c_n} \geq \frac{199}{100}$.

(You may use the following results without proof:
$\displaystyle\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$; $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right) = 1-\frac{1}{n+1}$)

19 Let *a*, *b* and *c* be positive real numbers. Show that

$$(a+b>c \text{ and } b+c>a \text{ and } c+a>b)$$

if and only if

$$(a>|b-c| \text{ and } b>|a-c| \text{ and } c>|a-b|).$$

20 2018/Q3
A triangle has sides of lengths *a*, *b* and *c* units. In each of the following cases, prove that there is a triangle having sides of the given lengths.

(i) $\dfrac{a}{1+a}$, $\dfrac{b}{1+b}$ and $\dfrac{c}{1+c}$ units. [4]

(ii) $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$ units. [3]

(iii) $\sqrt{a(b+c-a)}$, $\sqrt{b(c+a-b)}$ and $\sqrt{c(a+b-c)}$ units. [6]

Addendum

Proof of the Arithmetic Mean-Geometric Mean Inequality

For any nonnegative real numbers $x_1, x_2, \ldots, x_n$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n} \ ,$$

where the equality holds if and only if $x_1 = x_2 = \ldots = x_n$

Proof 1

Preamble: example 7 (in the notes) can be proved without using the AM-GM inequality as follows: Since $x^{n-1}-y^{n-1}$ is positive / zero / negative when $x-y$ is positive / zero / negative, we obtain $(x-y)(x^{n-1}-y^{n-1})\geq 0$ i.e. $x^n+y^n-xy^{n-1}-x^{n-1}y\geq 0$ and so $x^n+y^n\geq xy^{n-1}+x^{n-1}y$.

This result will be used later in the proof.

We will prove AM-GM by induction.

1. Let P_n be the statement $\dfrac{x_1+x_2+\cdots+x_n}{n}\geq\sqrt[n]{x_1x_2\ldots x_n}$ for all nonnegative real numbers $x_1,x_2,\ldots,x_n$.
2. (Base case) P_2 is true (shown in example 2).
3. Suppose that P_k is true, i.e. $\dfrac{x_1+x_2+\cdots+x_k}{k}\geq\sqrt[k]{x_1x_2\ldots x_k}$.
4. (Induction step) Consider the statement P_{k+1}. We have

$$LHS=\frac{1}{k+1}(x_1+x_2+\cdots+x_{k+1})$$

$$=\frac{1}{k+1}\left(y_1^{k+1}+y_2^{k+1}+\cdots+y_{k+1}^{k+1}\right)\quad\text{where } y_i=(x_i)^{\frac{1}{k+1}}\text{, i.e. } y_i^{k+1}=x_i$$

$$=\frac{k}{k(k+1)}\left(y_1^{k+1}+y_2^{k+1}+\cdots+y_{k+1}^{k+1}\right)$$

$$=\frac{1}{k(k+1)}\Big[\left(y_1^{k+1}+y_2^{k+1}\right)+\left(y_1^{k+1}+y_3^{k+1}\right)+\cdots+\left(y_1^{k+1}+y_{k+1}^{k+1}\right)$$

$$+\left(y_2^{k+1}+y_3^{k+1}\right)+\cdots+\left(y_2^{k+1}+y_{k+1}^{k+1}\right)$$

$$+\cdots$$

$$+\left(y_k^{k+1}+y_{k+1}^{k+1}\right)\Big]$$

$$\geq\frac{1}{k(k+1)}\Big[y_1y_2^k+y_1y_3^k+\cdots+y_1y_{k+1}^k$$

$$+y_2y_1^k+y_2y_3^k+\cdots+y_2y_{k+1}^k$$

$$+\cdots$$

$$+y_{k+1}y_1^k+y_{k+1}y_2^k+\cdots+y_{k+1}y_k^k\Big]$$

$$=\frac{1}{k(k+1)}\left[y_1\left(y_2^k+y_3^k+\cdots+y_{k+1}^k\right)+y_2\left(y_1^k+y_3^k+\cdots+y_{k+1}^k\right)+\cdots+y_{k+1}\left(y_1^k+y_2^k+\cdots+y_k^k\right)\right]$$

$$=\frac{1}{k+1}\left[y_1\frac{y_2^k+y_3^k+\cdots+y_{k+1}^k}{k}+y_2\frac{y_1^k+y_3^k+\cdots+y_{k+1}^k}{k}+\cdots+y_{k+1}\frac{y_1^k+y_2^k+\cdots+y_k^k}{k}\right]$$

$$\geq\frac{1}{k+1}\left[y_1\sqrt[k]{y_2^ky_3^k\cdots y_{k+1}^k}+y_2\sqrt[k]{y_1^ky_3^k\cdots y_{k+1}^k}+\cdots+y_{k+1}\sqrt[k]{y_1^ky_2^k\cdots y_k^k}\right]$$

$$=\frac{1}{k+1}\left[y_1y_2y_3\cdots y_{k+1}+y_1y_2y_3\cdots y_{k+1}+\cdots+y_1y_2y_3\cdots y_{k+1}\right]$$

$$=y_1y_2y_3\cdots y_{k+1}$$

$$=\sqrt[k+1]{x_1x_2\cdots x_{k+1}}=RHS\quad\text{so } P_{k+1}\text{ is true.}$$

5. Hence, by mathematical induction, P_n is true for all integer $n\geq 2$.

Proof 2

Again, this is a proof by induction.

1. Let P_n be the statement $\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}$ for all nonnegative real numbers $x_1, x_2, \ldots, x_n$.
2. (Base case) P_2 is true (shown in example 2).
3. Suppose that P_k is true, i.e. $\frac{x_1 + x_2 + \cdots + x_k}{k} \geq \sqrt[k]{x_1 x_2 \ldots x_k}$.
4. (Induction step) Consider the statement P_{k+1}, and let $\alpha = \frac{x_1 + x_2 + \ldots + x_{k+1}}{k+1}$.

If every x_i is equal to α, then $LHS = RHS$ and P_{k+1} is true.
Otherwise, we must be able to find one element that is less than α and one which is greater. Without loss of generality, let $x_k > \alpha$ and $x_{k+1} < \alpha$. Then since $(x_k - \alpha)(\alpha - x_{k+1}) > 0$, we have

$$\alpha x_k + \alpha x_{k+1} - \alpha^2 - x_k x_{k+1} > 0$$
$$\alpha(x_k + x_{k+1} - \alpha) > x_k x_{k+1}.$$

Now let $y = x_k + x_{k+1} - \alpha$. Then $\frac{1}{k}\left(y + \sum_{i=1}^{k-1} x_i\right) = \alpha = \frac{1}{k+1}\left(\sum_{i=1}^{k+1} x_i\right)$.

Hence,

$$\left(\frac{1}{k+1}\sum_{i=1}^{k+1} x_i\right)^{k+1} = \alpha^{k+1}$$
$$= \alpha^k \alpha$$
$$= \left(\frac{1}{k}\left(y + \sum_{i=1}^{k-1} x_i\right)\right)^k \alpha$$
$$\geq \left(y \prod_{i=1}^{k-1} x_i\right)\alpha \quad \text{(by induction hypothesis)}$$
$$= \alpha y \prod_{i=1}^{k-1} x_i$$
$$\geq x_k x_{k+1} \prod_{i=1}^{k-1} x_i \quad \text{(since } \alpha y = \alpha(x_k + x_{k+1} - \alpha)\text{)}$$
$$= \prod_{i=1}^{k+1} x_i$$

and so taking both sides to the power $\frac{1}{k+1}$ shows that P_{k+1} is true.

5. Hence, by mathematical induction, P_n is true for all integer $n \geq 2$.

Proof 3

Induction works by showing both a base case, and an induction step $P_k \Rightarrow P_{k+1}$. In this proof, since the induction step is not easy to show directly, we will instead replace the induction step with these two results: $P_k \Rightarrow P_{2k}$ and $P_k \Rightarrow P_{k-1}$. Together, these will also suffice us to show that P_n is true for all positive integer *n*, since we can always start from the base case P_2 to reach any positive integer *n* by following these results. For example, to show that P_5 is true, we follow the chain $P_2 \Rightarrow P_4 \Rightarrow P_3 \Rightarrow P_6 \Rightarrow P_5$.

This technique is sometimes called "forward-backward induction".

1. Let P_n be the statement $\dfrac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 x_2 \ldots x_n}$ for all nonnegative real numbers $x_1, x_2, \ldots, x_n$.

2. P_2 is true.

3. Suppose that P_k is true for some integer $k \ge 2$, i.e. $\dfrac{x_1 + x_2 + \cdots + x_k}{k} \ge \sqrt[k]{x_1 x_2 \ldots x_k}$.

4a. Consider the statement P_{2k}. We have

$$LHS = \frac{x_1 + x_2 + x_3 + x_4 + \cdots + x_{2k-1} + x_{2k}}{2k}$$

$$= \frac{\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2} + \cdots + \frac{x_{2k-1} + x_{2k}}{2}}{k}$$

$$\ge \frac{\sqrt{x_1 x_2} + \sqrt{x_3 x_4} + \cdots + \sqrt{x_{2k-1} x_{2k}}}{k} \qquad \text{since } P_2 \text{ is true}$$

$$\ge \sqrt[k]{\left(\sqrt{x_1 x_2}\right)\left(\sqrt{x_3 x_4}\right)\cdots\left(\sqrt{x_{2k-1} x_{2k}}\right)} \qquad \text{using the result in step 3}$$

$$= \sqrt[2k]{x_1 x_2 \ldots x_{2k}}$$

so P_{2k} is true.

4b. Consider the statement P_{k-1}, and let $\alpha = \dfrac{x_1 + x_2 + \ldots + x_{k-1}}{k-1}$. We have

$$\alpha = \frac{x_1 + x_2 + \cdots + x_{k-1}}{k-1}$$

$$= \frac{\frac{k}{k-1}\left(x_1 + x_2 + \cdots + x_{k-1}\right)}{k}$$

$$= \frac{x_1 + x_2 + \cdots + x_{k-1} + \left(\frac{x_1 + x_2 + \cdots + x_{k-1}}{k-1}\right)}{k}$$

$$= \frac{x_1 + x_2 + \cdots + x_{k-1} + \alpha}{k}$$

$$\ge \sqrt[k]{x_1 x_2 \cdots x_{k-1} \alpha} \qquad \text{using the result in step 3.}$$

What we have shown so far is $\alpha \ge \sqrt[k]{x_1 x_2 \cdots x_{k-1} \alpha}$, so raising both sides to the power of *k* yields $\alpha^k \ge x_1 x_2 \cdots x_{k-1} \alpha$. Dividing by α leads to $\alpha^{k-1} \ge x_1 x_2 \cdots x_{k-1}$ and finally raising both sides to the power of $\dfrac{1}{k-1}$ gives us P_{k-1}. So P_{k-1} is true.

5. Hence, P_n is true for all integer $n \geq 2$.

Proofs of the Cauchy-Schwarz Inequality

> For any real numbers $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$,
> $$\left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right) \geq \left(\sum_{i=1}^{n} u_i v_i\right)^2,$$
> where the equality holds if there exists a nonzero constant k such that $u_i = kv_i$ for all $i = 1, 2, \ldots, n$

<u>Proof 1</u>

Consider the sum $\sum_{i=1}^{n}(u_i - xv_i)^2$. Since this is a sum of squares, this sum is non-negative, i.e.

$$\sum_{i=1}^{n}(u_i - xv_i)^2 \geq 0$$

Expanding the sum, we get

$$\left(\sum_{i=1}^{n} v_i^2\right)x^2 - \left(2\sum_{i=1}^{n} u_i v_i\right)x + \left(\sum_{i=1}^{n} u_i^2\right) \geq 0$$

Interpreting this as a quadratic inequality in *x*, since the LHS is always ≥ 0, the discriminant must be non-positive, i.e.

$$\left(-2\sum_{i=1}^{n} u_i v_i\right)^2 - 4\left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} u_i^2\right) \leq 0$$

Simplifying, we get

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} u_i^2\right)$$

which is precisely the Cauchy-Schwarz inequality.

Proof 2

Consider the following sum of squares

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(u_i v_j - u_j v_i\right)^2 \geq 0$$

where the double sum is a sum over all pairs of integers $(i,\, j)$ where $1 \leq i < j \leq n$.

Expanding, we get

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(u_i^{\,2} v_j^{\,2} + u_j^{\,2} v_i^{\,2} - 2u_i v_i u_j v_j\right) \geq 0$$

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(u_i^{\,2} v_j^{\,2} + u_j^{\,2} v_i^{\,2}\right) \geq 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(u_i v_i u_j v_j\right)$$

Adding common terms to both LHS and RHS,

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(u_i^{\,2} v_j^{\,2} + u_j^{\,2} v_i^{\,2}\right) + \sum_{i=1}^{n}\left(u_i^{\,2} v_i^{\,2}\right) \geq 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(u_i v_i u_j v_j\right) + \sum_{i=1}^{n}\left(u_i^{\,2} v_i^{\,2}\right)$$

However, the LHS is now equal to $\left(\sum_{i=1}^{n} u_i^{\,2}\right)\left(\sum_{j=1}^{n} v_j^{\,2}\right)$ and the RHS is precisely $\left(\sum_{i=1}^{n} u_i v_i\right)^2$, so $\left(\sum_{i=1}^{n} u_i^{\,2}\right)\left(\sum_{i=1}^{n} v_i^{\,2}\right) \geq \left(\sum_{i=1}^{n} u_i v_i\right)^2$ as required.

H3 Inequalities Possible Solutions:

1	Find the minimum value of $\frac{4x^2+9}{x}$ where $x>0$.
	$\frac{4x^2+9}{x}=4x+\frac{9}{x}$ $\geq 2\sqrt{(4x)\left(\frac{9}{x}\right)}$ (AM-GM) $=2\sqrt{36}$ $=12$ Equality is attained when $4x=\frac{9}{x}\Leftrightarrow x^2=\frac{9}{4}\Leftrightarrow x=\frac{3}{2}$ since $x>0$ So minimum value is 12
2	Find the minimum value of $\frac{x^2}{x-9}$ where $x>9$.
	<u>**Method 1**</u> Let $y=x-9$ $\frac{x^2}{x-9}=\frac{(y+9)^2}{y}$ $=\frac{y^2+18y+81}{y}$ $=18+y+\frac{81}{y}$ $\geq 18+2\sqrt{y\left(\frac{81}{y}\right)}$ (AM-GM) $=36$ **Comment:** Substitutions defined by the solver come in two forms. There are "inspired substitutions" where an expression seemingly pulled from the air greatly simplifies matters, but it is hard to fathom why one might think of that expression. There are also "indicated substitutions", where the problem strongly hints at the substitution to be made. This solution is an example of the latter. The condition $x>9$ is highly unusual, since we are much more used to "positive" conditions, and suggests that working with $x-9$ might be easier. Once the substitution is made, the problem looks a lot like Q1. <u>**Method 2**</u>

	$\frac{x^2}{x-9}=x+9+\frac{81}{x-9}$ $=18+(x-9)+\frac{81}{x-9}$ $\geq 18+2\sqrt{(x-9)\left(\frac{81}{x-9}\right)}$ by AM-GM $=18+2(9)$ $=36$ Equality is attained when $x-9=\frac{81}{x-9} \Leftrightarrow (x-9)^2=81 \Leftrightarrow x=18$ since $x>9$ So minimum value is 36
3	Let *a*, *b*, *c* be positive real numbers such that $a+b+c=2019$. State the maximum possible value of $\sqrt{3a+12}+\sqrt{3b+12}+\sqrt{3c+12}$, and prove that this is a maximum.
	Since $a+b+c=2019$, $a+4+b+4+c+4=2031$. Thus $\sqrt{3a+12}+\sqrt{3b+12}+\sqrt{3c+12}$ $=\sqrt{3(a+4)}+\sqrt{3(b+4)}+\sqrt{3(c+4)}$ $\leq\sqrt{((a+4)+(b+4)+(c+4))(3+3+3)}$ (Cauchy-Schwarz) $=\sqrt{18279}$ This value is attained when $a=b=c=\frac{2019}{3}$, so it is the maximum. *Why does AM-GM not work?* $\sqrt{3a+12}+\sqrt{3b+12}+\sqrt{3c+12}$ $=\sqrt{3(a+4)}+\sqrt{3(b+4)}+\sqrt{3(c+4)}$ $\leq\frac{3+(a+4)}{2}+\frac{3+(b+4)}{2}+\frac{3+(c+4)}{2}$ (by AM-GM) $=\frac{a+b+c+21}{2}$ $=\frac{2040}{2}$ $=1020$ For equality to hold, we require $a+4=b+4=c+4=3 \Leftrightarrow a=b=c=-1$ but *a*, *b*, *c* are positive and $a+b+c=2019$

4	Prove that $x^3+y^3+z^3 \ge x^2y+y^2z+z^2x$ for positive numbers x, y, z.
	By AM-GM, the following 3 inequalities hold: $\begin{cases} x^3+x^3+y^3 \ge 3\sqrt[3]{x^6y^3}=3x^2y \\ y^3+y^3+z^3 \ge 3\sqrt[3]{y^6z^3}=3y^2z \\ z^3+z^3+x^3 \ge 3\sqrt[3]{z^6x^3}=3z^2x \end{cases}$ The required result is obtained by adding these together and dividing by 3.
5	Given that $p+q+r=1$, find the minimum value of $p^2+2q^2+r^2$.
	$p^2+2q^2+r^2$ $=\frac{2}{5}\left(p^2+2q^2+r^2\right)\left(1+\frac{1}{2}+1\right)$ $\ge\frac{2}{5}\left(\sqrt{\left(p^2\right)(1)}+\sqrt{\left(2q^2\right)\left(\frac{1}{2}\right)}+\sqrt{\left(r^2\right)(1)}\right)^2$ (by Cauchy-Schwarz) $=\frac{2}{5}\left(p+q+r\right)^2$ $=\frac{2}{5}$ Equality is attained when $p=\frac{q\sqrt{2}}{\sqrt{\frac{1}{2}}}=r \Leftrightarrow p=2q=r$ Subst into $p+q+r=1$ gives $5q=1$, then $p=\frac{2}{5}, q=\frac{1}{5}, r=\frac{2}{5}$ The minimum value is $\frac{2}{5}$
6	Harmonic Means. By considering the AM-GM inequality for positive real numbers $b_1, b_2, \cdots, b_n$ and the substitution $b_r=\frac{1}{a_r}$, show that $\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$.
	$\frac{b_1+b_2+\cdots+b_n}{n} \ge \sqrt[n]{b_1b_2\cdots b_n}$ $\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n} \ge \sqrt[n]{\frac{1}{a_1}\frac{1}{a_2}\cdots\frac{1}{a_n}}$ $\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n} \ge \frac{1}{\sqrt[n]{a_1a_2\cdots a_n}}$ Since both sides are positive,

	we can take reciprocal and obtain $\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} \le \sqrt[n]{a_1 a_2 \cdots a_n}$
7	Let $a,b,c \in \mathbb{R}^+$ such that $abc=1$. Find the minimum value of $\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ca}{1+c}$.
	Method 1 $\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ca}{1+c}=\frac{abc+ab}{1+a}+\frac{abc+bc}{1+b}+\frac{abc+ca}{1+c}$ $=\frac{ab(1+c)}{1+a}+\frac{bc(1+a)}{1+b}+\frac{ca(1+b)}{1+c}$ $\ge 3\times\sqrt[3]{\frac{ab(1+c)bc(1+a)ca(1+b)}{(1+a)(1+b)(1+c)}}$ (AM-GM) $=3\times\sqrt[3]{a^2b^2c^2}$ $=3$ (attained when $a=b=c=1$) **Method 2** $\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ca}{1+c}=\frac{1+1/c}{1+a}+\frac{1+1/a}{1+b}+\frac{1+1/b}{1+c}$ $=\frac{1+c}{c(1+a)}+\frac{1+a}{a(1+b)}+\frac{1+b}{b(1+c)}$ $\ge 3\sqrt[3]{\frac{1+c}{c(1+a)}\cdot\frac{1+a}{a(1+b)}\cdot\frac{1+b}{b(1+c)}}$ by AM-GM $=3\sqrt[3]{\frac{1}{abc}}$ $=3$ (attained when $a=b=c=1$)
8	Nesbitt's Inequality. Show that $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}$ for positive real numbers *a*, *b*, *c*.
	Method 1 $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$ $=\frac{a+b+c}{b+c}+\frac{b+c+a}{c+a}+\frac{c+a+b}{a+b}-3$ $=(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)-3$ $=\frac{1}{2}(2a+2b+2c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)-3$ $=\frac{1}{2}\left((b+c)+(c+a)+(a+b)\right)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)-3$

	$\geq \frac{1}{2}(1+1+1)^2 - 3$ (Cauchy-Schwarz) $= \frac{3}{2}$ **Method 2** Let $x = a+b, y = a+c, z = b+c$ $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} = \frac{x+y-z}{2z}+\frac{x+z-y}{2y}+\frac{y+z-x}{2x}$ $= \frac{1}{2}\left[\frac{x}{z}+\frac{y}{z}+\frac{x}{y}+\frac{z}{y}+\frac{y}{x}+\frac{z}{x}\right]-\frac{3}{2}$ $= \frac{1}{2}\left[\frac{x}{z}+\frac{z}{x}+\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}\right]-\frac{3}{2}$ $\geq \frac{1}{2}\left[2\sqrt{\frac{x}{z}\cdot\frac{z}{x}}+2\sqrt{\frac{x}{y}\cdot\frac{y}{x}}+2\sqrt{\frac{y}{z}\cdot\frac{z}{y}}\right]-\frac{3}{2}$ $= \frac{1}{2}(6)-\frac{3}{2}$ $= \frac{3}{2}$
9	Titu's Lemma. Let $a_1, a_2, \ldots, a_n$ be real numbers and $b_1, b_2, \ldots, b_n$ be positive numbers. Show that $\frac{a_1{}^2}{b_1}+\frac{a_2{}^2}{b_2}+\cdots+\frac{a_n{}^2}{b_n} \geq \frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}$.
	By Cauchy-Schwarz, $\left(\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n}\right)(b_1+b_2+\cdots+b_n) \geq (a_1+a_2+\cdots+a_n)^2$ Since all the *b* are positive, $b_1+b_2+\cdots+b_n$ is positive, and dividing both sides by this factor yields the required result $\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\cdots+\frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n}$.
10	Let $a, b, c > 0$. Prove that $\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \geq a+b+c$.
	Method 1 By AM-GM, $\frac{a^4+a^4+b^4+c^4}{4} \geq a^2bc$, and similarly $\frac{a^4+b^4+b^4+c^4}{4} \geq ab^2c$ and $\frac{a^4+b^4+c^4+c^4}{4} \geq abc^2$ The result follows by adding these three inequalities together and dividing by *abc*. **Method 2**

By AM-GM,

$$\frac{a^3}{bc}+b+c \ge 3\sqrt[3]{\left(\frac{a^3}{bc}\right)(b)(c)} = 3\sqrt[3]{a^3} = 3a$$

and similarly $\frac{b^3}{ca}+c+a \ge 3b$ and $\frac{c^3}{ab}+a+b \ge 3c$

Adding these three inequalities together produces

$$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}+2a+2b+2c \ge 3a+3b+3c$$

and hence

$\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab} \ge a+b+c$ as required.

Method 3

By AM-GM,

$$\left(\frac{a^3}{bc}+\frac{b^3}{ca}\right)+\left(\frac{b^3}{ca}+\frac{c^3}{ab}\right)+\left(\frac{c^3}{ab}+\frac{a^3}{bc}\right) \ge 2\left(\sqrt{\frac{a^3b^3}{abc^2}}+\sqrt{\frac{b^3c^3}{a^2bc}}+\sqrt{\frac{a^3c^3}{ab^2c}}\right)$$

$$2\left(\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\right) \ge 2\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)$$

Furthermore, by AM-GM,

$$\left(\frac{ab}{c}+\frac{bc}{a}\right)+\left(\frac{bc}{a}+\frac{ca}{b}\right)+\left(\frac{ca}{b}+\frac{ab}{c}\right) \ge 2\left(\sqrt{\frac{ab^2c}{ac}}+\sqrt{\frac{abc^2}{ab}}+\sqrt{\frac{a^2bc}{bc}}\right)$$

$$2\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right) \ge 2(a+b+c)$$

Combining the above, $2\left(\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\right) \ge 2(a+b+c)$ and the result follows.

Method 4

We have

$$2\left(\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\right) = \left(\frac{a^3}{bc}+\frac{b^3}{ca}\right)+\left(\frac{b^3}{ca}+\frac{c^3}{ab}\right)+\left(\frac{c^3}{ab}+\frac{a^3}{bc}\right)$$

$$\ge 2\left(\sqrt{\frac{a^3b^3}{abc^2}}+\sqrt{\frac{b^3c^3}{a^2bc}}+\sqrt{\frac{a^3c^3}{ab^2c}}\right) \qquad \text{by AM-GM}$$

$$= 2\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)$$

Therefore

$$\left(\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\right)^2 \ge \left(\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\right)\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)$$

$$\ge (a+b+c)^2 \qquad \text{by Cauchy-Schwarz}$$

	so taking square roots yields $\frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}\geq a+b+c$ as required.
11	$a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1+a_2+\ldots+a_n=s$. Show that $\sum_{i=1}^{n}\frac{a_i}{s-a_i}\geq\frac{n}{n-1}$ and $\sum_{i=1}^{n}\frac{s-a_i}{a_i}\geq n(n-1)$.
	By Cauchy-Schwarz, $\left(\sum_{i=1}^{n}\frac{s}{s-a_i}\right)\left(\sum_{i=1}^{n}(s-a_i)\right)\geq\left(\sum_{i=1}^{n}\sqrt{s}\right)^2$ $\left(\sum_{i=1}^{n}\frac{s}{s-a_i}\right)(ns-s)\geq n^2 s$ $\sum_{i=1}^{n}\frac{s}{s-a_i}\geq\frac{n^2 s}{ns-s}=\frac{n^2}{n-1}$ Therefore, $\sum_{i=1}^{n}\left(\frac{a_i}{s-a_i}+\frac{s-a_i}{s-a_i}\right)\geq\frac{n^2}{n-1}$ $\sum_{i=1}^{n}\frac{a_i}{s-a_i}\geq\frac{n^2}{n-1}-n=\frac{n}{n-1}$ Similarly, by Cauchy-Schwarz, $\left(\sum_{i=1}^{n}\frac{s}{a_i}\right)\left(\sum_{i=1}^{n}a_i\right)\geq\left(\sum_{i=1}^{n}\sqrt{s}\right)^2$ $\left(\sum_{i=1}^{n}\frac{s}{a_i}\right)\geq\frac{n^2 s}{s}=n^2$ Therefore, $\sum_{i=1}^{n}\left(\frac{s-a_i}{a_i}+\frac{a_i}{a_i}\right)\geq n^2$ $\sum_{i=1}^{n}\frac{s-a_i}{a_i}\geq n^2-n=n(n-1)$
12	Find the minimum value of $36^{x^2+y}+36^{y^2+x}$ where $x, y\in\mathbb{R}$.
	$36^{x^2+y}+36^{y^2+x}\geq 2\sqrt{\left(36^{x^2+y}\right)\left(36^{y^2+x}\right)}$ (AM-GM)(*) $=2\sqrt{6^{2\left(x^2+y+y^2+x\right)}}$ $=2\times 6^{x^2+x+y^2+y}$ $=2\times 6^{\left[\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2-\frac{1}{2}\right]}$ $\geq 2\times 6^{\left(-\frac{1}{2}\right)}$ $=\frac{\sqrt{6}}{3}$

	Since $6^{x^2+x+y^2+y}$ is an increasing function, $6^{x^2+x+y^2+y}$ attains its minimum when x^2+x+y^2+y attains its minimum By inspection, when $x=y=-\frac{1}{2}$, $36^{x^2+y}=36^{y^2+x}$ so equality is attained for the AM-GM (*)
13	x, y, z are positive real numbers. Show that $\frac{2x}{y}+\frac{y}{z}\geq\frac{3x}{\sqrt[3]{xyz}}$, and hence show that $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq\frac{x+y+z}{\sqrt[3]{xyz}}$.
	$\frac{2x}{y}+\frac{y}{z}\geq 3\times\sqrt[3]{\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)\left(\frac{y}{z}\right)}$ (AM-GM) $=3\times\sqrt[3]{\frac{x^2}{yz}}$ $=3\times\sqrt[3]{\frac{x^3}{xyz}}$ $=\frac{3x}{\sqrt[3]{xyz}}$ Hence, $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=\frac{1}{3}\left[\left(\frac{2x}{y}+\frac{y}{z}\right)+\left(\frac{2y}{z}+\frac{z}{x}\right)+\left(\frac{2z}{x}+\frac{x}{y}\right)\right]$ $\geq\frac{1}{3}\left(\frac{3x}{\sqrt[3]{xyz}}+\frac{3y}{\sqrt[3]{xyz}}+\frac{3z}{\sqrt[3]{xyz}}\right)$ (using result) $=\frac{x+y+z}{\sqrt[3]{xyz}}$
14	Let a, b, c be nonnegative real numbers such that $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2$. State the value of $\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}$ and hence show that $ab+bc+ca\leq\frac{3}{2}$.
	Firstly, note that $\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}=\frac{a^2+1-1}{a^2+1}+\frac{b^2+1-1}{b^2+1}+\frac{c^2+1-1}{c^2+1}$ $=3-\left(\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\right)$ $=1$ Now, by Cauchy-Schwarz, $\left(\frac{a^2}{a^2+1}+\frac{b^2}{b^2+1}+\frac{c^2}{c^2+1}\right)\left(\left(a^2+1\right)+\left(b^2+1\right)+\left(c^2+1\right)\right)\geq\left(a+b+c\right)^2$ Simplifying, $(1)\left(3+a^2+b^2+c^2\right)\geq a^2+b^2+c^2+2\left(ab+bc+ca\right)$ $3\geq 2\left(ab+bc+ca\right)$

	$ab+bc+ca \le \frac{3}{2}$
15	Let a, b, c be positive numbers such that $a^2+b^2+c^2=3$. (i) Show that $a^4+b^4+c^4 \ge 3$. (ii) Deduce from (i) that $a^2b^2+b^2c^2+c^2a^2 \le 3$. (iii) Hence, using your result in part (ii), show that $a^2b+b^2c+c^2a \le 3$.
	(i) By Cauchy-Schwarz, $\left(a^4+b^4+c^4\right)(1+1+1) \ge \left(a^2+b^2+c^2\right)^2$ $\left(a^4+b^4+c^4\right)(3) \ge (3)^2$ $a^4+b^4+c^4 \ge 3$ (ii) $\left(a^2+b^2+c^2\right)^2=9$ $a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2=9$ $a^2b^2+b^2c^2+c^2a^2=\frac{9-\left(a^4+b^4+c^4\right)}{2} \le \frac{9-3}{2}=3$ WRONG METHOD for (ii) From (i), $a^4+b^4+c^4 \ge 3$ $\left(a^2b^2+b^2c^2+c^2a^2\right)^2 \le (a^4+b^4+c^4)(b^4+c^4+a^4)$ (Cauchy-Schwarz) ≤ 9 (by part (i)) $a^2b^2+b^2c^2+c^2a^2 \le 3$ $a^4+b^4+c^4 \ge 3$ and $a^4+b^4+c^4 \ge a^2b^2+b^2c^2+c^2a^2$ does not imply $3 \ge a^2b^2+b^2c^2+c^2a^2$ (iii) $3\times 3 \ge \left(a^2+b^2+c^2\right)\left(a^2b^2+b^2c^2+c^2a^2\right)$ (using part (ii)) $\ge \left(a^2b+b^2c+c^2a\right)^2$ (Cauchy-Schwarz) so taking square roots on both sides we get $3 \ge a^2b+b^2c+c^2a$ as required.
16	(i) Let $p,q,r \in \mathbb{R}^+$ such that $pqr=1$. Show that $\frac{p^2}{q+r}+\frac{q^2}{r+p}+\frac{r^2}{p+q} \ge \frac{3}{2}$. (ii) Hence, by performing a suitable substitution, deduce that if a, b, c are positive real numbers such that $abc=1$, then $\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \ge \frac{3}{2}$
	(i) By Cauchy-Schwarz, $\left(\frac{p^2}{q+r}+\frac{q^2}{r+p}+\frac{r^2}{p+q}\right)\left((q+r)+(r+p)+(p+q)\right) \ge (p+q+r)^2$

	$\left(\frac{p^2}{q+r}+\frac{q^2}{r+p}+\frac{r^2}{p+q}\right)(2p+2q+2r)\geq(p+q+r)^2$ $\frac{p^2}{q+r}+\frac{q^2}{r+p}+\frac{r^2}{p+q}\geq\frac{p+q+r}{2}$ $\frac{p^2}{q+r}+\frac{q^2}{r+p}+\frac{r^2}{p+q}\geq\frac{3\times\sqrt[3]{pqr}}{2}$ (AM-GM) $\frac{p^2}{q+r}+\frac{q^2}{r+p}+\frac{r^2}{p+q}\geq\frac{3}{2}$ (ii) Let $a=\frac{1}{p}$, $b=\frac{1}{q}$, $c=\frac{1}{r}$. Then the result in part (i) becomes $\frac{1}{a^2\left(\frac{1}{b}+\frac{1}{c}\right)}+\frac{1}{b^2\left(\frac{1}{c}+\frac{1}{a}\right)}+\frac{1}{c^2\left(\frac{1}{a}+\frac{1}{b}\right)}\geq\frac{3}{2}$ $\frac{bc}{a^2(b+c)}+\frac{ca}{b^2(c+a)}+\frac{ab}{c^2(a+b)}\geq\frac{3}{2}$ $\frac{abc}{a^3(b+c)}+\frac{bca}{b^3(c+a)}+\frac{cab}{c^3(a+b)}\geq\frac{3}{2}$ $\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$
17	HCI 2020 Prelim Q7 modified (ii) Given that $x\geq1$, $y\geq1$, $z\geq1$, and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$, use the Cauchy-Schwarz Inequality to prove that $\sqrt{x+y+z}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$ [3] (iii) Given that $p+q+r+s=3$ and $p^2+2q^2+3r^2+6s^2=5$, show that $1\leq p\leq2$. [3]
	(ii) Observe that $\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=3-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=3-2=1$ By Cauchy-Schwarz, $\sqrt{\left(\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}\right)}\sqrt{x+y+z}\geq\sqrt{(x-1)}+\sqrt{(y-1)}+\sqrt{(z-1)}$ $(1)\sqrt{x+y+z}\geq\sqrt{(x-1)}+\sqrt{(y-1)}+\sqrt{(z-1)}$ (iii) From the given conditions, $q+r+s=3-p$ and $2q^2+3r^2+6s^2=5-p^2$. By Cauchy-Schwarz, $(2q^2+3r^2+6s^2)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)\geq(q+r+s)^2$ $\Rightarrow(5-p^2)(1)\geq(3-p)^2$ $\Rightarrow0\geq2p^2-6p+4$ $\Rightarrow0\geq(p-1)(p-2)$

	$\Rightarrow 1 \le p \le 2$
18	(Modified from SMO 2013 Senior Section 2nd Round) Let $c_1, c_2, \cdots$ be a sequence of positive real numbers such that for each $n \ge 1$, $$c_{n+1}^2 \ge \frac{c_1^2}{1^3} + \frac{c_2^2}{2^3} + \cdots + \frac{c_n^2}{n^3} .$$ Prove that $\sum_{n=1}^{199} \frac{c_{n+1}}{c_1 + c_2 + \cdots + c_n} \ge \frac{199}{100}$. (You may use the following results without proof: $\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$; $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right) = 1 - \frac{1}{n+1}$)
	By Cauchy-Schwarz, $\left(\frac{c_1^2}{1^3} + \frac{c_2^2}{2^3} + \cdots + \frac{c_n^2}{n^3}\right)\left(1^3 + 2^3 + \cdots + n^3\right) \ge \left(c_1 + c_2 + \cdots + c_n\right)^2$ $\left(\frac{c_1^2}{1^3} + \frac{c_2^2}{2^3} + \cdots + \frac{c_n^2}{n^3}\right)\left(\frac{n(n+1)}{2}\right)^2 \ge \left(c_1 + c_2 + \cdots + c_n\right)^2$ $\frac{c_1^2}{1^3} + \frac{c_2^2}{2^3} + \cdots + \frac{c_n^2}{n^3} \ge \left(\frac{2(c_1 + c_2 + \cdots + c_n)}{n(n+1)}\right)^2$ so $c_{n+1}^2 \ge \frac{c_1^2}{1^3} + \frac{c_2^2}{2^3} + \cdots + \frac{c_n^2}{n^3} \ge \left(\frac{2(c_1 + c_2 + \cdots + c_n)}{n(n+1)}\right)^2$ Taking square roots on both sides, $c_{n+1} \ge \frac{2(c_1 + c_2 + \cdots + c_n)}{n(n+1)}$ $\frac{c_{n+1}}{c_1 + c_2 + \cdots + c_n} \ge \frac{2}{n(n+1)}$ Taking summation, $\sum_{n=1}^{199} \frac{c_{n+1}}{c_1 + c_2 + \cdots + c_n} \ge \sum_{n=1}^{199} \frac{2}{n(n+1)}$ $= 2\sum_{n=1}^{199} \frac{1}{n(n+1)}$ $= 2\left(1 - \frac{1}{200}\right)$ $= \frac{199}{100}$
19	Let a, b and c be positive real numbers. Show that ($a + b > c$ and $b + c > a$ and $c + a > b$) if and only if ($a > \lvert b - c \rvert$ and $b > \lvert a - c \rvert$ and $c > \lvert a - b \rvert$).
	Proof:

	$(\Leftrightarrow)$ $\begin{array}{lclclcl} a+b>c & & a>c-b \text{ and } a>b-c & & a>c-b \text{ and } a>-(c-b) & & a>\lvert c-b\rvert \\ b+c>a & \Leftrightarrow & b>a-c \text{ and } b>c-a & \Leftrightarrow & b>a-c \text{ and } b>-(a-c) & \Leftrightarrow & b>\lvert a-c\rvert \\ c+a>b & & c>b-a \text{ and } c>a-b & & c>b-a \text{ and } c>-(b-a) & & c>\lvert b-a\rvert \end{array}$ Proof: $(\Rightarrow)$ Suppose $a+b>c$ and $b+c>a$ and $c+a>b$ From $a+b>c$, we get $a>c-b$ From $c+a>b$, we get $a>b-c$ Thus, $a>\max\{b-c,c-b\}=\lvert b-c\rvert$ By symmetry, From $a+b>c$ and $b+c>a$, we deduce that $b>\lvert a-c\rvert$ From $c+a>b$ and $b+c>a$, we deduce that $c>\lvert a-b\rvert$ Therefore, all three inequalities $a>\lvert b-c\rvert$, $b>\lvert a-c\rvert$ and $c>\lvert a-b\rvert$ hold $(\Leftarrow)$ Suppose $a>\lvert b-c\rvert$ and $b>\lvert a-c\rvert$ and $c>\lvert a-b\rvert$ From $a>\lvert c-b\rvert$, we get $a>c-b$ and $a>b-c$ by definition Thus, $a+b>c$ and $c+a>b$ By symmetry, From $b>\lvert a-c\rvert$, we deduce that $b+c>a$ Therefore, all three inequalities $a+b>c$, $b+c>a$ and $c+a>b$ hold We show that ($a+b>c$ and $b+c>a$ and $c+a>b$) if and only if ($a>\lvert b-c\rvert$ and $b>\lvert a-c\rvert$ and $c>\lvert a-b\rvert$).
20	2018/Q3 A triangle has sides of lengths *a*, *b* and *c* units. In each of the following cases, prove that there is a triangle having sides of the given lengths. (i) $\dfrac{a}{1+a}$, $\dfrac{b}{1+b}$ and $\dfrac{c}{1+c}$ units. [4] (ii) $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$ units. [3] (iii) $\sqrt{a(b+c-a)}$, $\sqrt{b(c+a-b)}$ and $\sqrt{c(a+b-c)}$ units. [6]
	The existence of the given triangle implies $a+b>c$ and $b+c>a$ and $a+c>b$ (*) (i) WLOG let $a\geq b\geq c$. By triangle inequality, we have $a<b+c$.

	Moreover, $\frac{x}{x+1}=1-\frac{1}{x+1}$ is an increasing function for $x>0$. So, $\frac{b}{b+1}+\frac{a}{a+1}>\frac{c}{c+1}$ and $\frac{c}{c+1}+\frac{a}{a+1}>\frac{b}{b+1}$, we only need to show $\frac{b}{b+1}+\frac{c}{c+1}>\frac{a}{a+1}$. Now, $\frac{b}{b+1}+\frac{c}{c+1}>\frac{b}{b+1}+\frac{c}{b+1}$ $=\frac{b+c}{b+1}$ $>\frac{a}{b+1}$ (by triangle inequality) $>\frac{a}{a+1}$. (ii) $\left(\sqrt{b}+\sqrt{c}\right)^2=b+c+2\sqrt{bc}$ $>a+2\sqrt{bc}$ by (*) $>a+0$ $=\left(\sqrt{a}\right)^2$ Implying $\sqrt{b}+\sqrt{c}>\sqrt{a}$. Similarly, $\sqrt{a}+\sqrt{b}>\sqrt{c}$ and $\sqrt{a}+\sqrt{c}>\sqrt{b}$ (iii) From (ii) it is sufficient to show that $a(b+c-a)$, $b(c+a-b)$ and $c(a+b-c)$ are sides of a triangle. By symmetry of the expressions, it suffices to show that $a(b+c-a)+b(c+a-b)>c(a+b-c)$. $a(b+c-a)+b(c+a-b)-c(a+b-c)$ $=ab+ac-a^2+bc+ab-b^2-ac-bc+c^2$ $=c^2-a^2-b^2+2ab$ $=c^2-(a-b)^2$ $=[c-a+b][c+a-b]$ >0 by (*) So $a(b+c-a)+b(c+a-b)>c(a+b-c)$. $a(b+c-a)+c(a+b-c)>b(c+a-b)$ and $c(a+b-c)+b(c+a-b)>a(b+c-a)$ can be shown similarly.