9225/1

Anglo-Chinese Junior College

SCHOOL PRELIMINARY EXAMINATIONS

FURTHER MATHEMATICS

SYLLABUS C

PAPER 1

Wednesday 28 August 1991 3 hours

Additional Materials provided by the College

1. Formulae (List MF6)
2. Mathematical Tables (on request)
3. Graph paper

Instructions to candidates:

Answer any **seven** *questions.*

Each question carries **14** *marks.*

The intended marks for questions or parts of questions are given in brackets [ ].

Begin each answer on a FRESH SHEET *of paper and arrange your answers in* NUMERICAL ORDER .

THE COVER SHEET PROVIDED MUST BE TIED TOGETHER WITH YOUR ANSWER SCRIPTS.

This Question Paper consists of 5 printed pages

[Turn over

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1a. Given that the roots of the equation

$$x^3 + px^2 + r = 0 \qquad (r \neq 0)$$

are α, β and γ, show that

(i) $\alpha^3 + \beta^3 + \gamma^3 = -p^3 - 3r$ [2]

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = 0$ [1]

(iii) $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3} + \dfrac{1}{\gamma^3} = \dfrac{-3}{r}$ [2]

Given also that $\alpha^3 + \beta^3 + \gamma^3 = 0$, show that

$$(\alpha + \beta + \gamma)^3\left(\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3}\right) = -9$$ [3]

1b. The curve C is defined by the equation

$$y = 2x + 1 + \frac{2}{(x + 2)}$$

State the equations of the asymptotes of C. Give the coordinates of any stationary points and points of intersection with the coordinate axes. [6]

2a. Verify that

$$n^2 + 6n + 4 = (n+2)(n+1) + 3(n+2) - 4$$

Making use of the method of differences, prove that

$$\sum_{n=1}^{k} \frac{n^2 + 6n + 4}{(n+2)!} = 3 - \frac{1}{(k+1)!} - \frac{4}{(k+2)!}$$

Find the sum as $k \rightarrow \infty$. [8]

2b. Prove by the method of mathematical induction that

$$3^{4n} + 2^{4n+1} \equiv 3 \pmod 5, \; n \in \mathbb{N}$$

[$x \equiv 3 \pmod 5$ if x leaves a remainder of 3 when divided by 5] [6]

3. If $u_n = \displaystyle\int_0^x \frac{dt}{(1 + t^2)^n}$ where $n \in \mathbb{N}$, prove that

$$2n\, u_{n+1} = (2n-1)\, u_n + \frac{x}{(1 + x^2)^n}$$ [6]

By using a suitable transformation $t = f(\theta)$, show that

$$\int_0^{\infty} \operatorname{sech}^{2n-1}\theta \, d\theta = \int_0^{\infty} \frac{dt}{(1 + t^2)^n}$$ [4]

Hence prove that

$$\int_0^{\infty} \operatorname{sech}^{2n-1}\theta \, d\theta = \frac{(2n-3)(2n-5)\ldots 3.1\, \pi}{2^n (n-1)!}$$ [4]

4. A curve $y = f(x)$ is defined parametrically as follows :

$$3x = t^3 - 3t\ , \quad 3y = t^3$$

(i) Find $\frac{dy}{dx}$. Hence find the stationary points of the curve and the points where $\frac{dy}{dx}$ is undefined. [2]

(ii) For what values of t is $x \geq 0$? Find the corresponding range of values of y. [1]

(iii) Sketch the graph of $y = f(x)$. [3]

Find the mean value of y with respect to x between the points $t = -1$ and $t = 0$. [2]

The arc of this curve between the points where $t = -1$ and $t = 0$ is rotated completely about the y-axis. Prove that the area of the curved surface thus formed is

$$\frac{5\sqrt{2}}{24}\pi\ \{\ln(1+\sqrt{2}) + \sqrt{2}\} \qquad [6]$$

(You may assume that the area of the surface of revolution generated by rotating a curve $y = f(x)$ between $y = y_0$ *and* $y = y_1$ about the y-axis is

$$2\pi \int_{y_0}^{y_1} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy)$$

5a. Find y in terms of x if

$$x(x+1)\frac{dy}{dx} + y = (x+1)^2 \cosh^{-1}\left(\frac{1}{x}\right), \qquad (0 < x < 1)$$

[5]

5b. By writing $u = yx^2$, transform the differential equation

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2(y - 2yx^2 - 6) = 0$$

into the form $\frac{d^2u}{dx^2} - au = b$ where a and b are positive integers. [4]

Hence express y in terms of x, given that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = \ln 2$.

[3]

Deduce that $y \geq 0$ for every non zero value of x.

[2]

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6a. Let P and Q represent the complex numbers z and w respectively. z and w are connected by the equation

$$w = 2\left(\frac{1+z}{1-z}\right).$$

Describe completely with the aid of clear diagrams, indicating directions, the locus of Q in each of the following cases :

a) P moves along the imaginary axis from ∞ to 0. [3]

b) P moves along the real axis in the positive direction from the origin. [2]

c) P moves once anticlockwise along the circle $|z| = 2$ from $z = 2$. [4]

6b. By using complex numbers prove that

$$\sum_{r=0}^{n} \cos r\theta \sec^r \theta = \frac{\sin (n+1)\theta \sec^n \theta}{\sin \theta}$$

where $\sin 2\theta \neq 0$. [5]

7. The linear transformation $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ is represented by the matrix **A** where

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 5 & 0 & 9 \\ 3 & -2 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 \\ 2 & 3 & 5 & 1 & 8 \end{pmatrix}$$

a. Find in either order

(i) A basis for the range space of T and its dimension. [5]

(ii) A basis for the null space of T and its dimension. [5]

b) Let **W** be the set of vectors which are solutions of

$$\mathbf{Ax} = \mathbf{b} \text{ where } \mathbf{b} = \begin{pmatrix} 15 \\ 3 \\ -3 \\ 15 \end{pmatrix}.$$

Find **W**. Explain why **W** is not a subspace of $\mathbb{R}^5$. [4]

8. Find the eigenvalues and corresponding eigenvectors of the matrix **A** where

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & -3 \\ -8 & 6 & -3 \\ 8 & -2 & 7 \end{pmatrix}$$ [7]

Hence, write down a matrix **P** and **D** such that $\mathbf{P^{-1}AP} = \mathbf{D}$, and calculate $\mathbf{P^{-1}}$. [4]

Write down a matrix **C** such that $\mathbf{C}^2 = \mathbf{D}$. Hence or otherwise find a matrix **B** such that $\mathbf{B}^2 = \mathbf{A}$. [3]

9a. Prove that the line through the point (x_1, y_1, z_1) perpendicular to the plane $ax + by + cz = d$ meets the plane at a point whose coordinates are $(x_1+ta,\ y_1+tb,\ z_1+tc)$ where

$$t = \frac{d - (ax_1+by_1+cz_1)}{a^2 + b^2 + c^2}$$

Hence, show that the perpendicular distance from the point to the plane is

$$\left| \frac{d - (ax_1+by_1+cz_1)}{\sqrt{(a^2 + b^2 + c^2)}} \right|$$

[6]

Hence, find the perpendicular distance of (1,2,3) from the plane passing through the points (0,1,1), (1,2,1) and (1,3,2). [3]

9b. Given that $\underset{\sim}{\mathbf{a}}$ is a constant vector which is perpendicular to a unit vector $\underset{\sim}{\hat{\mathbf{u}}}$, and that R is any point on the line $\underset{\sim}{\mathbf{r}} = \underset{\sim}{\mathbf{a}} + t\underset{\sim}{\hat{\mathbf{u}}}$. Show that the distance to R from a fixed point P whose position vector is $\underset{\sim}{\mathbf{p}}$ is given by

$$PR^2 = (\underset{\sim}{\mathbf{a}} - \underset{\sim}{\mathbf{p}}).(\underset{\sim}{\mathbf{a}} - \underset{\sim}{\mathbf{p}}) - 2t(\underset{\sim}{\hat{\mathbf{u}}} . \underset{\sim}{\mathbf{p}}) + t^2$$

Hence, show that the least value of PR^2 as t varies is

$$(\underset{\sim}{\mathbf{a}} - \underset{\sim}{\mathbf{p}}).(\underset{\sim}{\mathbf{a}} - \underset{\sim}{\mathbf{p}}) - (\underset{\sim}{\hat{\mathbf{u}}} . \underset{\sim}{\mathbf{p}})^2$$

Prove that PR is perpendicular to the given line.

[5]

10. Referred to a fixed origin O, the planes Π_1 and Π_2 have equations $\underset{\sim}{\mathbf{r}} . (2\underset{\sim}{\mathbf{i}} - \underset{\sim}{\mathbf{j}} + 2\underset{\sim}{\mathbf{k}}) = 9$ and $\underset{\sim}{\mathbf{r}} . (4\underset{\sim}{\mathbf{i}} + 3\underset{\sim}{\mathbf{j}} - \underset{\sim}{\mathbf{k}}) = 8$ respectively.

(a) Determine the shortest distance from O to the line of intersection of Π_1 and Π_2. [5]

(b) Find the cartesian equation of the plane Π_3 which is perpendicular to both Π_1 and Π_2 and passes through the point with position vector $2\underset{\sim}{\mathbf{j}} + \underset{\sim}{\mathbf{k}}$. [4]

(c) Find the position vector of the point which lies in Π_1, Π_2 and Π_3. [5]

End of Paper