

SUMMARY: Probability

For uniform sample space

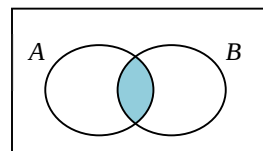
$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}$$

Combination of events

A and B

This means outcomes satisfying both A and B , i.e. $A \cap B$

- To calculate $P(A \cap B)$, determine the actual outcomes in $A \cap B$



A or B

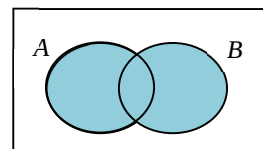
This means all outcomes in A or B or both, i.e. $A \cup B$

- To calculate $P(A \cup B)$, usually easier to use

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Formula can be extended to $P(A \cup B \cup C)$,

$$\begin{aligned} P(A \cup B \cup C) = & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(B \cap C) - P(C \cap A) \\ & + P(A \cap B \cap C) \end{aligned}$$



Conditional Probability

$P(A|B)$ is the probability of A given that B occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

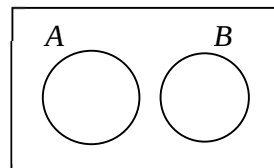
- Can be found by 'reduced sample space' method

Relationship of Two Events

Mutually Exclusive

No intersection of events A and B , i.e. $P(A \cap B) = 0$.

- Can be seen from Venn Diagram



Independence

Two events A and B are independent if the *probability* of A is not affected by whether B occurs or not.

- It cannot be seen on Venn diagram
- A and B are independent if and only if

$$P(A \cap B) = P(A)P(B) \quad \text{or} \quad P(A|B) = P(A)$$

- A and B independent implies A and B' are independent, etc.

Methods and Approaches

Students should remember the preferred method or approach for different types of questions.

(i) Venn Diagram

(ii) Tree Diagram

(iii) P & C / Probability Method

Example (Venn Diagram)

For events A and B , it is given that $P(A) = \frac{11}{20}$ and $P(B) = \frac{1}{2}$.

- (i) Find the greatest and least possible values of $P(A \cap B)$.

It is given in addition that $P(B|A') = \frac{7}{9}$.

- (ii) Find $P(A \cup B)$.

- (iii) Determine if A and B are independent events. Justify your answer.

Solution

- (i) $P(A \cup B) \leq 1$

$$P(A) + P(B) - P(A \cap B) \leq 1$$

$$P(A \cap B) \geq P(A) + P(B) - 1 = \frac{1}{20}$$

$$P(A \cap B)_{\min} = \frac{1}{20}$$

The biggest $P(A \cap B)$ occurs when B is a subset of A . Hence $P(A \cap B)_{\max} = \frac{1}{2}$.

$$\text{(ii)} \quad P(B|A') = \frac{P(B \cap A')}{P(A')} = \frac{7}{9}$$

$$\Rightarrow P(B \cap A') = \frac{7}{9} P(A') = \left(\frac{7}{9}\right) \left(\frac{9}{20}\right) = \frac{7}{20}$$

$$P(A \cup B) = P(B \cap A') + P(A) = \frac{7}{20} + \frac{11}{20} = \frac{9}{10} \quad (\text{from Venn diagram})$$

(iii) $P(B|A') = \frac{7}{9} \neq \frac{1}{2} = P(B)$, therefore B and A' are not independent and hence B and A are not independent.

OR

Since $P(B)P(A') = \left(\frac{1}{2}\right) \left(\frac{9}{20}\right) = \frac{9}{40} \neq \frac{7}{20} = P(B \cap A')$, therefore B and A' are not independent and hence B and A are not independent.

OR

$$P(A \cap B) = P(B) - P(B \cap A') = \frac{1}{2} - \frac{7}{20} = \frac{3}{20}$$

$$\text{Since } P(A)P(B) = \left(\frac{11}{20}\right) \left(\frac{1}{2}\right) = \frac{11}{40} \neq \frac{3}{20} = P(A \cap B),$$

A and B are not independent.

Example (Tree Diagram)

John and Peter play a game of chess. It is equally likely for either player to make the first move. If John makes the first move, the probability of him winning the game is 0.3 while the probability of Peter winning the game is 0.2. If Peter makes the first move, the probability of him winning the game is 0.5 while the probability of John winning the game is 0.4. If there is no winner, then the game ends in a draw.

- (i) Find the probability of a game ending in a draw.
 (ii) Find the probability that Peter made the first move given that he won the game.

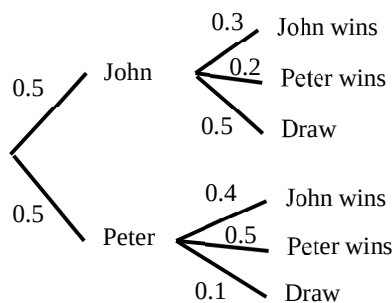
Solution

(i) $P(\text{draw}) = (0.5)(0.5) + (0.5)(0.1) = 0.3$

(ii) $P(\text{Peter first move} \mid \text{Peter won})$

$$= \frac{P(\text{Peter first move and Peter won})}{P(\text{Peter won})}$$

$$= \frac{(0.5)(0.5)}{(0.5)(0.2) + (0.5)(0.5)} = \frac{5}{7}$$



Example (P&C/Probability Method)

A bag contains 5 red, 3 green and 4 blue balls. Three balls are drawn randomly from the bag (without replacement). Find the probability that

- (i) all three balls are of different colour,
 (ii) all three balls are of the same colour,
 (iii) there are 2 green balls and 1 blue ball,
 (iv) there are more green balls given that there is at least 1 blue ball drawn.

Solution

(i) $P(\text{all different}) = \frac{{}^5C_1 {}^3C_1 {}^4C_1}{{}^{12}C_3} = \frac{3}{11}$ or $\frac{5}{12} \times \frac{3}{11} \times \frac{4}{10} \times 3! = \frac{3}{11}$

(ii) $P(\text{all same}) = P(\text{RRR}) + P(\text{GGG}) + P(\text{BBB})$

$$= \frac{{}^5C_3 + {}^3C_3 + {}^4C_3}{{}^{12}C_3} = \frac{3}{44}$$
 or $\frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} + \frac{3}{12} \times \frac{2}{11} \times \frac{1}{10} + \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{3}{44}$

(iii) $P(2 \text{ green } 1 \text{ blue}) = \frac{{}^3C_2 {}^4C_1}{{}^{12}C_3} = \frac{3}{55}$ or $\frac{3}{12} \times \frac{2}{11} \times \frac{4}{10} \times \frac{3!}{2!} = \frac{3}{55}$

(iv) $P(\text{more green} \mid \geq 1 \text{ blue})$

$$= \frac{P(\text{more green and } \geq 1 \text{ blue})}{P(\geq 1 \text{ blue})} = \frac{P(2 \text{ green and } 1 \text{ blue})}{P(\geq 1 \text{ blue})} = \frac{\frac{3}{55}}{1 - \frac{{}^8C_3}{{}^{12}C_3}} = \frac{3}{41}$$