MOE H3 Math   Mathematical Proofs and Reasoning

Problem Set 3

1. Use Mathematical Induction to prove the following:
   (a) For each positive integer n, $1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$.
   (b) For each positive integer n, $(-7)^n - 9^n$ is divisible by 16.
   (c) For each positive integer n with $n \geq 3$, $\left(1 + \frac{1}{n}\right)^n < n$.
   (d) For each positive integer n with $n \geq 6$, $n^3 < n!$.

2. Let $f_1, f_2, \ldots, f_n, \ldots$ be the Fibonacci sequence.
   i.e. The sequence is defined recursively by

$$f_1 = 1 \text{ and } f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3$$

   Prove each of the following:
   (a) For each positive integer n, f_{5n} is a multiple of 5.
   (b) For each positive integer n, $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$.
   (c) For each positive integer n, $2f_n + 3f_{n+1} = f_{n+4}$.
   (d) For each positive integer n, f_n is even if and only if $3 \mid n$.

3. Use mathematical induction together with the following result:

   For any prime p and any integers a, b, if $p \mid ab$ and $p \nmid a$, then $p \mid b$.

   prove that, for any prime p and any integers $q_1, q_2, \cdots, q_n$, if $p \mid q_1 q_2 \cdots q_n$, then $p \mid q_i$ for some i.

4. Suppose you want to prove $P(n)$ is true (only) for all integers $n \geq 7$ that are <u>not divisible by 4</u> using a version of mathematical induction as follow:

   (i) Basis step: $P(a), P(b), P(c)$ are true; and
   (ii) Inductive step: $(\forall\ k \in \mathbb{Z}^+)\ \ P(k) \rightarrow P(k+d)$ is true.

   What should be the values for a, b, c, d?

5. Find the mistake in the following "proof" that purports to show that:
   Every nonnegative integer power of every nonzero real number is 1.

Proof. Let r be any nonzero real number and let the predicate $P(n)$ be

$$P(n) : r^n = 1.$$

Basis step: $P(0)$ is true because $r^0 = 1$ by definition of 0-th power.
Inductive step: $P(k-1)$ and $P(k) \to P(k+1)$

Suppose that $r^{k-1} = 1$ and $r^k = 1$. This is the induction hypothesis. We must show that $r^{k+1} = 1$. Now

$$\begin{aligned} r^{k+1} &= r^{k+k-(k-1)} \\ &= \frac{r^k \cdot r^k}{r^{k-1}} \\ &= \frac{1 \cdot 1}{1} \quad \text{(by induction hypothesis)} \\ &= 1 \end{aligned}$$

Thus $r^{k+1} = 1$.
Hence the inductive step is proven. □

6. Prove that for integers $n \geq 2$,

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}.$$

7. Suppose x is a real number with $x \neq 0$ and $x + \dfrac{1}{x} \in \mathbb{Z}$. Prove by induction on n that

$$x^n + \frac{1}{x^n} \in \mathbb{Z} \quad \text{for all } n \in \mathbb{Z}_{>0}.$$

8. Prove that for positive integers n and positive real numbers $x_1, \ldots, x_n$,

$$\frac{1}{n} \sum_{i=1}^{n} x_i \geq \left(\prod_{i=1}^{n} x_i \right)^{1/n}.$$

First prove the statement for $n = 2^m$ for $m \geq 0$ by induction on m. The general result now follows by proving the converse of the usual inductive step: if the result holds for $n = k+1$, where k is a positive integer, then it holds for $n = k$.

Hints

1. *Hint for Question 1.* (a) In the inductive step, add an additional term to both sides of $P(k)$.
   (b) In the inductive step, multiply 9 to both sides of $P(k)$.
   (c) Basis step is $n = 3$. You need to use the inequality $\left(1 + \frac{1}{k+1}\right)^{k+1} < \left(1 + \frac{1}{k}\right)^{k+1}$ in the inductive step.
   (d) Basis step is $n = 3$. You need to use the inequality $(k+1)^2 \leq k^3$ for $k \geq 3$ in the inductive step.

2. *Hint for Question 2.* (a) Let $P(n)$ be the predicate 'f_{5n} is a multiple of 5'. In the inductive step, try to show $f_{5(k+1)} = 5f_{5k+1} + 3f_{5k}$. (Use the recursive relation of Fibonacci sequence.)
   (b) In the inductive step, $P(k) : f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}$. Add f_{2k+1} to both sides of $P(k)$.
   (c) Prove $P(1)$ and P(2) in the basis step; and use $P(k-1)$ and $P(k)$ in the induction hypothesis.
   (d) Note that this is a biconditional statement that can be rephrased as:
   (i) if $3 \mid n$, then f_n is even; (ii) if $3 \nmid n$, then f_n is odd.
   For each case, use a variation of PMI to prove it. The inductive step should be $P(k) \to P(k+3)$.

3. *Hint for Question 3.* Use PMI on the number of terms in the product.

4. *Hint for Question 7.* Use Principle of Strong Induction.
   For the inductive step consider $\left(x^k + \frac{1}{x^k}\right)\left(x + \frac{1}{x}\right)$.

5. *Hint for Question 8.* Assuming statement holds for $n = 2^m$, to show that statement holds for 2^{m+1}, define
   $$y_j = \frac{x_{2j-1} + x_{2j}}{2} \qquad (1 \leq j \leq 2^m).$$
   Use the $n = 2$ case to show that $y_j \geq (x_{2j-1}x_{2j})^{1/2}$.
   To extend beyond powers of two, assume AM$\geq$GM holds for some $k+1 \geq 2$. For $x_1, \ldots, x_k > 0$, let $A = \frac{1}{k}\sum_{i=1}^{k} x_i$. Apply the $k+1$ case to $x_1, \ldots, x_k, A$.