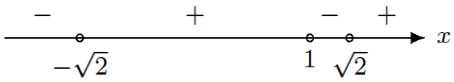
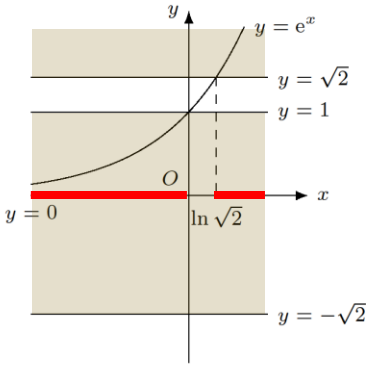
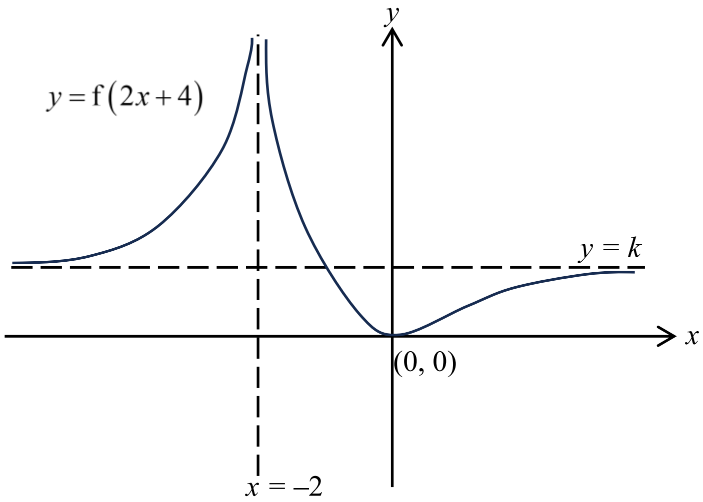
EJC 9758 2025 Prelim P1 Solutions

	Solution
1(a)	$x+1>\dfrac{1}{x-1}$ $x+1-\dfrac{1}{x-1}>0$ $\dfrac{(x+1)(x-1)-1}{x-1}>0$ $\dfrac{\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)}{x-1}>0$  $-\sqrt{2}<x<1$ or $x>\sqrt{2}$
(b)	 $x<0$ or $x>\ln\sqrt{2}=\dfrac{1}{2}\ln 2$

	Solution
2	Perimeter of outline = 180 $3a+2a+2b+\frac{1}{2}\pi(a)=180$ $\left(5+\frac{\pi}{2}\right)a+2b=180$ $\Rightarrow b=90-\left(\frac{5}{2}+\frac{\pi}{4}\right)a$ Area enclosed by wire, $A=3ab+\frac{1}{2}\pi\left(\frac{a}{2}\right)^2$ $=3a\left[90-\left(\frac{5}{2}+\frac{\pi}{4}\right)a\right]+\frac{\pi}{8}a^2$ $=270a-\frac{5\pi}{8}a^2-\frac{15}{2}a^2$ $=270a-\left(\frac{5\pi}{8}+\frac{15}{2}\right)a^2$ $\frac{\mathrm{d}A}{\mathrm{d}a}=270-\left(\frac{5\pi}{4}+15\right)a$ When $\frac{\mathrm{d}A}{\mathrm{d}a}$=0, $a=\frac{270}{\frac{5\pi}{4}+15}$ or 14.265 $\frac{\mathrm{d}^2A}{\mathrm{d}a^2}=-\left(\frac{5\pi}{4}+15\right)<0$ Hence the area is maximum when $a=\frac{270}{\frac{5\pi}{4}+15}$. Maximum area $=270\left(\frac{270}{\frac{5\pi}{4}+15}\right)-\left(\frac{5\pi}{8}+\frac{15}{2}\right)\left(\frac{270}{\frac{5\pi}{4}+15}\right)^2$ $=1925.82\text{ cm}^2$

	Solution
3(a)	$AC=\sqrt{13^2-5^2}=12$ $\cos\angle BAC=\dfrac{AC}{AX}=\dfrac{12}{13}$
(b)	$AX=\dfrac{12}{\cos\left(\angle BAC-\alpha\right)}$ $=\dfrac{12}{\cos\angle BAC\cos\alpha+\sin\angle BAC\sin\alpha}$ $=\dfrac{12}{\frac{12}{13}\cos\alpha+\frac{5}{13}\sin\alpha}$ $=\dfrac{156}{12\cos\alpha+5\sin\alpha}$ (shown)
(c)	Using small angle approximation, $AX\approx\dfrac{156}{12\left(1-\frac{\alpha^2}{2}\right)+5\alpha}$ $=156\left[12\left(1+\frac{5}{12}\alpha-\frac{\alpha^2}{2}\right)\right]^{-1}$ $=156\left[12\right]^{-1}\left[1+\left(\frac{5}{12}\alpha-\frac{\alpha^2}{2}\right)\right]^{-1}$ $=13\left[1-\left(\frac{5}{12}\alpha-\frac{\alpha^2}{2}\right)+\left(\frac{5}{12}\alpha-\frac{\alpha^2}{2}\right)^2+\ldots\right]$ $=13\left[1-\frac{5}{12}\alpha+\frac{\alpha^2}{2}+\left(\frac{5}{12}\alpha\right)^2+\ldots\right]$ $\approx13-\frac{65}{12}\alpha+\frac{1261}{144}\alpha^2$ $p=-\frac{65}{12},\quad q=\frac{1261}{144}$ (shown)

<table>
<tr><td></td><td>Solution</td></tr>
<tr><td>4
(a)</td><td>Translation of 2 units in the negative x-direction.

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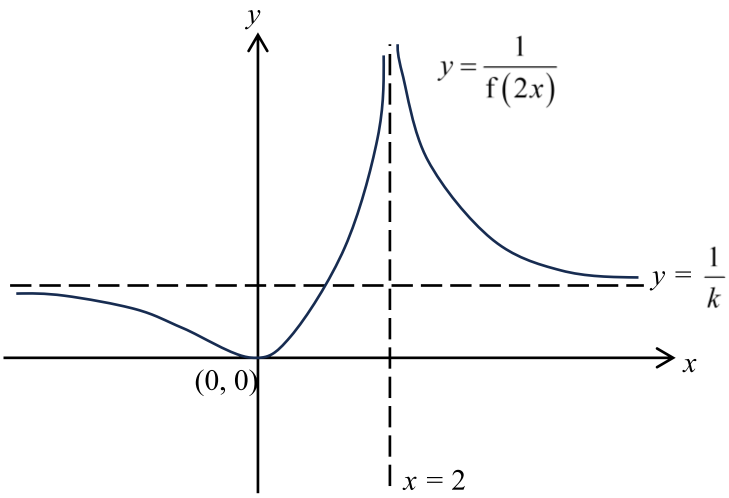
<tr><td>(b)</td><td>
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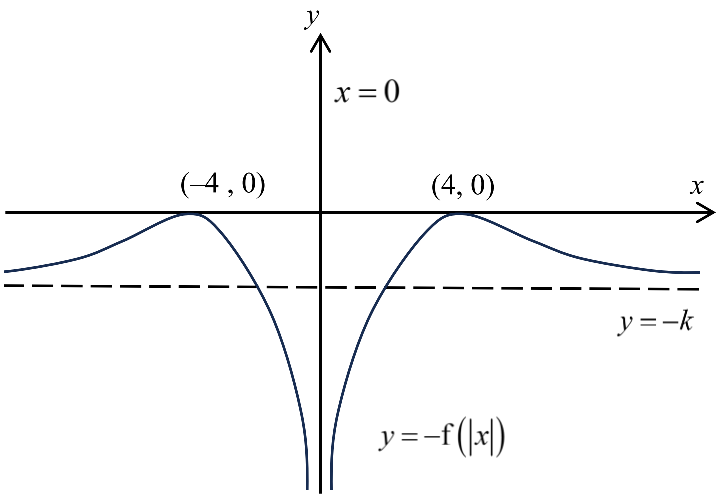
<tr><td>(c)</td><td>
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	Solution

5(a)

$$S_2 = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{7}{12}$$

Area under curve between $x = 1$ and $x = 2$ is $\int_1^2 \frac{1}{x}\,\mathrm{d}x = \left[\ln x\right]_1^2 = \ln 2$

Since the two rectangles lie under the curve, the area of the two rectangles must be less than the area under curve, so

$$\frac{7}{12} < \ln 2$$

(b)

We can use any integer larger than 2 for n.

Some possible answers are:

n	S_n
3	$\frac{37}{60}$
4	$\frac{533}{840}$
5	$\frac{1627}{2520}$
6	$\frac{15797}{27720}$

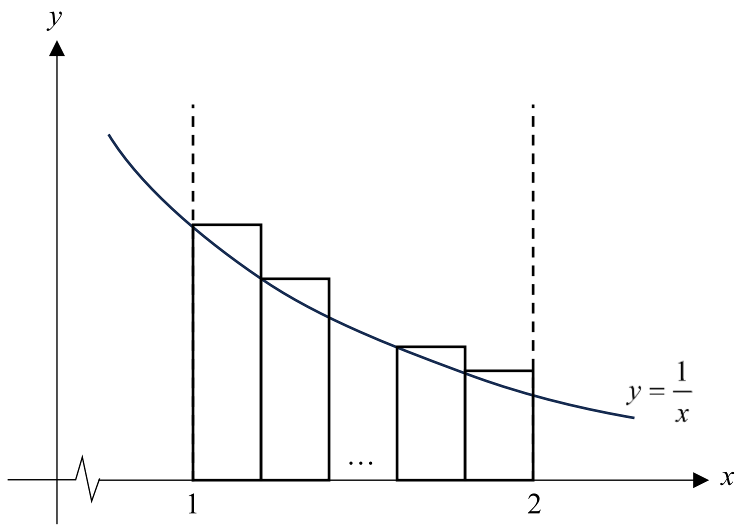
(c)

$$S_n = \left(\frac{1}{n}\right)\left(\frac{1}{\left(\frac{n+1}{n}\right)}\right) + \left(\frac{1}{n}\right)\left(\frac{1}{\left(\frac{n+2}{n}\right)}\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{1}{\left(\frac{2n}{n}\right)}\right)$$

$$= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$$

$$= \sum_{r=1}^{n} \frac{1}{n+r}$$

(d)



Consider n rectangles of equal width in the same interval, drawn above the curve.

Then

$\ln 2 <$ Area of rectangles

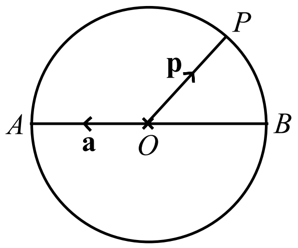
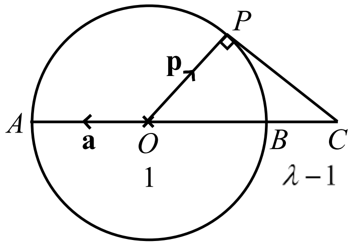
$$=\left(\frac{1}{n}\right)\left(\frac{1}{\frac{n}{n}}\right)+\left(\frac{1}{n}\right)\left(\frac{1}{\frac{n+1}{n}}\right)+\ldots+\left(\frac{1}{n}\right)\left(\frac{1}{\frac{2n-1}{n}}\right)$$

$$=\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n-1}$$

$$=\frac{1}{n}+\quad\left(S_n-\frac{1}{2n}\right)$$

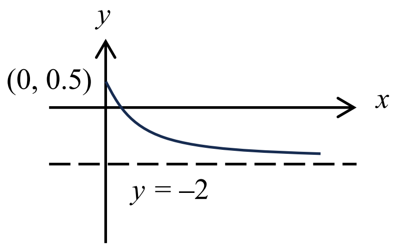
$$=S_n+\frac{1}{n}-\frac{1}{2n}$$

$$=S_n+\frac{1}{2n}$$

	Solution
6(a)	 $\overrightarrow{AP} = \mathbf{p} - \mathbf{a}$ $\overrightarrow{BP} = \mathbf{p} + \mathbf{a}$ $\overrightarrow{AP}.\overrightarrow{BP} = (\mathbf{p} - \mathbf{a}).(\mathbf{p} + \mathbf{a})$ $= \lvert\mathbf{p}\rvert^2 - \lvert\mathbf{a}\rvert^2$ $= \lvert\mathbf{a}\rvert^2 - \lvert\mathbf{a}\rvert^2$ (since OA and OP are radius, $\therefore \lvert\mathbf{p}\rvert = \lvert\mathbf{a}\rvert$) $= 0$
(b)	 By Ratio Theorem, $\mathbf{b} = \dfrac{(\lambda - 1)\mathbf{a} + \mathbf{c}}{\lambda}$ $\mathbf{c} = \lambda\mathbf{b} - (\lambda - 1)\mathbf{a}$ $\mathbf{c} = \lambda(-\mathbf{a}) - (\lambda - 1)\mathbf{a}$ $= (1 - 2\lambda)\mathbf{a}$
(c)	$\overrightarrow{PC} = (1 - 2\lambda)\mathbf{a} - \mathbf{p}$ $\overrightarrow{OP}.\overrightarrow{PC} = 0$ $\mathbf{p}.\left((1 - 2\lambda)\mathbf{a} - \mathbf{p}\right) = 0$ $(1 - 2\lambda)\mathbf{p}.\mathbf{a} - \mathbf{p}.\mathbf{p} = 0$ $(1 - 2\lambda)\lvert\mathbf{p}\rvert\lvert\mathbf{a}\rvert\cos 120^\circ - \lvert\mathbf{p}\rvert^2 = 0$ $-\dfrac{1}{2}(1 - 2\lambda)\lvert\mathbf{a}\rvert^2 - \lvert\mathbf{a}\rvert^2 = 0$ $-\dfrac{1}{2}(1 - 2\lambda) = 1$ $\lambda = \dfrac{3}{2}$

	Solution
7(a)	$u_2 = \frac{u_1 + u_3}{2} = \frac{9+b}{2}$
(b)	$v_2 = \pm\sqrt{9b} = \pm 3\sqrt{b}$
(c)	$u_2 - v_2 = 8$ $v_2 = u_2 - 8$ $v_2^{\,2} = (u_2 - 8)^2$ $9b = \frac{b^2 - 14b + 49}{4}$ $b^2 - 50b + 49 = 0$ $(b-1)(b-49) = 0$ $b = 1$ or $b = 49$ If $b = 49$, $\lvert r\rvert = \sqrt{\frac{49}{9}} = \frac{7}{3}$ but we need $\lvert r\rvert < 1$ since the sequence is convergent. So $b = 1$. $u_2 = \frac{9+1}{2} = 5$ $v_2 = 5 - 8 = -3$

	Solution
8(a)	$(a+b\mathrm{i})^2 = 2\mathrm{i}$ $\mathrm{a}^2 - b^2 + 2ab\mathrm{i} = 2\mathrm{i}$ $\mathrm{a}^2 - b^2 = 0 - (1)$ $2ab = 2 - (2)$ From (2), $a = \dfrac{1}{b}$ $(\dfrac{1}{b})^2 - b^2 = 0$ $1 - b^4 = 0$ $(1+b^2)(1-b^2) = 0$ $b^2 = 1$ ($\because$ b is real) So $b = 1, a = 1$ or $b = -1, a = -1$ (reject since $0 \le \arg(w) \le \dfrac{\pi}{2}$) $w = 1 + \mathrm{i}$
(b)	Since $1+\mathrm{i}$ is a root, we can write $z^3 - z^2 + (1-\mathrm{i})z + s = \left[z - (1+\mathrm{i})\right]\left(z^2 + Bz + C\right)$ for some (possibly complex) constants B and C. Comparing coefficients, z^2 : $-1 = -1 - \mathrm{i} + B \Rightarrow B = \mathrm{i}$ z : $1 - \mathrm{i} = -B - B\mathrm{i} + C \Rightarrow C = 0$ constant: $s = -(1+\mathrm{i})C \Rightarrow s = 0$ $z^2 + \mathrm{i}z = 0$ $z(z+\mathrm{i}) = 0$ $z = 0$ or $z = -\mathrm{i}$
(c)	Replace z with iv: $(\mathrm{i}v)^3 - (\mathrm{i}v)^2 + (1-\mathrm{i})(\mathrm{i}v) + s = 0$ $-\mathrm{i}v^3 + v^2 + (1+\mathrm{i})v + s = 0$ Hence, $z = \mathrm{i}v \Rightarrow v = -\mathrm{i}z$ $v = -\mathrm{i}(1+\mathrm{i}) = 1 - \mathrm{i}$, $v = -\mathrm{i}(0) = 0$ or $v = -\mathrm{i}(-\mathrm{i}) = -1$

	Solution
9(ai)	From GC, coordinates are $(-1,10)$ and $(3,-22)$
(aii)	Largest possible value is $a=-1$ Domain of f^{-1} is equal to range of f, which is $(-\infty,10]$
(bi)	$\mathrm{g}^2(x)=\dfrac{1-2\left(\dfrac{1-2x}{x+2}\right)}{\left(\dfrac{1-2x}{x+2}\right)+2}$ $=\dfrac{x+2-2(1-2x)}{1-2x+2(x+2)}$ $=\dfrac{5x}{5}$ $=x$
(bii)	$\mathrm{g}^n(x)=\begin{cases} x & \text{if } n \text{ is even} \\ \dfrac{1-2x}{x+2} & \text{if } n \text{ is odd} \end{cases}$
(ci)	$\mathrm{R_h}=[0,\infty)$ $\mathrm{D_g}=\mathbb{R}\backslash\{-2\}$ OR $\mathrm{D_g}=(-\infty,-2)\cup(-2,\infty)$ Since $\mathrm{R_h}\subseteq\mathrm{D_g}$, then gh exists.
(cii)	From graph of $y=\mathrm{g}(x)$ with domain restricted to $[0,\infty)$  $\mathrm{R_{gh}}=\left(-2,\tfrac{1}{2}\right]$

	Solution
10(a)	$\int_0^4 e^{\lvert x-1\rvert}\ dx$ $=\int_0^1 e^{1-x}\ dx+\int_1^4 e^{x-1}\ dx$ $=\left[-e^{1-x}\right]_0^1+\left[e^{x-1}\right]_1^4$ $=-1+e+e^3-1$ $=e^3+e-2$
(b)	$\int \sin^2 3x+\tan^2 3x\ dx$ $=\int \frac{1-\cos 6x}{2}+\sec^2 3x-1\ dx$ $=\int -\frac{1}{2}\cos 6x+\sec^2 3x-\frac{1}{2}\ dx$ $=-\frac{1}{12}\sin 6x+\frac{1}{3}\tan 3x-\frac{1}{2}x+C$
(c)	Since $\frac{dx}{du}=6u^5$, $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\ dx$ $=\int \frac{1}{u^3+u^2}\cdot 6u^5\ du$ $=6\int \frac{u^3}{u+1}\ du$ $=6\int u^2-u+1-\frac{1}{u+1}\ du$ $=6\left[\frac{1}{3}u^3-\frac{1}{2}u^2+u-\ln(u+1)\right]+C$ $=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln\left(\sqrt[6]{x}+1\right)+C$

	Solution
11(a)	$\frac{dy}{dt} = -k\sqrt{y}$ $\frac{1}{\sqrt{y}}\frac{dy}{dt} = -k$ $\int \frac{1}{\sqrt{y}}\,dy = \int -k\,dt$ $2\sqrt{y} = -kt + C$ When $t = 0$, $y = h$: $C = 2\sqrt{(h)} + k(0) = 2\sqrt{h}$ $2\sqrt{y} = -kt + 2\sqrt{h}$ $\sqrt{y} = \sqrt{h} - \frac{kt}{2}$
(b)	When $y = 0$, $\sqrt{h} - \frac{kt}{2} = 0$ $\sqrt{h} = \frac{kt}{2}$ $t = \frac{2\sqrt{h}}{k}$
(c)	Since $\cos t$ varies between -1 and 1, $\frac{1+\cos t}{2}$ varies between 0 and 1. So the flow is now slower.
(d)	$\frac{dy}{dt} = -k\left(\frac{1+\cos t}{2}\right)\sqrt{y}$ $\frac{1}{\sqrt{y}}\frac{dy}{dt} = -\frac{k(1+\cos t)}{2}$ $\int \frac{1}{\sqrt{y}}\,dy = -\frac{k}{2}\int 1+\cos t\,dt$ $2\sqrt{y} = -\frac{k}{2}(t+\sin t) + D$ When $t = 0$, $y = h$: $D = 2\sqrt{(h)} + \frac{k}{2}(0) = 2\sqrt{h}$ $2\sqrt{y} = -\frac{k}{2}(t+\sin t) + 2\sqrt{h}$

	$\sqrt{y}=\sqrt{h}-\frac{kt+k\sin t}{4}$
(e)	When $h=10$ and $k=0.1$, time taken in the original model is 63.25 (seconds). For the revised model, from GC, time taken is 126.08 (seconds). $\frac{126.08...}{63.25...}=1.99$ so the ratio is 1.99.