1

(a) $$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\right)=\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}\right)=\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}$$

Normal vector of tangent plane: $\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\\ -1\end{pmatrix}=\begin{pmatrix}\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}\\ \frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}\\ -1\end{pmatrix}$

Since tangent plane at the point $P(a, b, c)$ is parallel to the plane with equation $x-z=1$,

$$\begin{pmatrix}\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}\\ \frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}\\ -1\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\Rightarrow \frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}=1$$

$$\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}=0\Rightarrow\frac{\partial f}{\partial u}=\frac{\partial f}{\partial v}$$

$$\therefore 2\frac{\partial f}{\partial u}=1\Rightarrow\frac{\partial f}{\partial u}=\frac{\partial f}{\partial v}=\frac{1}{2}$$, shown

Alternative Method

Since tangent plane at the point $P(a, b, c)$ is parallel to the plane with equation $x-z=1$,

$$\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\\ -1\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}\Rightarrow\frac{\partial f}{\partial x}=1,\frac{\partial f}{\partial y}=0$$

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u}\right)+\frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u}\right)=1\left(\frac{1}{2}\right)+0\left(\frac{1}{2}\right)=\frac{1}{2}$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial v}\right)+\frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial v}\right)=1\left(\frac{1}{2}\right)+0\left(-\frac{1}{2}\right)=\frac{1}{2}$$

$$\Rightarrow\frac{\partial f}{\partial u}=\frac{\partial f}{\partial v}=\frac{1}{2}$$, shown

(b) $$z=x+y-\sqrt{\frac{x+y}{x-y}}\Rightarrow z=u-\sqrt{\frac{u}{v}}$$

	$\frac{\partial f}{\partial u}=1-\frac{1}{2\sqrt{uv}}$, $\frac{\partial f}{\partial v}=\frac{\sqrt{u}}{2\left(\sqrt{v}\right)^3}$ $1-\frac{1}{2\sqrt{uv}}=\frac{1}{2}\Rightarrow\sqrt{v}=\frac{1}{\sqrt{u}}$ $\frac{\sqrt{u}}{2\left(\sqrt{v}\right)^3}=\frac{1}{2}\Rightarrow\frac{\sqrt{u}}{\left(\frac{1}{\sqrt{u}}\right)^3}=1\Rightarrow u^2=1\Rightarrow u=\pm1\Rightarrow u=1$ (reject $u=-1$) $\therefore v=1$ $x+y=1---(1)$ $x-y=1---(2)$ Solving gives $x=1, y=0$ Coordinates of point P is $(1,0,0)$.
2	**(a)** Let U be the specified subset. $\mathbf{u}_1=\begin{pmatrix}1\\-1\\0\end{pmatrix}\in U$, $\mathbf{u}_2=\begin{pmatrix}1\\1\\0\end{pmatrix}\in U$. However, $\mathbf{u}_1+\mathbf{u}_2=\begin{pmatrix}2\\0\\0\end{pmatrix}\notin U$, since $2^2\neq0^2$. Not closed under vector addition. Therefore, it does not form a linear space. **(b)** Let W be the specified subset and let $\mathbf{w}_1=\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}$ and $\mathbf{w}_2=\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}$ be two arbitrary vectors in W, i.e. $2a_1-b_1+c_1=0$ and $2a_2-b_2+c_2=0$. For any real constants $\lambda,\mu\in\mathbb{R}$, $\lambda\mathbf{w}_1+\mu\mathbf{w}_2=\begin{pmatrix}\lambda a_1+\mu a_2\\\lambda b_1+\mu b_2\\\lambda c_1+\mu c_2\end{pmatrix}$. Here, $2(\lambda a_1+\mu a_2)-(\lambda b_1+\mu b_2)+(\lambda c_1+\mu c_2)=\lambda(2a_1-b_1+c_1)+\mu(2a_2-b_2+c_2)=0$. Closed under vector addition and scalar multiplication. W is a subspace. **(b)** $\mathbf{M}^2+2\mathbf{M}+3\mathbf{I}=\mathbf{0}\Rightarrow\mathbf{M}^3+2\mathbf{M}^2+3\mathbf{M}=\mathbf{0}$ $\mathbf{M}^3=-2\mathbf{M}^2-3\mathbf{M}$ $=-2(-2\mathbf{M}-3\mathbf{I})-3\mathbf{M}$ $=\mathbf{M}+6\mathbf{I}$ (shown)

	$\mathbf{M}^8+2\mathbf{M}^7+4\mathbf{M}^6+\mathbf{M}=\mathbf{M}^6\left(\mathbf{M}^2+2\mathbf{M}+4\mathbf{I}\right)+\mathbf{M}$ $=\left(\mathbf{M}^3\right)^2\left((\mathbf{M}^2+2\mathbf{M}+3\mathbf{I})+\mathbf{I}\right)+\mathbf{M}$ $=\left(\mathbf{M}+6\mathbf{I}\right)^2\left(\mathbf{I}\right)+\mathbf{M}$ $=\mathbf{M}^2+12\mathbf{M}+36\mathbf{I}+\mathbf{M}$ $=\left(-2\mathbf{M}-3\mathbf{I}\right)+13\mathbf{M}+36\mathbf{I}$ $=11\mathbf{M}+33\mathbf{I}$ $=11(\mathbf{M}+3\mathbf{I})$
3	$w=\frac{\mathrm{d}y}{\mathrm{d}x}+2x\Rightarrow\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\Rightarrow\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ $\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+4\left(\frac{\mathrm{d}w}{\mathrm{d}x}-2\right)+4\left(w-2x\right)+8\left(x+1\right)=\mathrm{e}^{-2x}$ $\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+4\frac{\mathrm{d}w}{\mathrm{d}x}+4w=\mathrm{e}^{-2x}$ Consider Homogeneous Equation: $\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+4\frac{\mathrm{d}w}{\mathrm{d}x}+4w=0$ Characteristic Equation: $m^2+4m+4=0$ $m=\frac{-4\pm\sqrt{4^2-4(1)\left(4\right)}}{2}=-2$ Complementary Function: $w_c=\left(Ax+B\right)\mathrm{e}^{-2x}$ where A and B are arbitrary constants. Let P.I. be $w_p=Cx^2\mathrm{e}^{-2x}$ $w'_p=2Cx\mathrm{e}^{-2x}-2Cx^2\mathrm{e}^{-2x}$ $w''_p=2C\mathrm{e}^{-2x}-4Cx\mathrm{e}^{-2x}-4Cx\mathrm{e}^{-2x}+4Cx^2\mathrm{e}^{-2x}$ $2C\mathrm{e}^{-2x}-4Cx\mathrm{e}^{-2x}-4Cx\mathrm{e}^{-2x}+4Cx^2\mathrm{e}^{-2x}+4\left(2Cx\mathrm{e}^{-2x}-2Cx^2\mathrm{e}^{-2x}\right)+4Cx^2\mathrm{e}^{-2x}=\mathrm{e}^{-2x}$ $2C\mathrm{e}^{-2x}-8Cx\mathrm{e}^{-2x}+4Cx^2\mathrm{e}^{-2x}+8Cx\mathrm{e}^{-2x}-8Cx^2\mathrm{e}^{-2x}+4Cx^2\mathrm{e}^{-2x}=\mathrm{e}^{-2x}$ $2C\mathrm{e}^{-2x}=\mathrm{e}^{-2x}$ Comparing coefficients, $2C=1\Rightarrow C=\frac{1}{2}$ $\therefore w_p=\frac{1}{2}x^2\mathrm{e}^{-2x}$ $w=\left(Ax+B\right)\mathrm{e}^{-2x}+\frac{1}{2}x^2\mathrm{e}^{-2x}$ where A and B are arbitrary constants $\frac{\mathrm{d}y}{\mathrm{d}x}+2x=\left(Ax+B\right)\mathrm{e}^{-2x}+\frac{1}{2}x^2\mathrm{e}^{-2x}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\left(Ax+B\right)\mathrm{e}^{-2x}+\frac{1}{2}x^2\mathrm{e}^{-2x}-2x$

	$\frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{1}{2}x^2+Ax+B\right)\mathrm{e}^{-2x}-2x$ $y=\int\left[\left(\frac{1}{2}x^2+Ax+B\right)\mathrm{e}^{-2x}-2x\right]\,\mathrm{d}x$ $=-\frac{1}{2}\mathrm{e}^{-2x}\left(\frac{1}{2}x^2+Ax+B\right)-\int-\frac{1}{2}\mathrm{e}^{-2x}\left(x+A\right)\,\mathrm{d}x-x^2$ $=-\frac{1}{2}\mathrm{e}^{-2x}\left(\frac{1}{2}x^2+Ax+B\right)-\frac{1}{4}\mathrm{e}^{-2x}\left(x+A\right)+\int\frac{1}{4}\mathrm{e}^{-2x}\,\mathrm{d}x-x^2$ $=-\frac{1}{2}\mathrm{e}^{-2x}\left(\frac{1}{2}x^2+Ax+B\right)-\frac{1}{4}\mathrm{e}^{-2x}\left(x+A\right)-\frac{1}{8}\mathrm{e}^{-2x}-x^2+D$ $=-\frac{1}{8}\mathrm{e}^{-2x}\left(2x^2+4Ax+2x+4B+2A+1\right)-x^2+D$ $=-\frac{1}{8}\mathrm{e}^{-2x}\left(2x^2+C_1x+C_2\right)-x^2+D$ where C_1, C_2 and D are arbitrary constants.
4	**(a)** $\mathbf{v}_{n+1}=\begin{pmatrix}0 & 1\\-10 & 7\end{pmatrix}\mathbf{v}_n$ **(b)** $\det\left(\mathbf{A}-\lambda\mathbf{I}\right)=\det\begin{pmatrix}-\lambda & 1\\-10 & 7-\lambda\end{pmatrix}=-7\lambda+\lambda^2+10=0$ $(\lambda-5)(\lambda-2)=0$ $\lambda=5$ or 2 For $\lambda=5$: $\mathbf{A}-5\mathbf{I}=\begin{pmatrix}-5 & 1\\-10 & 2\end{pmatrix}\longrightarrow\begin{pmatrix}-5 & 1\\0 & 0\end{pmatrix}$ $-5x+y=0\Rightarrow y=5x$ $\begin{pmatrix}x\\y\end{pmatrix}=x\begin{pmatrix}1\\5\end{pmatrix},\ x\in\mathbb{R}$ An eigenvector is $\begin{pmatrix}1\\5\end{pmatrix}$. For $\lambda=2$: $\mathbf{A}-2\mathbf{I}=\begin{pmatrix}-2 & 1\\-10 & 5\end{pmatrix}\longrightarrow\begin{pmatrix}-2 & 1\\0 & 0\end{pmatrix}$ $-2x+y=0\Rightarrow y=2x$ $\begin{pmatrix}x\\y\end{pmatrix}=x\begin{pmatrix}1\\2\end{pmatrix},\ x\in\mathbb{R}$ An eigenvector is $\begin{pmatrix}1\\2\end{pmatrix}$. **(c)** $\mathbf{v}_n=\mathbf{A}\mathbf{v}_{n-1}=\mathbf{A}^2\mathbf{v}_{n-2}=\ldots=\mathbf{A}^n\mathbf{v}_0$ $=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}\mathbf{v}_0$ $=\begin{pmatrix}1 & 1\\5 & 2\end{pmatrix}\begin{pmatrix}5^n & 0\\0 & 2^n\end{pmatrix}\begin{pmatrix}1 & 1\\5 & 2\end{pmatrix}^{-1}\mathbf{v}_0$ $=-\frac{1}{3}\begin{pmatrix}5^n & 2^n\\5^{n+1} & 2^{n+1}\end{pmatrix}\begin{pmatrix}2 & -1\\-5 & 1\end{pmatrix}\mathbf{v}_0$ $=-\frac{1}{3}\begin{pmatrix}2(5^n)-5(2^n) & -5^n+2^n\\\cdots & \cdots\end{pmatrix}\begin{pmatrix}2\\9\end{pmatrix}$

	$a_n = -\frac{1}{3}\left[4(5^n)-10(2^n)-9(5^n)+9(2^n)\right]$ $= -\frac{1}{3}\left[-5(5^n)-2^n\right]$ $= \frac{1}{3}\left(5^{n+1}+2^n\right)$ **(d)** If the matrix associated with the recurrence relation is not diagonalisable (only one unique eigenvalue and the dimension of its eigenspace is 1, e.g. $b_{n+2} = 6b_{n+1} - 9b_n$), then it is not possible to adopt the previous approach directly.
5	(a) $\int_0^1 \frac{1}{e^x + e^{-x}}\,dx = \int_0^1 e^x \cdot \frac{1}{1+(e^x)^2}\,dx$ $= \left[\tan^{-1}\left(e^x\right)\right]_0^1$ $= \tan^{-1} e - \tan^{-1} 1$ Let $C = \tan^{-1} e$, $D = \tan^{-1} 1$. $\tan(C-D) = \frac{\tan C - \tan D}{1+\tan C \tan D} = \frac{e-1}{1+e(1)}$. $\therefore \int_0^1 \frac{1}{e^x+e^{-x}}\,dx = \tan^{-1}\left(\frac{e-1}{e+1}\right)$ (shown) (Note that in this case, $\tan^{-1} 1 < \tan^{-1} e < \frac{\pi}{2}$) (b) <u>Simpson's Rule</u> Step size, $h = \frac{1}{6}$.

n	0	1	2	3	4	5	6
x_n	0	1/6	1/3	1/2	2/3	5/6	1
y_n	0.5	0.493135	0.473453	0.443409	0.406314	0.365554	0.324027

$$\int_0^1 \frac{1}{e^x+e^{-x}}\,dx \approx \frac{h}{3}\left[y_0 + y_6 + 4\left(y_1+y_3+y_5\right)+2\left(y_2+y_4\right)\right] \approx 0.432886337$$

<u>Series Expansion</u>

$$\frac{1}{e^x+e^{-x}} = \left(1+x+\frac{x^2}{2!}+\ldots+1-x+\frac{x^2}{2!}-\ldots\right)^{-1}$$
$$= \left(2+x^2+\ldots\right)^{-1}$$
$$= \frac{1}{2}\left(1+\frac{1}{2}x^2+\ldots\right)^{-1}$$
$$= \frac{1}{2}-\frac{1}{4}x^2+\ldots$$
$$\int_0^1 \frac{1}{e^x+e^{-x}}\,dx \approx \int_0^1 \left(\frac{1}{2}-\frac{1}{4}x^2\right)dx = \frac{5}{12}$$

Let the exact answer be A.

Percentage error using Simpson's rule $= \frac{\left|0.432886337 - A\right|}{A} \times 100\% = 0.000369\%$

Percentage error using series expansion $= \frac{\left|5/12 - A\right|}{A} \times 100\% = 3.75\%$

	(c) Student A (using Simpson's rule) would obtain a better approximation. With the same step-size, Simpson's rule performs similarly for a wider interval. As for series expansion, it is only accurate near $x = 0$ and becomes increasingly inaccurate in approximating the original curve as we move away from this point.
6	(a) Using GP sum, $z+z^2+\ldots+z^n=\dfrac{z(1-z^n)}{1-z}$ Let $z=\mathrm{e}^{\mathrm{i}\theta}$. So $\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{\mathrm{i}2\theta}+\ldots+\mathrm{e}^{\mathrm{i}n\theta}=\dfrac{\mathrm{e}^{\mathrm{i}\theta}(1-\mathrm{e}^{\mathrm{i}n\theta})}{1-\mathrm{e}^{\mathrm{i}\theta}}$ $=\dfrac{\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}(n\theta/2)}(\mathrm{e}^{\mathrm{i}(-n\theta/2)}-\mathrm{e}^{\mathrm{i}(n\theta/2)})}{\mathrm{e}^{\mathrm{i}(\theta/2)}(\mathrm{e}^{\mathrm{i}(-\theta/2)}-\mathrm{e}^{\mathrm{i}(\theta/2)})}$ $=\dfrac{\mathrm{e}^{\mathrm{i}((n+1)\theta/2)}\left(-2\mathrm{i}\sin\dfrac{n\theta}{2}\right)}{-2\mathrm{i}\sin\dfrac{\theta}{2}}$ $=\dfrac{\left(\cos\dfrac{(n+1)\theta}{2}+\mathrm{i}\sin\dfrac{(n+1)\theta}{2}\right)\left(\sin\dfrac{n\theta}{2}\right)}{\sin\dfrac{\theta}{2}}$ Comparing real and imaginary parts, $\displaystyle\sum_{r=1}^{n}\cos r\theta=\cos\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2}$ and $\displaystyle\sum_{r=1}^{n}\sin r\theta=\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2}$. (b) Taking modulus and squaring: $(\cos\theta+\cos2\theta+\ldots+\cos n\theta)^2+(\sin\theta+\sin2\theta+\ldots+\sin n\theta)^2=\left(\dfrac{\sin\dfrac{n\theta}{2}}{\sin\dfrac{\theta}{2}}\right)^2$ (c) Take $\theta=\dfrac{2\pi}{n+1}$. From (b), the expression is equal to $\left(\dfrac{\sin\left(\dfrac{n\pi}{n+1}\right)}{\sin\left(\dfrac{\pi}{n+1}\right)}\right)^2=\left(\dfrac{\sin\left(\pi-\dfrac{n\pi}{n+1}\right)}{\sin\left(\dfrac{\pi}{n+1}\right)}\right)^2=\left(\dfrac{\sin\left(\dfrac{\pi}{n+1}\right)}{\sin\left(\dfrac{\pi}{n+1}\right)}\right)^2=1$ (constant independent of n)
7	(a) $\int\sin^n x\,\mathrm{d}x=-\cos x\sin^{n-1}x-\int(-\cos x)(n-1)\sin^{n-2}x\cos x\,\mathrm{d}x$ $=-\cos x\sin^{n-1}x+(n-1)\int(1-\sin^2 x)\sin^{n-2}x\,\mathrm{d}x$ $n\int\sin^n x\,\mathrm{d}x=-\cos x\sin^{n-1}x+(n-1)\int\sin^{n-2}x\,\mathrm{d}x$ Dividing throughout by n, $\int\sin^n x\,\mathrm{d}x=-\dfrac{\cos x\sin^{n-1}x}{n}+\left(1-\dfrac{1}{n}\right)\int\sin^{n-2}x\,\mathrm{d}x$ (shown). (b)(i) Let $I_n=\int_0^{\frac{\pi}{2}}\sin^n\theta\,\mathrm{d}\theta$. Then $I_n=\left(1-\dfrac{1}{n}\right)I_{n-2}-\left[\dfrac{\cos x\sin^{n-1}x}{n}\right]_0^{\frac{\pi}{2}}=\left(\dfrac{n-1}{n}\right)I_{n-2}$, and $I_0=\dfrac{\pi}{2}$. Then $I_2=\dfrac{1}{2}I_0=\dfrac{\pi}{4}$, $I_4=\dfrac{3}{4}I_2=\dfrac{3\pi}{16}$, $I_6=\dfrac{5}{6}I_4=\dfrac{5\pi}{32}$, $I_8=\dfrac{7}{8}I_6=\dfrac{35\pi}{256}$.

	Required area = $2\times\int_0^{\frac{\pi}{2}}\frac{1}{2}\left(\sin^4\theta\right)^2\ \mathrm{d}\theta = I_8 = \frac{35\pi}{256}$. (ii) At intersection, $\sin^4\theta = \cot^4\theta$ $\sin^8\theta = \cos^4\theta$ $\left(\sin^2\theta\right)^4 = (1-\sin^2\theta)^2$ Let $u=\sin^2\theta$. Then $u^4=(1-u)^2$. Since $u\geq 0$, $u^2 = 1-u \Rightarrow u = \frac{\sqrt{5}-1}{2}$. $\sin^2\theta = \frac{\sqrt{5}-1}{2} \Rightarrow \sin\theta = \sqrt{\frac{\sqrt{5}-1}{2}}$ (since sine is positive in the first 2 quadrants) Note that $\frac{1}{\phi} = \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right)} = \frac{2\left(1-\sqrt{5}\right)}{\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)} = \frac{\sqrt{5}-1}{2}$. So $\sin\theta = \frac{1}{\sqrt{\phi}}$. y-coordinate $= r\sin\theta = \sin^5\theta = \phi^{-\frac{5}{2}}$. **(iii)** Length $=2\int_{\sin^{-1}\frac{1}{\sqrt{\phi}}}^{\frac{\pi}{2}}\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$ $=2\int_{\sin^{-1}\frac{1}{\sqrt{\phi}}}^{\frac{\pi}{2}}\sqrt{\cot^8\theta+\left(-4\operatorname{cosec}^2\theta\cot^3\theta\right)^2}\ \mathrm{d}\theta$ ≈ 0.76861 $=0.769$ (3 d.p.)
8	(a) $1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2$ $=1+\frac{x^2}{4}-\frac{1}{2}+\frac{1}{4x^2}$ $=\frac{1}{4}\left(x^2+2+\frac{1}{x^2}\right)$ $=\frac{1}{4}\left(x+\frac{1}{x}\right)^2$ Arc length $=\int_2^4\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \frac{1}{2}\int_2^4\left(x+\frac{1}{x}\right)\mathrm{d}x$ $=\frac{1}{2}\left[\frac{1}{2}x^2+\ln x\right]_2^4$ $=\frac{1}{2}\left[(8+\ln 4)-(2+\ln 2)\right]$ $=3+\frac{\ln 2}{2} = 3+\ln\sqrt{2}$ (shown)

(b) $S=2\pi\int_2^4 y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$

$=2\pi\int_2^4 \frac{1}{4}\left(x^2-2\ln x\right)\cdot\frac{1}{2}\left(x+\frac{1}{x}\right)\mathrm{d}x$

$=\frac{\pi}{4}\int_2^4\left(x^3+x-2x\ln x-\frac{2\ln x}{x}\right)\mathrm{d}x$

$=\frac{\pi}{4}\left(\left[\frac{1}{4}x^4+\frac{1}{2}x^2\right]_2^4-2\left(\left[\frac{1}{2}x^2\ln x\right]_2^4-\int_2^4\frac{1}{2}x\,\mathrm{d}x\right)-2\left[\frac{1}{2}(\ln x)^2\right]_2^4\right)$

$=\frac{\pi}{4}\left[66-(16\ln 4-4\ln 2)+6-3(\ln 2)^2\right]$

$=\frac{\pi}{4}\left[72-28\ln 2-3(\ln 2)^2\right]$

$=\pi\left(18-7\ln 2-\frac{3}{4}(\ln 2)^2\right)$

(c) Need to use shell method. First find the x-coordinates when y = 1, 3.
When $y=1$, $x=2.3977138$. When $y=3$, $x=3.8323570$.

Required volume $=\overbrace{\pi(2.3977138)^2(2)}^{\text{Volume of cylinder}}+2\pi\int_{2.3977138}^{3.8323570}x(3-y)\,\mathrm{d}x$

$=36.12223+2\pi\int_{2.3977138}^{3.8323570}x\left(3-\frac{1}{4}\left(x^2-2\ln x\right)\right)\mathrm{d}x$

$=64.708788=64.7$ (3 s.f.)

9

(a) Let $\mathrm{f}(x,y)=\sin x\cos x-y\cos x$

$y\left(\frac{h}{2}\right)\approx y_0+\frac{h}{2}\left(\mathrm{f}(0,y_0)\right)=y_0\left(1-\frac{h}{2}\right)$

$y(h)\approx y_0\left(1-\frac{h}{2}\right)+\frac{h}{2}\left(\mathrm{f}\left(\frac{h}{2},y\left(\frac{h}{2}\right)\right)\right)$

$=y_0\left(1-\frac{h}{2}\right)+\frac{h}{2}\left(\sin\frac{h}{2}\cos\frac{h}{2}-y_0\left(1-\frac{h}{2}\right)\cos\frac{h}{2}\right)$

$=y_0\left(1-\frac{h}{2}-\frac{h}{2}\cos\frac{h}{2}+\frac{h^2}{4}\cos\frac{h}{2}\right)+\frac{h}{2}\sin\frac{h}{2}\cos\frac{h}{2}$

$\approx y_0\left(1-\frac{h}{2}-\frac{h}{2}\left(1-\frac{h^2}{8}\right)+\frac{h^2}{4}\left(1-\frac{h^2}{8}\right)\right)+\frac{h}{2}\left(\frac{h}{2}\right)\left(1-\frac{h^2}{8}\right)$ ($\because h$ is small)

$\approx y_0\left(1-h+\frac{h^2}{4}\right)+\frac{h^2}{4}$

(b) $y^*(h)=y_0+h\left(\mathrm{f}(0,y_0)\right)=y_0(1-h)$

	$y(h) \approx y_0 + \frac{h}{2}\left(\mathrm{f}(0, y_0) + \mathrm{f}\left(h, y^*(h)\right)\right)$ $= y_0 + \frac{h}{2}\left(-y_0 + \sin h \cos h - y_0(1-h)\cos h\right)$ $= y_0\left(1 - \frac{h}{2} - \frac{h}{2}(1-h)\cos h\right) + \frac{h}{2}\sin h \cos h$ $\approx y_0\left(1 - \frac{h}{2} - \frac{h}{2}(1-h)\left(1 - \frac{h^2}{2}\right)\right) + \frac{h}{2}(h)\left(1 - \frac{h^2}{2}\right)$ $\approx y_0\left(1 - h + \frac{h^2}{2}\right) + \frac{h^2}{2}$ (c) $\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = \sin x \cos x$ Integrating Factor: $\mathrm{e}^{\int \cos x \,\mathrm{d}x} = \mathrm{e}^{\sin x}$ $\mathrm{e}^{\sin x}\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{\sin x} y\cos x = \mathrm{e}^{\sin x}\sin x \cos x$ $\mathrm{e}^{\sin x} y = \int\left(\cos x \mathrm{e}^{\sin x}\right)\sin x \,\mathrm{d}x$ $= \mathrm{e}^{\sin x}\sin x - \int \cos x \mathrm{e}^{\sin x} \,\mathrm{d}x$ $= \mathrm{e}^{\sin x}\sin x - \mathrm{e}^{\sin x} + c$ $\Rightarrow y = \sin x - 1 + c\mathrm{e}^{-\sin x}$ $y(0) = y_0 \Rightarrow y_0 = -1 + c \Rightarrow c = y_0 + 1$ $\therefore y = \sin x - 1 + \left(y_0 + 1\right)\mathrm{e}^{-\sin x}$ $y(h) = \sin h - 1 + \left(y_0 + 1\right)\mathrm{e}^{-\sin h}$ $\approx h - 1 + \left(y_0 + 1\right)\mathrm{e}^{-h}$ $\approx h - 1 + \left(y_0 + 1\right)\left(1 - h + \frac{h^2}{2}\right)$ $= y_0\left(1 - h + \frac{h^2}{2}\right) + \frac{h^2}{2}$ (d) The results show that the improved Euler's method with one step leads to a much more accurate answer as compared to the Euler's method even with two steps.
10	(a) $a + bP_{n-1} = c - dP_n$ $P_n = \frac{c-a}{d} - \frac{b}{d}P_{n-1}$ $P_n = \alpha - \beta P_{n-1}$

$$P_n = \alpha - \beta\left(\alpha - \beta P_{n-2}\right)$$
$$= \alpha - \beta\alpha + \beta^2 P_{n-2}$$
$$= \alpha - \beta\alpha + \beta^2\alpha - \beta^3 P_{n-3} = \ldots = \alpha\left(1 - \beta + \beta^2 - \ldots + (-1)^{n-1}\beta^{n-1}\right) + (-1)^n \beta^n P_0$$

$$P_n = \alpha\left(\frac{1-(-\beta)^n}{1-(-\beta)}\right) + (-\beta)^n P_0$$

$$= \frac{\alpha}{1+\beta}\left(1-(-\beta)^n\right) + (-\beta)^n P_0$$

$$= \frac{\alpha}{1+\beta} + \left(P_0 - \frac{\alpha}{1+\beta}\right)(-\beta)^n$$

(b) $0 < \beta < 1 \Rightarrow (-\beta)^n \to 0$ when $n \to \infty$

$$\therefore P_n \to \frac{\alpha}{1+\beta}$$

The price of coffee approaches $\dfrac{\alpha}{1+\beta}$ in the long run.

(c) $S_n = a + \dfrac{b}{2}\left(P_{n-1} + P_{n-2}\right)$

$$a + \frac{b}{2}\left(P_{n-1} + P_{n-2}\right) = c - dP_n$$

$$\Rightarrow P_n + \frac{b}{2d}\left(P_{n-1} + P_{n-2}\right) = \frac{c-a}{d}$$

$$\Rightarrow P_n + \frac{1}{2}\left(P_{n-1} + P_{n-2}\right) = 2P_0$$

$$\Rightarrow 2P_n + P_{n-1} + P_{n-2} = 4P_0$$

(d) Consider $P_n + \dfrac{1}{2}\left(P_{n-1} + P_{n-2}\right) = 0$

A.E: $\quad m^2 + \dfrac{1}{2}m + \dfrac{1}{2} = 0$

$$m = \frac{-\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)}}{2} = \frac{1}{4}\left(-1 \pm \sqrt{7}i\right) = \frac{1}{\sqrt{2}}\mathrm{e}^{\pm i\theta} \text{ where } \theta = \pi - \tan^{-1}\sqrt{7}$$

Since $\dfrac{\alpha}{1+\beta} = \dfrac{2P_0}{1+1} = P_0$ is a solution,

$$P_n = \left(\frac{1}{\sqrt{2}}\right)^n \left(A\cos n\theta + B\sin n\theta\right) + P_0$$

$$P_0 = \left(\frac{1}{\sqrt{2}}\right)^0 \left(A\cos 0 + B\sin 0\right) + P_0 \Rightarrow A = 0$$

$$P_1 = \frac{6}{5}P_0 \Rightarrow \frac{6}{5}P_0 = \left(\frac{1}{\sqrt{2}}\right)\left(B\sin\theta\right) + P_0$$

$$\Rightarrow \frac{1}{5}P_0 = \left(\frac{1}{\sqrt{2}}\right)\left(B\frac{\sqrt{7}}{2\sqrt{2}}\right)$$

$$\Rightarrow B = \frac{4}{5\sqrt{7}}P_0$$

$$\therefore P_n = \left(\frac{1}{\sqrt{2}}\right)^n \left(\frac{4}{5\sqrt{7}}P_0 \sin n\theta\right) + P_0 \text{ where } H = \frac{1}{\sqrt{2}}, K = \frac{4}{5\sqrt{7}}, \text{ shown}$$

(e) First model: $P_n = P_0$

The 2nd model is better because the 1st model stays constant at P_0 while the second model oscillates about P_0 before eventually approaching it in the long run and this is more realistic as compared to the 1st model where prices fluctuations are common.