2

FUNCTIONS

SYLLABUS

- Concept of function, domain and range
- Finding inverse functions and composite functions
- Conditions for the existence of inverse functions and composite functions
- Domain restriction to obtain an inverse function
- Relationship between graphs of a one-to-one function and its inverse

CONTENTS

LECTURE 1

Lesson Outline

- Basic concepts on set language and notation
- Concept of function including mapping, definition, domain and range of a function
- Types of functions including basic algebraic, even and odd functions

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0 RECAP: SET LANGUAGE AND NOTATION

A **set** is a collection of objects such as numbers, letters, people etc. The objects in a set are called **elements** of the set.

Conventions and Notations

- Uppercase letters such as A, B, C etc. are often used to denote a set.
- Lowercase letters such as a, b, c etc. are often used to denote the elements of a set.
- We write $a \in A$ to mean that the element a belongs to the set A, or ***a* is an element of *A***.
- We write $b \notin B$ to mean that the element b does not belong to the set B, or ***b* is not an element of *B*.** The set of all elements that are not in B is denoted by B'.
- We write $A \subseteq B$ if **set *A* is a subset of set *B*** i.e. every element of A is an element of B.

Notations for important sets of numbers

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

$\mathbb{N}$	the set of natural numbers = $\{1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers = $\{... -3, -2, -1, 0, 1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers. For example, an element in $\mathbb{Q}$ may be $\frac{1}{2}$, $-\frac{5}{4}$, 0 or $\frac{9}{8}$.
$\mathbb{R}$	the set of real numbers. For example, an element in $\mathbb{R}$ may be 1, π, e, $\frac{1}{2}$, 0.1212... or ln 9.
$\mathbb{C}$	the set of complex numbers. For example, an element in $\mathbb{C}$ may be 1, π, e, $\frac{1}{2}$, $\sqrt{-1}$, $3+\sqrt{-2}$.

Other Common Set Notations

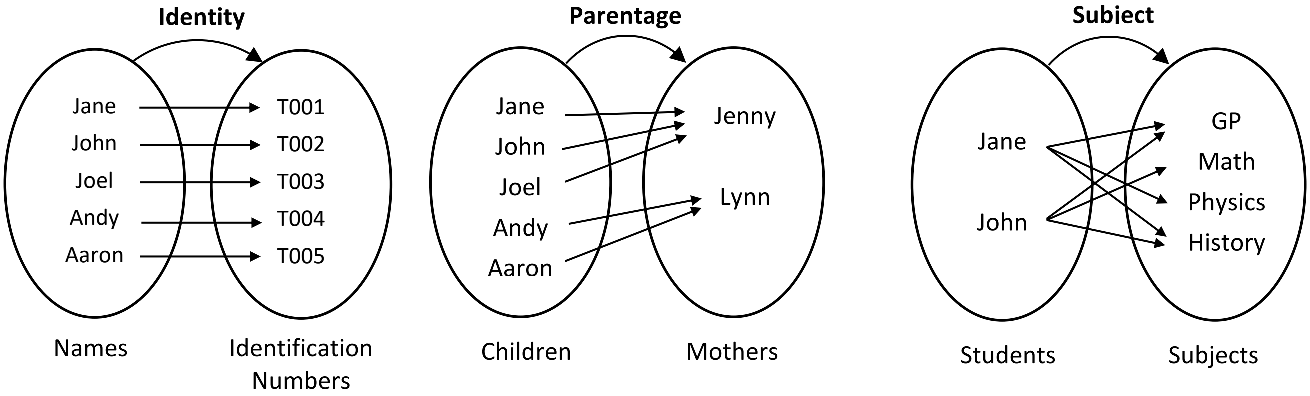
Interval notation	$x \in [a, b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$
	$x \in [a, b)$	$\{x \in \mathbb{R} : a \leq x < b\}$
	$x \in (a, b]$	$\{x \in \mathbb{R} : a < x \leq b\}$
	$x \in (a, b)$	$\{x \in \mathbb{R} : a < x < b\}$
Superscript	$x \in \mathbb{R}^+$	$\{x \in \mathbb{R} : x > 0\}$
	$x \in \mathbb{R}^-$	$\{x \in \mathbb{R} : x < 0\}$
Union (of sets)	$x \in \mathbb{R}^+ \cup \{0\}$	$\{x \in \mathbb{R} : x \geq 0\}$
	$x \in \mathbb{R}^- \cup \{0\}$	$\{x \in \mathbb{R} : x \leq 0\}$
Difference (of sets)	$x \in \mathbb{R} \setminus \{0\}$	$\{x \in \mathbb{R} : x < 0 \text{ or } x > 0\}$
	$x \in \mathbb{R} \setminus \{0, 1\}$	$\{x \in \mathbb{R} : x \neq 0, 1\}$

1 CONCEPT OF FUNCTION

In Mathematics, we attempt to explain physical phenomenon by relating a physical quantity x to another physical quantity y. Ordered pairs (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) form a relation. This relationship can be represented in the form of a formula, graph or table. The concept of function is essentially a mathematical expression that serves to describe this relationship.

1.1 Relations

A **relation** or mapping is any rule that associates two sets of items. There are many examples of relations in real life:

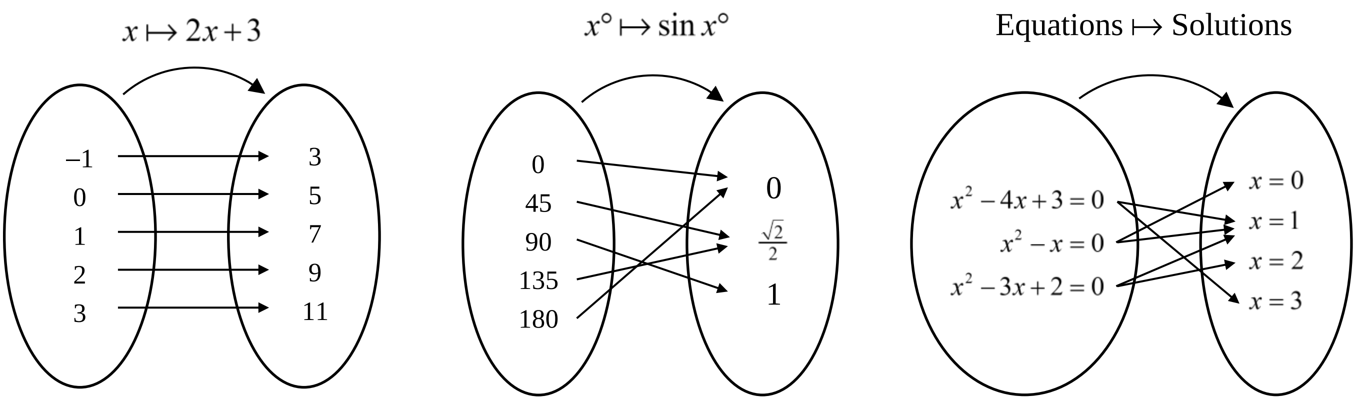


One-to-one relation   Many-to-one relation   One-to-many relation

Diagrams illustrating the various types of relationship between elements of sets.

A relation can be one-to-one, one-to-many, many-to-one, or many-to-many.

Here are some examples of mathematical relations. In each example, the items on the left form an input, and the items on the right form an output.

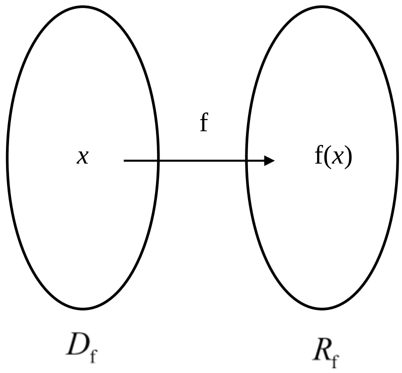


1.2 Definition and Domain of Function

A relation that is one-to-one or many-to-one is called a **function**. The set of inputs is called the **domain** of the function.

More formally, a function f is a rule or relation which assigns **each and every** element x in the domain, D_{f}, to a **unique** image, f(x). In other words, for each $x \in D_{\mathrm{f}}$, there corresponds one and only one element f(x).

The set of images is called the **range**, R_{f}, of the function.



Notes

1 If an element a is mapped to an element b, we write $\mathrm{f}(a) = b$.

2 When defining a function, **both** the **rule** and **domain** must be stated. Note that the functions $\mathrm{f}: x \mapsto \ln x, x > 0$ and $\mathrm{g}: x \mapsto \ln x, x > 1$ are different functions as they have different __________________ (although the ____________________ is the same).

3 There are several different but equivalent ways of writing a function. For example, the function which maps x to $2x+5$, where the domain is the set of positive real numbers, can be written in any of the following ways:

$$\mathrm{f}(x) = 2x+5,\ x \in \mathbb{R}^{+} \qquad \mathrm{f}(x) = 2x+5,\ x > 0$$

$$\mathrm{f}: x \mapsto 2x+5,\ x \in \mathbb{R}^{+} \qquad \mathrm{f}: x \mapsto 2x+5,\ x > 0$$

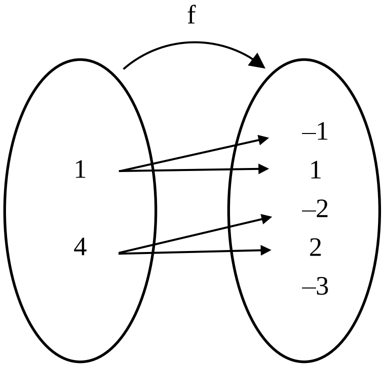
4 A function can be represented visually by a graph, a table or mappings between sets.

5 The "vertical line test" can be used as a quick check to see if a graph represents a function. If the line $x = k$, for **all** $k \in D_{\mathrm{f}}$, cuts the graph of $y = \mathrm{f}(x)$ **exactly once**, then the graph represents a function.

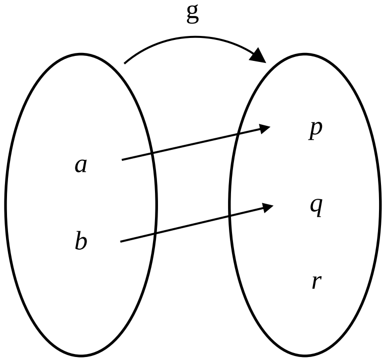
Example 1

Relations f, g, h and m are defined by the following diagrams. Identify which of them are functions.

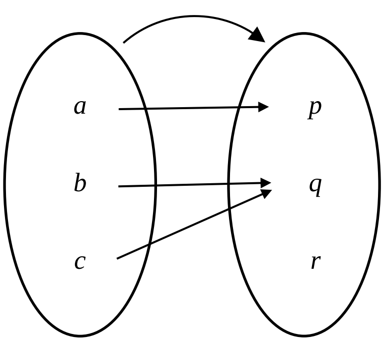
(a)



(b)



(c)



(d)

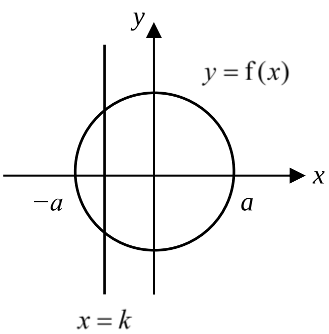
x	m(x)
1	−1
1	1
4	−2
4	2
9	−3
9	3

■

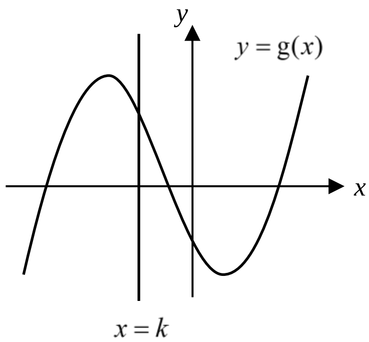
Example 2

Which of the following are graphs of functions?

(a)



(b)



Solution

No, this is not a function.

The vertical line $x = k$ (see graph) cuts the graph of $y = \mathrm{f}(x)$ twice. This means one value of x is mapped to two values of y, hence f is not a function.

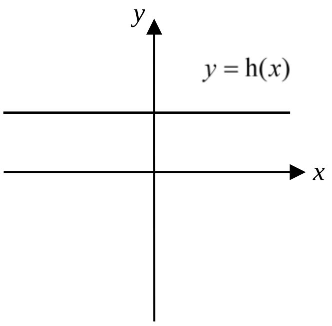
Yes, this is a function.

The vertical line $x = k$ (see graph), for any real value of k, cuts the graph of $y = \mathrm{g}(x)$ exactly once. This means each value of x in the domain is mapped to only one value of y, hence g is a function. ■

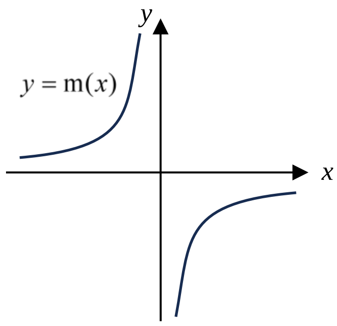
Self-Practice Exercise 1

Which of the following are graphs of functions?

(a)



(b)



Yes/No, this is /not a function.

Explanation:

Yes/No, this is /not a function.

Explanation:

■

Self-Practice Exercise 2

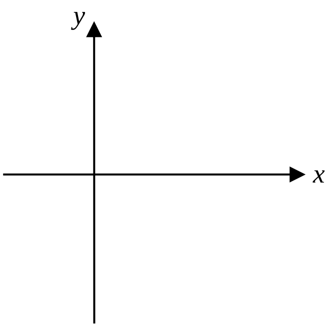
Determine if the functions f and h defined as follows are functions.

(a) $\mathrm{f} : x \mapsto \ln x, \quad x \in \mathbb{R}$

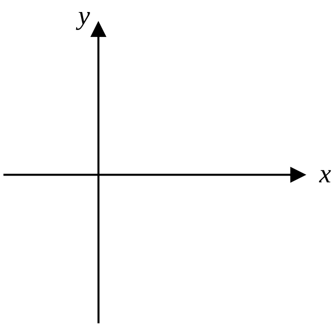
(b) $\mathrm{h} : x \mapsto \ln x, \quad x > 0$

Solution

(a)



(b)



■

1.3 Range of a Function

The **range** of f, denoted by R_f , is the **set of images** of all the elements in the domain under f.

The range of a function can be read from the graph of the function. It is the set of all possible y coordinates of the graph. Sometimes it can also be determined algebraically (see Chapter 3: Equations and Inequalities, Example 8).

Example 3

State the ranges for the following functions.

(a) $f_1 : x \mapsto x+2, x \in \mathbb{R}^+$

(b) $f_2 : x \mapsto |x|, x \in [-1,1)$

(c) $f_3 : x \mapsto 2, x \in \mathbb{R},\ x < 0$

(d) $f_4 : x \mapsto x^2 - x - 6, -1 \le x < 4$

(e) $f_5 : x \mapsto e^x, x \in \mathbb{R}^- \cup \{0\}$

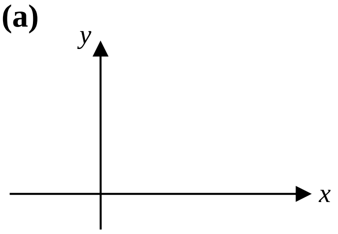
(f) $f_6 : x \mapsto \dfrac{x}{x+1}, x \in \mathbb{R}, x \ne -1$

(g) $f_7 : x \mapsto \sin 2x, x \in \mathbb{R}$

(h) $f_8 : x \mapsto \tan x, x \in \left(\frac{1}{3}\pi, \pi\right) \backslash \left\{\frac{1}{2}\pi\right\}$

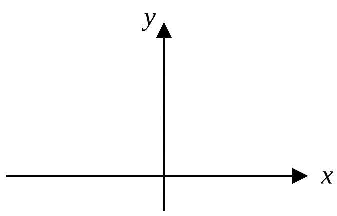
Solution

(a)



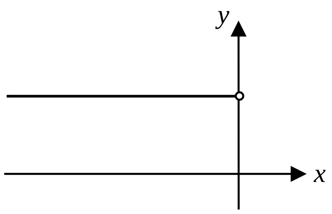
$R_{f_1} = (2, \infty)$

(b)



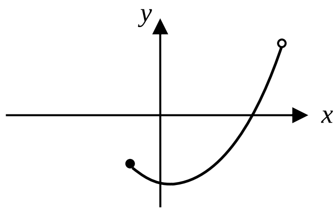
$R_{f_2} = [0,1]$

(c)



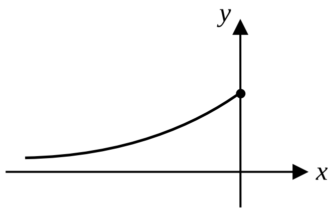
$R_{f_3} = \{2\}$

(d)



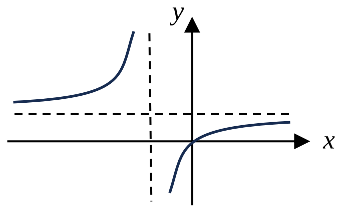
$R_{f_4} = [-6.25, 6)$

(e)



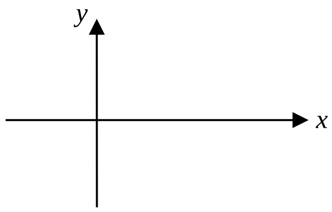
$R_{f_5} = (0,1]$

(f)



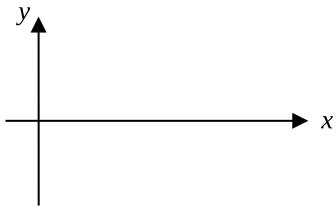
$R_{f_6} = \mathbb{R} \setminus \{1\}$

(g)



$R_{f_7} = [-1,1]$

(h)



$R_{f_8} = (-\infty, 0) \cup (\sqrt{3}, \infty)$

2 TYPES OF FUNCTIONS

2.1 Some Basic Algebraic Functions (Refer to Annex A)

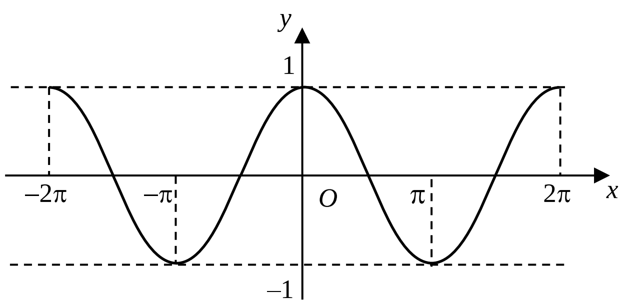
2.2 Even and Odd Functions

An even function f is one such that $\mathrm{f}(x) = \mathrm{f}(-x)$ for all $x \in D_{\mathrm{f}}$.

The graph of an even function is **symmetrical about the y-axis**.

Example

$\mathrm{f} : x \mapsto \cos x, x \in \mathbb{R}$

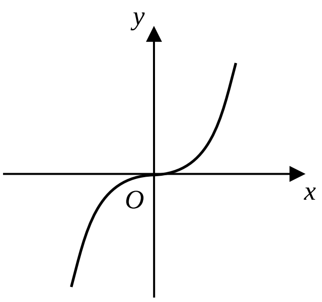


An odd function f is one such that $f(-x) = -f(x)$ for all $x \in D_f$.

The graph of an odd function is **symmetrical about the origin**.

Example

$f : x \mapsto x^3, x \in \mathbb{R}$



What are some other examples of even and odd functions?

What if a graph is symmetrical about the *x*-axis?

Extra Reading

Annex A

LECTURE 2

Lesson Outline

- Periodic Functions
- One-one Functions
- Inverse function including existence, definition, domain and range

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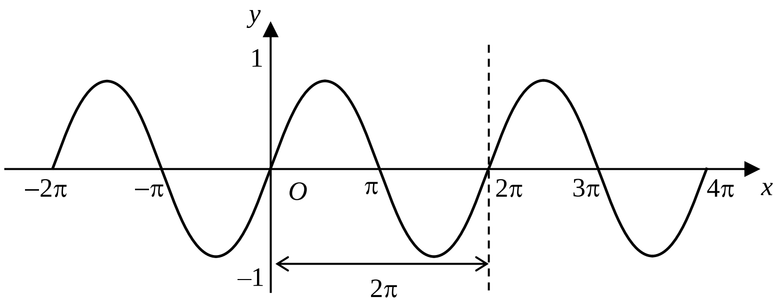
2.3 Periodic Functions

A function whose graph consists of a basic pattern which repeats at regular intervals is said to be **periodic**. The width of the basic pattern is the **period** of the function.

For instance, sin x is periodic with a period of 2π.



Note that $\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \dots$

$$= \sin(x-2\pi) = \sin(x-4\pi) = \dots$$

Definition

A function f is said to be **periodic** and has a **period of *k*** if $\mathrm{f}(x+k) = \mathrm{f}(x)$ for all real values of x.

Example 4

It is given that

$$\mathrm{f} : x \mapsto \begin{cases} 8-3x & \text{for } 0 < x \le 1,\ x \in \mathbb{R} \\ x^2+4 & \text{for } 1 < x \le 2,\ x \in \mathbb{R} \end{cases}$$

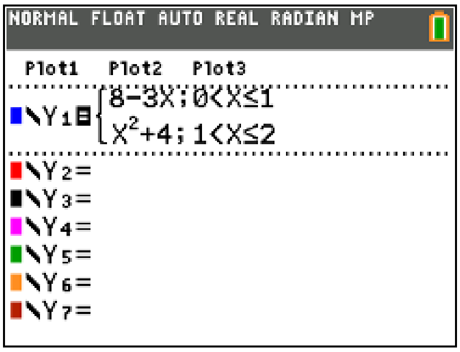
and that $\mathrm{f}(x) = \mathrm{f}(x+2)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $0 < x \le 2$ and state its range.

(ii) Evaluate $\mathrm{f}(5) + \mathrm{f}\left(11\tfrac{1}{2}\right)$.

Deduce the graph of $y = \mathrm{f}(x)$ for $-1 < x \le 7$.

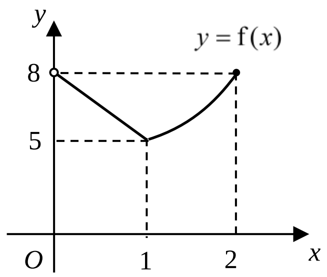
1) Press **[Y=]**.
2) For **Y1**, press **[MATH] [B:piecewise].** Select 2 pieces and **OK.**
3) Type **[8] [-] [3] [X,T,θ,n]**. Type **[0] [2ND] [MATH] [5:<] [X,T,θ,n] [2ND] [MATH] [6:≤] [1] [)]**. Press **[ENTER]**.



4) Type **[X,T,θ,n] [x²] [+] [4]**.Type **[1] [2ND] [MATH] [5:<] [X,T,θ,n] [2ND] [MATH] [6:≤] [2]**

Solution

(i)



For $0 < x \leq 1$, $y = 8 - 3x$ (a straight line).

When $x = 0$ $y = 8 - 0 = 8$.

When $x = 1$ $y = 8 - 3 = 5$.

For $1 < x \leq 2$, $y = x^2 + 4$ (a parabola).

When $x = 1$, $y = 1^2 + 4 = 5$.

When $x = 2$ $y = 2^2 + 4 = 8$.

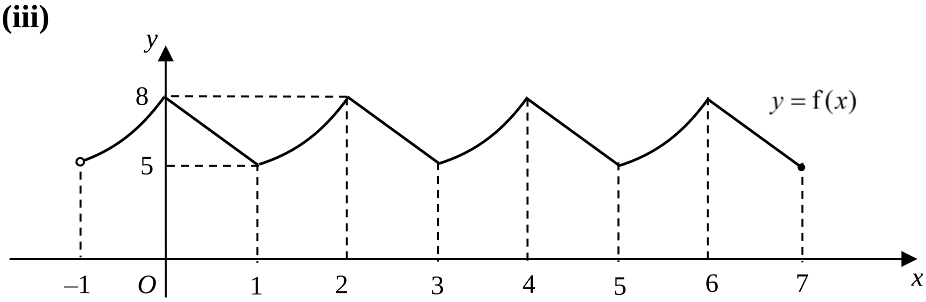
$R_f = [5,8]$

(ii) $\mathrm{f}(5) = \mathrm{f}(3) = \mathrm{f}(1) = 8 - 3 = 5$

$\mathrm{f}(11\tfrac{1}{2}) = \mathrm{f}(9\tfrac{1}{2}) = \mathrm{f}(7\tfrac{1}{2}) = \ldots = \mathrm{f}(\tfrac{3}{2}) = (\tfrac{3}{2})^2 + 4 = 6.25$

$\therefore \mathrm{f}(5) + \mathrm{f}(11\tfrac{1}{2}) = 5 + 6.25 = 11.25$.

(iii)



Do take note of the open and closed circles indicating if the end points are included or excluded from the sketch.

■

2.4 One-One Functions

A function f is **one to one** if no two elements in its domain have the same image.

We can prove that a function is one-one graphically using the horizontal line test:

To show that f is a one-one function, we sketch the graph of $y = \mathrm{f}(x)$ and a horizontal line $y = k$. If the horizontal line $y = k$ for all $k \in \mathbb{R}$ cuts the graph at most once, then f is a one-one function.	To show that f is not a one-one function, it is enough to give one counterexample. That is, if there is one horizontal line that cuts the graph more than once, then f is not a one-one function.

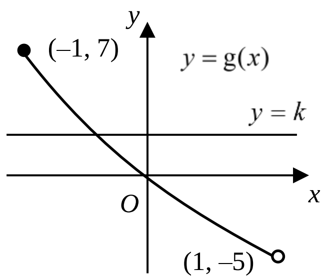
Example 5

Determine if the following are one-one functions.

(a) $\mathrm{g}: x \mapsto x^2 - 6x,\ x \in [-1, 1)$ **(b)** $\mathrm{h}: x \mapsto x^2,\ x \in \mathbb{R}$

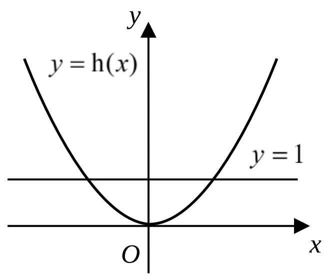
Solution

(a)



Since the horizontal line $y = k$, for **all** $k \in \mathbb{R}$, cuts the graph of $y = \mathrm{g}(x)$ **at most once**, g is a one-one function.

(b)



The horizontal line $y = 1$ cuts the graph of $y = \mathrm{h}(x)$ **at two points**. Hence h is not a one-one function.

Alternatively, $\mathrm{h}(-1) = \mathrm{h}(1) = 1$, hence this is a many-to-one function and not a one-one function.

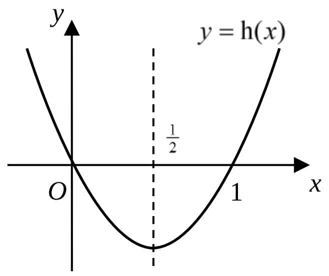
■

Example 6

The function h is defined by $\mathrm{h}: x \mapsto x^2 - x,\ \ x \in \mathbb{R}$. Give a reason why h is not one-one.

If the domain of h is restricted to $\mathbb{R}$ for which $x \geq A$, find the least value of A for which h is one-one.

Solution



The horizontal line __________ cuts the graph of $y = \mathrm{h}(x)$ **at two points**.

Hence h is **not** a one-one function.

The least value of A is $\frac{1}{2}$.

$y = x^2 - x = x(x-1)$ is a quadratic curve with x-intercepts at $x = 0$ and $x = 1$.

■

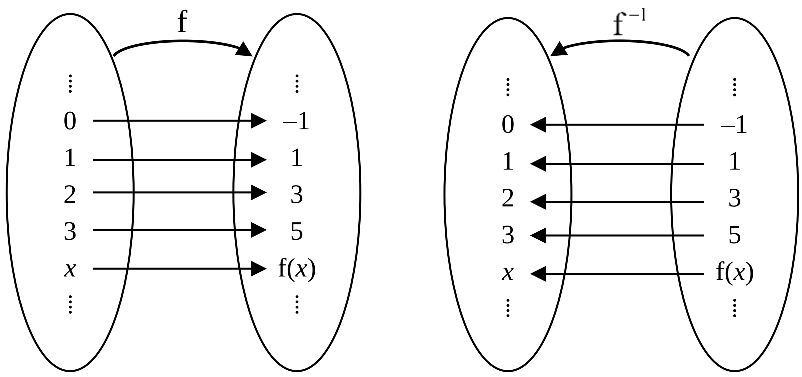
3 INVERSE FUNCTIONS

Consider the function $\mathrm{f}(x) = \dfrac{5}{9}(x - 32)$, where x is the temperature in degrees Fahrenheit. This function converts degrees Fahrenheit to degrees Celsius. The function that converts degrees Celsius to degrees Fahrenheit is the inverse function of f.

Degree Fahrenheit, x	Degree Celsius, $\mathrm{f}(x)$
	28°C
140°F	

3.1 Existence of an Inverse Function

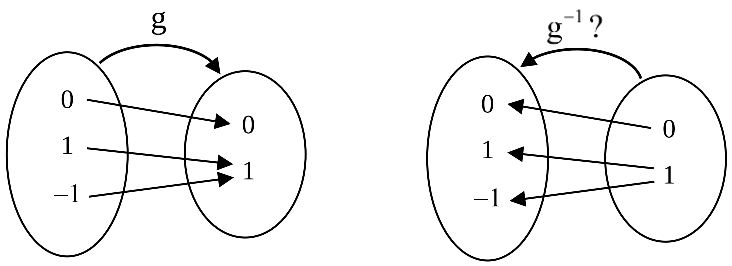
Consider the function $\mathrm{f}: x \mapsto 2x - 1,\ x \in \mathbb{Z}$. Notice that it is a one-one function.



We can reverse the arrows to form another relation, called inverse relation. This relation is also a function, because for any input $f(x)$, there is one and only one output x. This inverse relation is called the **inverse function** and is denoted by f^{-1}.

The inverse function f^{-1} is such that if $y = f(x)$, then $x = f^{-1}(y)$.

On the other hand, consider the function $g: x \mapsto x^2, x \in \mathbb{Z}$. Does g^{-1} exist?

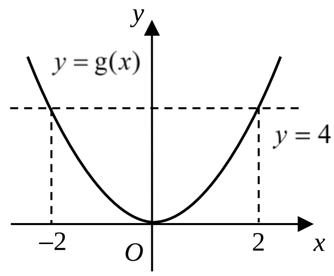


Note that the inverse relation is one-to-many, and hence g^{-1} does not exist as a function.

Thus, the **inverse function** of f exists if and only if f is **one-one**.

Consider $g: x \mapsto x^2, x \in \mathbb{R}$

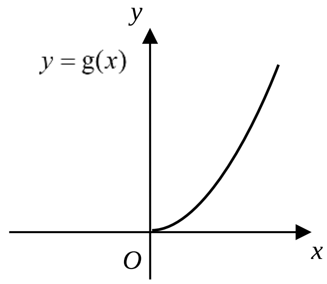
Since $g(2) = g(-2) = 4$, g is not one-one $\therefore g^{-1}$ does not exist.



However, if the domain of g is **restricted to** $\mathbb{R}^+$, the function g is now one-one, and hence the function g^{-1} exists.

i.e. if $g: x \mapsto x^2, x \in \mathbb{R}^+$,

then g is a one-one function, $\therefore$ g^{-1} exists.

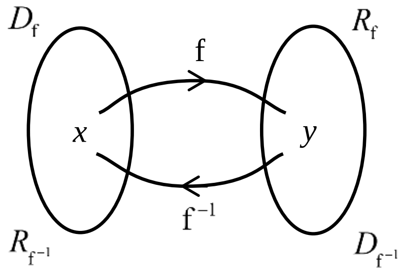


3.2 Definition of an Inverse Function

Let f be a one-one function.

Then f has an inverse function f^{-1} defined by

$$y=\mathrm{f}(x) \Leftrightarrow x=\mathrm{f}^{-1}(y) \text{ for all } x \in D_{\mathrm{f}}.$$



Note that f^{-1} and $\mathrm{f}^{-1}(x)$ are the notation for an inverse function and they are not to be confused with $\left(\mathrm{f}(x)\right)^{-1}=\dfrac{1}{\mathrm{f}(x)}$. For example, $\sin^{-1} x$ is the inverse trigonometric function of $\sin x$, while $\dfrac{1}{\sin x}=\operatorname{cosec} x$.

To obtain the rule of f^{-1}:

Step 1: Let $y=\mathrm{f}(x)$.

Step 2: Express x in terms of y i.e. $x=\mathrm{f}^{-1}(y)$.

Step 3: Replacing y by x, we obtain the rule for $\mathrm{f}^{-1}(x)$.

3.3 Domain and Range of an Inverse Function

Domain of f^{-1} = Range of function f, i.e.

$$D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}$$

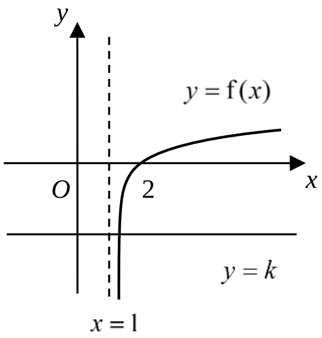
Range of f^{-1} = Domain of function f, i.e.

$$R_{\mathrm{f}^{-1}}=D_{\mathrm{f}}.$$

Example 7

The function f is defined by $\mathrm{f}: x \mapsto \ln(x-1),\ x \in \mathbb{R},\ x > 1$. Determine if f^{-1} exists. If it does, define f^{-1} and state its range.

Solution



Since the horizontal line $y = k$, for all $k \in \mathbb{R}$, cuts the graph of $y = \mathrm{f}(x)$ at most once, f is a one-one function. Hence f^{-1} exists.

Step 1: Let $y = \ln(x-1)$

Step 2 (make *x* the subject):

$\mathrm{e}^{y} = x - 1$

$x = \mathrm{e}^{y} + 1 \ \left(= \mathrm{f}^{-1}(y)\right)$

Step 3: $\mathrm{f}^{-1}(x) = \mathrm{e}^{x} + 1,\ x \in \mathbb{R}$.

$R_{\mathrm{f}^{-1}} = D_{\mathrm{f}} = (1, \infty)$. ■

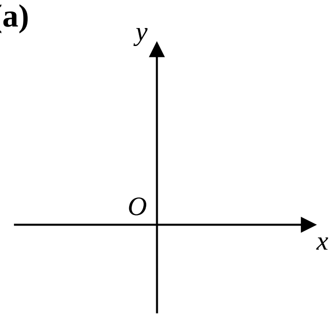
Self-Practice Exercise 3

For each of the following functions, determine if the inverse function exists. If it does, define the inverse function.

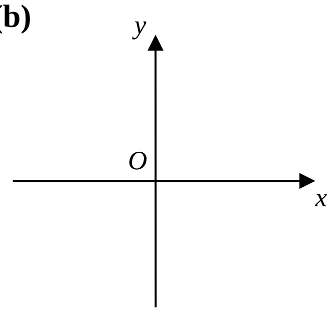
(a) $\mathrm{g}: x \mapsto \mathrm{e}^{x} - 2,\ x \in \mathbb{R}$ **(b)** $\mathrm{h}: x \mapsto \sin x,\ x \in [-\pi, \pi]$

Solution

(a)



(b)



■

LECTURE 3

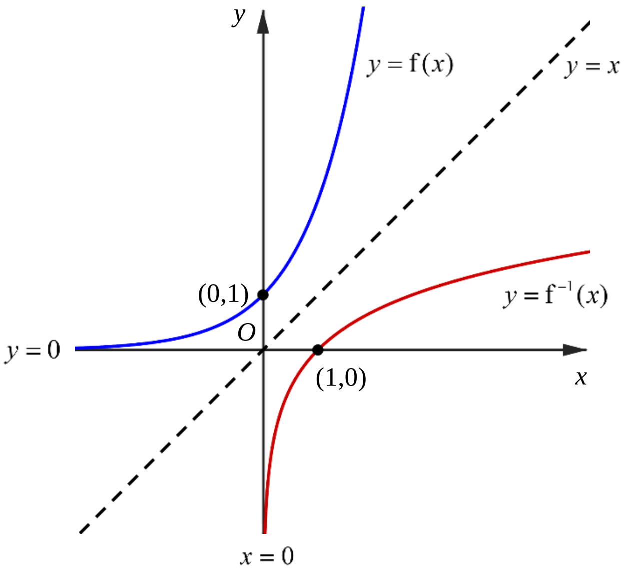
Lesson Outline

- Graph of an inverse function
- Composite function including existence and definition

3.4 Graph of an Inverse Function

Recall that the inverse function f^{-1} is such that $y=\mathrm{f}(x) \Leftrightarrow x=\mathrm{f}^{-1}(y)$.

Consider the graphs of the function $\mathrm{f}(x)=\mathrm{e}^{x}$ and its inverse function $\mathrm{f}^{-1}(x)=\ln x$. We note that $\mathrm{f}(0)=1$ and $\mathrm{f}^{-1}(1)=0$, and this is represented in the diagram by the points $(0,1)$ on the graph of $y=\mathrm{f}(x)$ and $(1,0)$ on the graph of $y=\mathrm{f}^{-1}(x)$.



In general, if the point (a,b) lies on the curve $y=\mathrm{f}(x)$, then the point $(\quad,\quad)$ lies on the curve $y=\mathrm{f}^{-1}(x)$.

Graphically, the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ are ____________________________________.

When sketching the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ on the same diagram, the same scale should be used for both the x- and y-axes.

Any asymptotes will also be reflected in the same line, i.e. a vertical asymptote $x=k$ in the graph of $y=\mathrm{f}(x)$ corresponds to a horizontal asymptote $y=k$ in the graph of $y=\mathrm{f}^{-1}(x)$.

Example 8

The function f is defined by $\mathrm{f} : x \mapsto (x-4)^2 + 1,\ x \in \mathbb{R}$.

(i) Show that f^{-1} does not exist.

(ii) The function f has an inverse if its domain is restricted to $x \le k$. Find the greatest value of k.

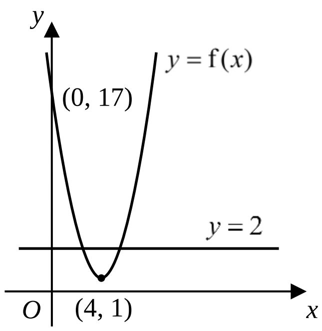
(iii) Hence, define f^{-1} and state its domain.

(iv) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram. Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$, and hence find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

[A Level 2008/P2/Q4 Modified]

Solution

(i)

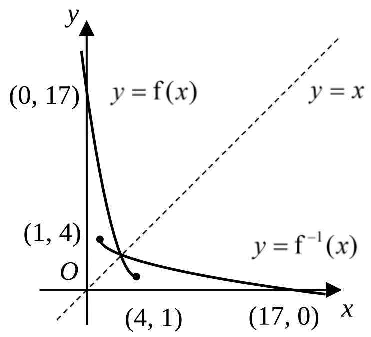


The line $y = 2$ cuts the graph of $y = \mathrm{f}(x)$ at two points. Hence f is not a one-one function and f^{-1} does not exist.

(ii) When $x \le 4$, f is a one-one function and f^{-1} exists.
$\therefore$ Greatest value of k is 4.

(iii) Let $\boxed{y = \qquad\qquad}$.
Then $y - 1 = (x-4)^2 \Rightarrow x - 4 = \pm\sqrt{y-1}$.
$\therefore\ x = 4 \pm \sqrt{y-1}$.
But check that $D_{\mathrm{f}} = (-\infty, 4]$, i.e. $x \le 4$.
Hence $\mathrm{f}^{-1}(x) = 4 - \sqrt{x-1},\ x \in [1, \infty)$.

(iv)



The equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$ is $y = x$.

To find the intersection of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, we just need to find the intersection of $y = \mathrm{f}(x)$ and $y = x$, i.e. solve $\mathrm{f}(x) = x$.

$$(x-4)^2 + 1 = x \Rightarrow x^2 - 9x + 17 = 0 \Rightarrow x = \frac{9 \pm \sqrt{13}}{2}$$

But check that the intersection required (see graph) is such that $x \le 4$ (the domain of f).

Hence $x = \dfrac{9-\sqrt{13}}{2}$. ■

In this example, solving the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ algebraically is challenging as it results in solving a quartic equation.

Graphically, any point(s) of intersection of $y = \mathrm{f}(x)$ and the line of reflection $y = x$ would also lie on $y = \mathrm{f}^{-1}(x)$. Therefore, an easier method of finding the roots of $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ would be to solve the (easier) equation $\mathrm{f}(x) = x$ or $\mathrm{f}^{-1}(x) = x$.

Example 9

Sketch the graph of $y = \mathrm{f}(x)$, where $\mathrm{f} : x \mapsto x^2,\ x \in \mathbb{R},\ x \geq 0$.

Sketch also the graph of $y = \mathrm{f}^{-1}(x)$ on the same diagram.

Solution

1) Press **[Y=]**.

2) For **Y1**, type **[X,T,θ,n] [x²]**. Press **[÷] [(] [X,T,θ,n] [2ND] [MATH] [4:≥] [0] [)]**. Press **[ENTER]**.

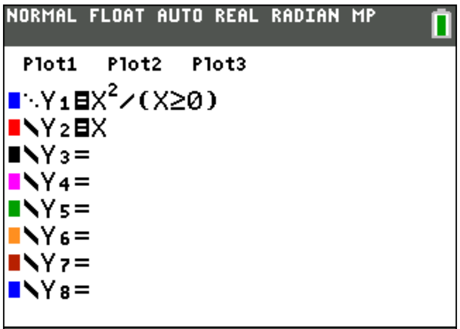
3) For **Y2**, type **[X,T,θ,n]**. Press **[ENTER]**.

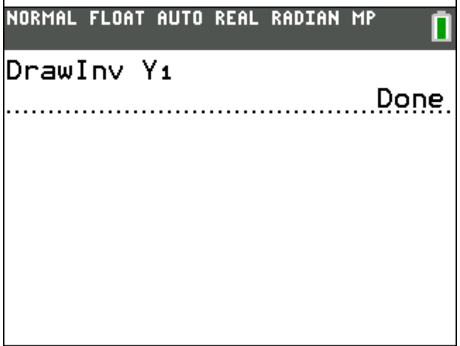
4) Move the cursor to the graph style icon to view different graph styles. Press **[ENTER]** until the desired style is chosen.

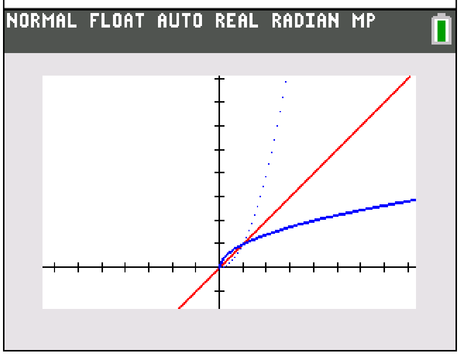
5) Press **[WINDOW]** to change the window settings, if necessary.

6) Press **[ZOOM]**, select **[5:ZSquare]** to ensure that the same scale is used on both axes. It ensures that the line $y = x$ drawn makes an angle of 45° with the vertical. A circle will not look distorted.

7) Press **[2ND] [PRGM]**, select **[8:DrawInv]**. Press **[ALPHA] [TRACE]**, select **[Y1]**. Press **[ENTER]**.







■

Extra Practice

Annex B

Annex C: Practice Question 1

4 COMPOSITE FUNCTIONS

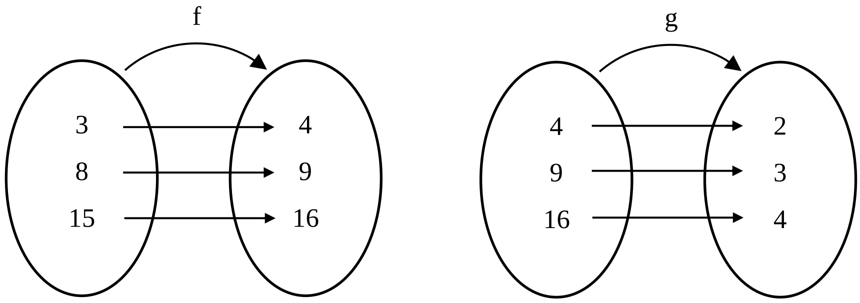
The function $\mathrm{g}(x) = x + 273.15$ converts a temperature in degrees Celsius to a temperature in Kelvins, the SI unit of temperature. Recall from Section 3 that the function $\mathrm{f}(x) = \frac{5}{9}(x - 32)$ converts a temperature in degrees Fahrenheit to degrees Celsius. By combining (or composing) the two functions f and g, we can form a single function, known as a composite function, that allows us to convert a temperature in degrees Fahrenheit directly to a temperature in Kelvins

4.1 Existence of a Composite Function

Consider the functions

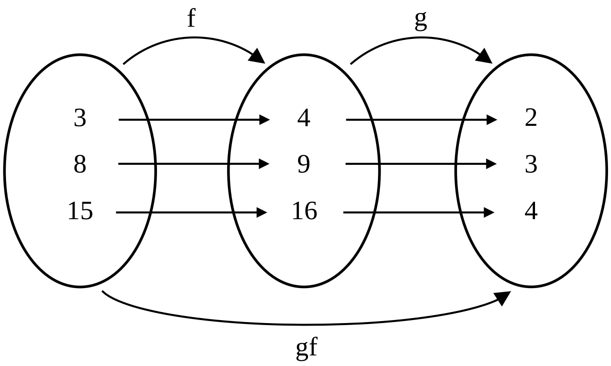
$$\mathrm{f} : x \mapsto x + 1,\ x \in \{3, 8, 15\},$$

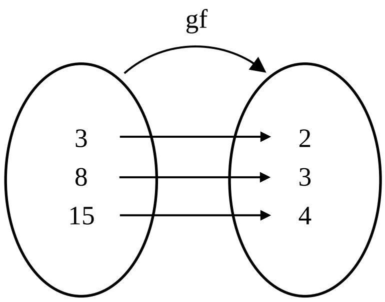
$$\mathrm{g} : x \mapsto \sqrt{x},\ x \in \{4, 9, 16\}.$$



The function f maps the element ‘3’ to ‘4’ and the function g then maps ‘4’ to ‘2’.

The two functions, f followed by g, is known as the **composite function** gf. The function gf maps ‘3’ to ‘2’.





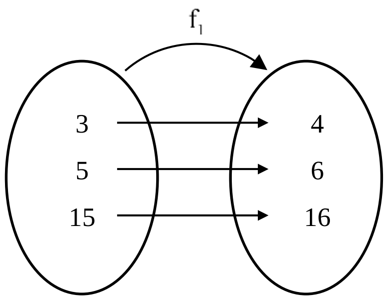
For any $x \in D_{\mathrm{f}}$, $x \xrightarrow{\mathrm{f}} x+1 \xrightarrow{\mathrm{g}} \sqrt{x+1}$.

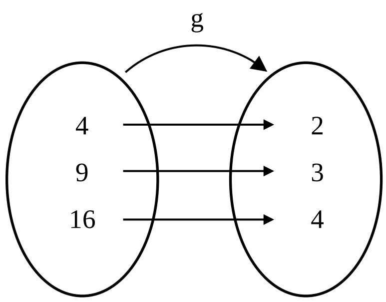
In other words, $\mathrm{gf}(x) = \mathrm{g}(x+1) = \sqrt{x+1}$.

Now consider the following functions where f has a different domain

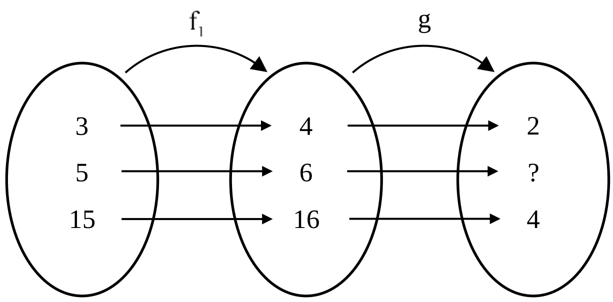
$$\mathrm{f}_1 : x \mapsto x+1,\ x \in \{3, 5, 15\},$$

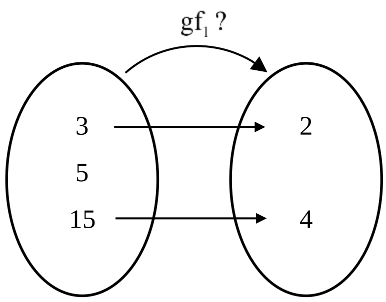
$$\mathrm{g} : x \mapsto \sqrt{x},\ x \in \{4, 9, 16\}.$$





Clearly f maps the element '5' to '6' but '6' has no image under the function g as it is not in the domain of g:





Thus gf_1 (i.e. f_1 followed by g) is not a function as

____.

Hence, in general, for any given functions f and g, the **composite function gf exists** if and only if $R_{\mathrm{f}} \subseteq D_{\mathrm{g}}$.

Similarly, fg (i.e. g followed by f) is a **composite function** if and only if $R_{\mathrm{g}} \subseteq D_{\mathrm{f}}$.

4.2 Definition of a Composite Function

Let f and g be two functions such that $R_{\mathrm{f}} \subseteq D_{\mathrm{g}}$.

The composite function gf is defined as $\mathrm{gf}(x) = \mathrm{g}\left(\mathrm{f}(x)\right)$ for all $x \in D_{\mathrm{f}}$.

$$\begin{tikzcd} x \arrow[r, "\mathrm{f}"] \arrow[rr, bend right, "\mathrm{gf}"'] & \mathrm{f}(x) \arrow[r, "\mathrm{g}"] & \mathrm{g}\left(\mathrm{f}(x)\right) \end{tikzcd}$$

Notes

1. gf may or may not be a function, even if it is possible to find the rule for gf.
2. In general, $\mathrm{gf} \neq \mathrm{fg}$, i.e. composition of functions is **not** commutative.
3. The composite function ff, if it exists, is written as f^2 i.e. $\mathrm{f}^2(x) = \mathrm{ff}(x) = \mathrm{f}\left(\mathrm{f}(x)\right)$. This is not to be confused with $\left(\mathrm{f}(x)\right)^2$, the square of the function $\mathrm{f}(x)$.

   In general, $\mathrm{f}^n(x) = \mathrm{ff}^{n-1}(x) = \mathrm{f}\left(\mathrm{f}^{n-1}(x)\right)$.

LECTURE 4

Lesson Outline

- Domain and range of a composite function

4.3 Domain and Range of a Composite Function

Domain of composite function gf, $D_{\mathrm{gf}} = D_{\mathrm{f}}$.

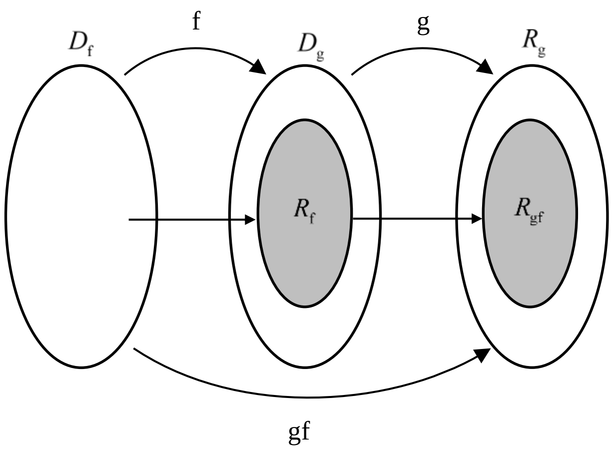
Similarly, domain of composite function fg, $D_{\mathrm{fg}} = D_{\mathrm{g}}$.

There are two methods for finding the range of the function gf:

(i) Sketch the graph of gf and read off its range.

(ii) If the graphs of g and f are available (and especially if the graph of gf is very difficult to sketch), then we can use the range of f as the restricted domain of g and read off the corresponding range to find R_{gf}, i.e. $R_{\mathrm{gf}} = \{\mathrm{g}(x) : x \in R_{\mathrm{f}}\}$.

i.e. $D_{\mathrm{f}} \xrightarrow{\mathrm{f}} R_{\mathrm{f}} \xrightarrow{\mathrm{g}} R_{\mathrm{gf}}$



Note that the rules of $\mathrm{f}^{-1}\mathrm{f}$ and ff^{-1} are the same: $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{ff}^{-1}(x) = x$.

However, $\mathrm{f}^{-1}\mathrm{f}$ and ff^{-1} are often different functions as they have different domains:

$$\mathrm{f}^{-1}\mathrm{f} : x \mapsto x,\ x \in \_\_\_\_\_\_\_\_$$

$$\mathrm{ff}^{-1} : x \mapsto x,\ x \in \_\_\_\_\_\_\_\_$$

Example 10

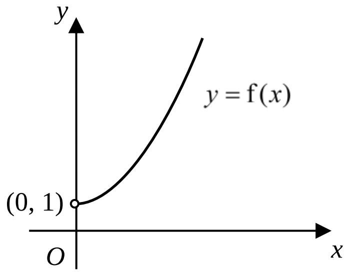
Let f and g be defined as

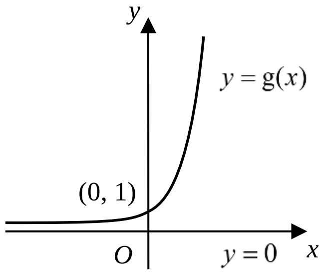
$$\mathrm{f}: x \mapsto x^2+1, \qquad x \in \mathbb{R},\ x>0$$
$$\mathrm{g}: x \mapsto \mathrm{e}^x, \qquad x \in \mathbb{R}$$

Sketch the graphs of f and g and state their respective ranges.

Define fg, gf and f^2. State their corresponding ranges.

Solution





$R_\mathrm{f} = (1, \infty)$ $\qquad$ $R_\mathrm{g} = (0, \infty)$ or $\mathbb{R}^+$

$$\mathrm{fg}(x) = \mathrm{f}\left(\mathrm{e}^x\right) = \left(\mathrm{e}^x\right)^2 + 1 = \mathrm{e}^{2x}+1, \quad D_{\mathrm{fg}} = D_\mathrm{g} = \mathbb{R}$$

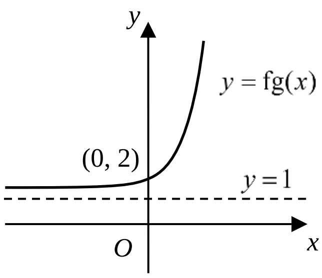
$\therefore \quad \mathrm{fg}: x \mapsto \mathrm{e}^{2x}+1,\ x \in \mathbb{R}$

To find the range of fg,

Method 1:

$$\underset{D_\mathrm{g}}{\mathbb{R}} \xrightarrow{\ \mathrm{g}\ } \underset{R_\mathrm{g}}{(0, \infty)} \xrightarrow{\ \mathrm{f}\ } \underset{R_{\mathrm{fg}}}{(1, \infty)}$$

Method 2: Sketch the graph of $y = \mathrm{fg}(x)$



From the graph, $R_{\mathrm{fg}} = (1, \infty)$.

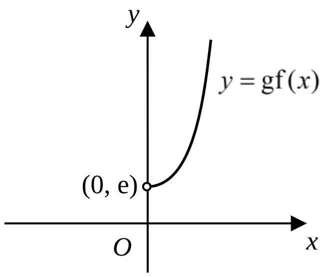
$\mathrm{gf}(x)=\mathrm{g}\left(x^2+1\right)=\mathrm{e}^{x^2+1}$, $D_{\mathrm{gf}}=D_{\mathrm{f}}=\left(0,\infty\right)$

$\therefore \mathrm{gf}: x \mapsto \mathrm{e}^{x^2+1}$, $x\in\left(0,\infty\right)$.

To find the range of gf,

Method 1:

Method 2: Sketch the graph of $y=\mathrm{gf}(x)$



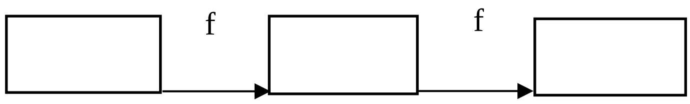
From the graph, $R_{\mathrm{gf}}=\left(\mathrm{e},\infty\right)$.

$\mathrm{ff}(x)=\mathrm{f}\left(x^2+1\right)=\left(x^2+1\right)^2+1$, $D_{\mathrm{ff}}=D_{\mathrm{f}}=\left(0,\infty\right)$

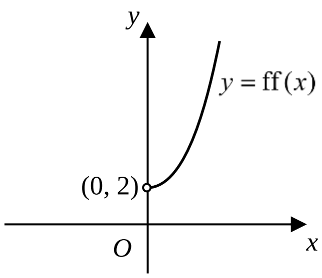
$\therefore \mathrm{ff}: x \mapsto \left(x^2+1\right)^2+1$, $x\in\left(0,\infty\right)$.

To find the range of ff,

Method 1:



Method 2: Sketch the graph of $y=\mathrm{ff}(x)$



From the graph, $R_{\mathrm{f}^2}=\left(2,\infty\right)$.

■

Extra Practice

Annex C: Question 2

Example 11

The function f is defined by $\mathrm{f}: x \mapsto \dfrac{ax}{bx-a}, x \in \mathbb{R}, x \neq \dfrac{a}{b}$, where a and b are non-zero constants.

(i) Find $\mathrm{f}^{-1}(x)$. Hence or otherwise find $\mathrm{f}^{2}(x)$ and state the range of f^{2}.

(ii) Deduce also $\mathrm{f}^{11}(x)$. [N09/II/3 Modified]

Solution

(i) Let $y = \dfrac{ax}{bx-a}$. Making x the subject,

$$x = \frac{ay}{by-a}$$

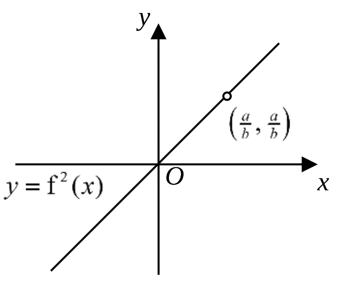
$$\therefore\ \mathrm{f}^{-1}(x) = \frac{ax}{bx-a}.$$

Note that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

Hence $\mathrm{ff}(x) = \mathrm{ff}^{-1}(x)$

$\Rightarrow \mathrm{f}^{2}(x) = x$ --------(*)

To find range of f^{2}, sketch the graph of $y = \mathrm{f}^{2}(x)$:



Note that the graph of $y = \mathrm{f}^{2}(x)$ is the graph of $y = x$ where $x \neq \frac{a}{b}$.

Hence $R_{\mathrm{f}^2} = \mathbb{R} \setminus \left\{\frac{a}{b}\right\}$.

(ii) From (*), since $\mathrm{f}^{2}(x) = x$, then $\mathrm{f}^{4}(x) = \mathrm{f}^{2}\left(\mathrm{f}^{2}(x)\right) = \mathrm{f}^{2}(x) = x$.

Similarly, $\mathrm{f}^{6}(x) = \mathrm{f}^{4}\left(\mathrm{f}^{2}(x)\right) = \mathrm{f}^{4}(x) = x$.

Hence $\mathrm{f}^{10}(x) = x$.

Therefore $\mathrm{f}^{11}(x) = \mathrm{f}\left(\mathrm{f}^{10}(x)\right) = \mathrm{f}(x) = \dfrac{ax}{bx-a}$, $x \in \mathbb{R}$, $x \neq \dfrac{a}{b}$. ■

Example 12

Functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto -\sqrt{x-2}, \quad x \in [2, \infty)$$
$$\mathrm{g}: x \mapsto x^2 + 1, \qquad x \in \mathbb{R}$$

(a) Sketch the graphs and state their ranges respectively.

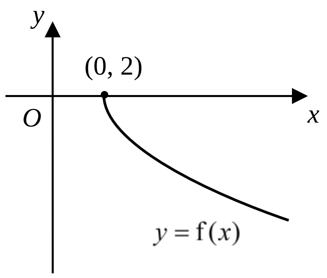
(b) Show with reasons that gf exists. Define gf and find its range.

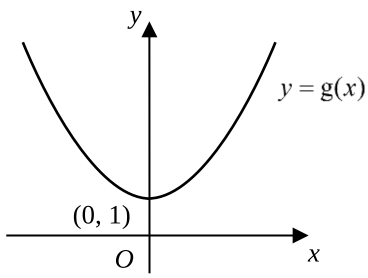
(c) Show with reasons that fg does not exist.

(d) Define the composite functions ff^{-1} and $\mathrm{f}^{-1}\mathrm{f}$, and sketch their graphs.

Solution

(a)





$R_\mathrm{f} =$ _______ $\qquad R_\mathrm{g} =$ _______

(b) $R_\mathrm{f} = (-\infty, 0]$, $D_\mathrm{g} = \mathbb{R}$

Since $R_\mathrm{f} \subseteq D_\mathrm{g}$, gf exists. (shown)

$$\begin{aligned}\mathrm{gf}(x) &= \mathrm{g}\left(-\sqrt{x-2}\right)\\ &= \left(-\sqrt{x-2}\right)^2 + 1\\ &= x - 2 + 1\\ &= x - 1\end{aligned}$$

$\therefore \mathrm{gf}: x \mapsto x - 1,\ x \in [2, \infty)$

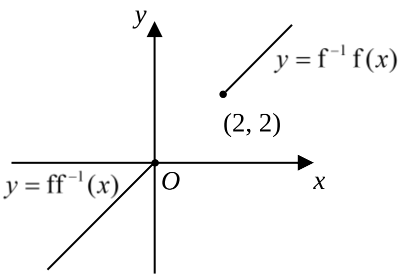
To find R_gf,

$[2, \infty) \xrightarrow{\mathrm{f}} (-\infty, 0] \xrightarrow{\mathrm{g}} [1, \infty) = R_\mathrm{gf}$.

(c) For fg to exist, _______________. Now $R_\mathrm{g} = [1, \infty)$ and $D_\mathrm{f} = [2, \infty)$.

Since ________________, fg does not exist.

(d)



$ff^{-1}(x) = x,\ x \in (-\infty, 0]$ since

$D_{ff^{-1}} = D_{f^{-1}} = (-\infty, 0]$.

$f^{-1}f(x) = x,\ x \in [2, \infty)$

since $D_{f^{-1}f} = D_f = [2, \infty)$.

■

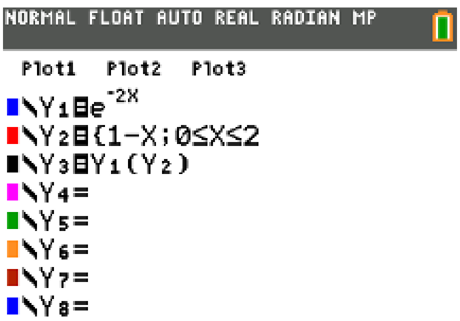
Example 13

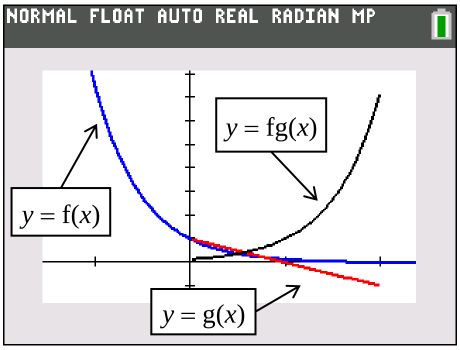
The functions f and g are defined by $f(x) = e^{-2x},\ x \in \mathbb{R}$ and $g(x) = 1 - x,\ x \in \mathbb{R},\ 0 \le x \le 2$.

Sketch $y = fg(x)$ and find R_{fg}.

Solution

1) Press **[Y=]**.
2) For **Y1**, type **[2ND] [LN] [(-)] [2] [X,T,θ,n]**. Press **[ENTER]**.
3) For **Y2**, press **[MATH] [B:piecewise]**. Select 1 piece and **OK**. In the first box, type **[1] [–] [X,T,θ,n]**. In the second box, type **[0] [2ND] [MATH] [6:≤] [X,T,θ,n] [2ND] [MATH] [6:≤] [2]**. Press **[ENTER]**.
4) For **Y3**, press **[ALPHA] [TRACE]** for the YVAR shortcut menu F4. Select **[Y1]**. Type **[(]**, press **[ALPHA] [TRACE]**, select **[Y2]**, type **[)]**. Press **[ENTER]**.
5) Move the cursor to the graph style icon to view different graph styles. Press **[ENTER]** until the desired style is chosen.
6) Press **[WINDOW]** to change the window settings, if necessary.
7) Press **[GRAPH]**.





$R_{fg} =$ ____________ ■

Example 14

The function f is defined by

$$\mathrm{f} : x \mapsto \ln(x-a),\ x \in \mathbb{R},\ a < x \leq a+2,$$

where a is a positive constant.

Given that f^{-1} exists, find an expression for $\mathrm{g}(x)$ for each of the following cases.

(a) $\mathrm{fg}(x) = x$, **(b)** $\mathrm{gf}(x) = x^2$.

[HCI/2013/P1/6(iii) (modified)]

Solution

Let $y = \mathrm{f}(x) = \ln(x-a) \Rightarrow \mathrm{e}^y = x - a$

$\Rightarrow x = \mathrm{e}^y + a = \mathrm{f}^{-1}(y)$

$\Rightarrow \mathrm{f}^{-1}(x) = \mathrm{e}^x + a$

(a) $\mathrm{fg}(x) = x$

$\mathrm{f}^{-1}\mathrm{fg}(x) = \mathrm{f}^{-1}(x)$

$\mathrm{g}(x) = \mathrm{f}^{-1}(x) = \mathrm{e}^x + a$

(b) $\mathrm{gf}(x) = x^2$

$\mathrm{gf}\left(\mathrm{f}^{-1}(x)\right) = \left(\mathrm{f}^{-1}(x)\right)^2$

$\mathrm{g}\left(\mathrm{ff}^{-1}(x)\right) = \left(\mathrm{f}^{-1}(x)\right)^2$

$\mathrm{g}(x) = \left(\mathrm{e}^x + a\right)^2$ ■

Example 15

The functions f, g and h are defined by

$$\mathrm{f}: x \mapsto \ln x, \quad x \in (0, \infty),$$
$$\mathrm{g}: x \mapsto \frac{1}{x}, \quad x \in (0, \infty),$$
$$\mathrm{h}: x \mapsto x^2, \quad x \in (-\infty, \infty).$$

(i) Give definitions of each of the functions fg and f^{-1}, and state, in each of the following cases, a geometrical relationship between the graphs of **(a)** f and fg **(b)** f and f^{-1}.

(ii) Sketch the graph of $y = \mathrm{hf}(x)$ for $x > 0$. State how the sketch shows that hf is not one-one, and prove that, if α and β, where $0 < \alpha < \beta$, are such that $\mathrm{hf}(\alpha) = \mathrm{hf}(\beta)$, then $\alpha = \mathrm{g}(\beta)$.

(iii) The function p is defined by $\mathrm{p}: x \mapsto \mathrm{hf}(x)$, $x \in \mathbb{R}$, $0 < x \le 1$. Sketch the graph of p^{-1}, and give an explicit expression in terms of x for $\mathrm{p}^{-1}(x)$. [J88/I/16 (modified)]

Solution

(i)(a) $\mathrm{fg}(x) = \mathrm{f}\left(\frac{1}{x}\right)$

$= \ln\left(\frac{1}{x}\right) = -\ln x$

$D_{\mathrm{fg}} = D_{\mathrm{g}} = (0, \infty)$

$\therefore \mathrm{fg}: x \mapsto -\ln x,\ x \in (0, \infty)$

Graphs of f and fg are reflections of each other about the x-axis.

(i)(b) Let $y = \ln x$

$\Rightarrow x = \mathrm{e}^y$

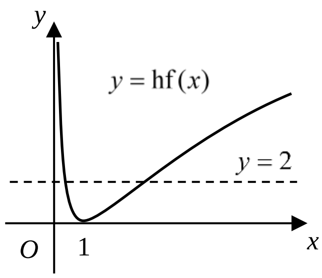
$\therefore \mathrm{f}^{-1}(x) = \mathrm{e}^x$, $D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = \mathbb{R}$

$\therefore \mathrm{f}^{-1}: x \mapsto \mathrm{e}^x,\ x \in \mathbb{R}$

Graphs of f and f^{-1} are reflections of each other about the line $y = x$.

(ii) $\mathrm{hf}(x) = (\ln x)^2$

From the diagram, since the line $y = 2$ cuts the graph of $y = \mathrm{hf}(x)$ twice, hf is not one-one.



Given $\mathrm{hf}(\alpha) = \mathrm{hf}(\beta)$,

$$(\ln \alpha)^2 = (\ln \beta)^2$$
$$\Rightarrow \ln \alpha = \pm \ln \beta$$
$$\Rightarrow \ln \alpha = \ln \beta \text{ or } \ln \alpha = -\ln \beta = \ln \frac{1}{\beta}$$
$$\Rightarrow \alpha = \beta \text{ (NA since } 0 < \alpha < \beta) \text{ or } \alpha = \frac{1}{\beta}$$
$$\therefore \alpha = \mathrm{g}(\beta) \quad \text{(Shown)}$$

Note that the graph of $y = \mathrm{p}(x)$ has a vertical asymptote $x = 0$.

Thus the graph of $y = \mathrm{p}^{-1}(x)$ has a horizontal asymptote $y = 0$.

(iii) $\mathrm{p}(x) = (\ln x)^2, 0 < x \le 1$

Let $y = (\ln x)^2$

$\Rightarrow \pm\sqrt{y} = \ln x$

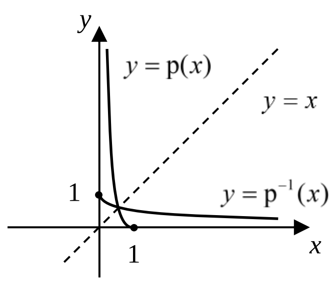
Since $0 < x \le 1, \ln x \le 0$,

$\therefore -\sqrt{y} = \ln x$

$\Rightarrow x = \mathrm{e}^{-\sqrt{y}}$

$\therefore \mathrm{p}^{-1}(x) = \mathrm{e}^{-\sqrt{x}}$,

$D_{\mathrm{p}^{-1}} = R_{\mathrm{p}} = [0, \infty)$



■

More about Functions

The term "function" was introduced by Gottfried Wilhelm Leibniz who was a German mathematician and philosopher. Born on 1 July 1646, Leibniz lost his father at the age of six and so he was largely self-taught by constant reading in his father's library. Before the age of fifteen, he came up with the first clues to his metaphysics while attempting to reform logic. He graduated from the University of Leipzig at the age of seventeen and received a Doctor of Law degree from the University of Altdorf when he was just twenty-one years old for his essay on a new method of teaching law.

In 1673, Leibniz introduced the word function into mathematics, to describe a quantity related to a curve, such as the gradient of the curve at a particular point. This concept was further developed by Johann Bernoulli and Leonhard Euler, who introduced the notation $y = \mathrm{f}(x)$ to represent the fact that y can be written in terms of x.

With his published papers on the "Fundamental Theorem of Calculus", Leibniz was said to be one of the two men (the other being Isaac Newton) who discovered Calculus. Though their approaches to Calculus were made independently, some followers of Newton accused Leibniz of plagiarism, resulting in a controversy that continued to rage on long after Leibniz's death in 1716.

Leibniz's contributions to the various fields of mathematics were remarkable. He was known to be the first to see that the coefficients of a system of linear equations could be arranged into an array, now called a matrix, which can be manipulated to find the solution of the system. Leibniz also invented a calculating machine which handled addition, subtraction, multiplication, division and extraction of roots. Prior to Leibniz, calculating machines could only add and subtract. He was elected a foreign member of the Royal Society for his work on his calculating machine. He and his mathematical rival, Isaac Newton, became the first foreign members of the French Academy of Science. For many centuries after his death, Leibniz remains a universal genius responsible for numerous discoveries in many fields of study, especially mathematics.

Further reading on the history of functions:

https://mathshistory.st-andrews.ac.uk/HistTopics/Functions/

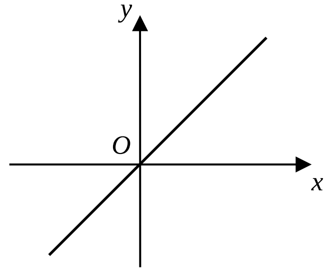
ANNEX A

Some Basic Algebraic Functions

Identity Function

$\mathrm{f} : x \mapsto x, \in \mathbb{R}$

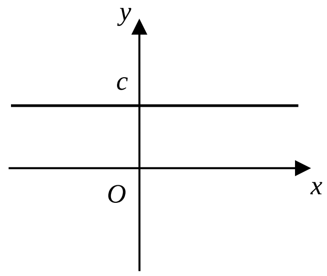
$R_{\mathrm{f}} = \mathbb{R}$



Constant Function

$\mathrm{f} : x \mapsto c, x \in \mathbb{R},$ where c is a constant.

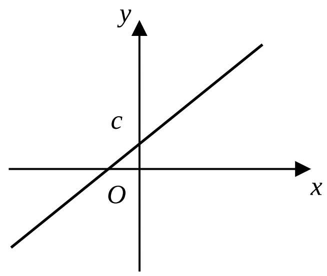
$R_{\mathrm{f}} = \{c\}$



Linear Function

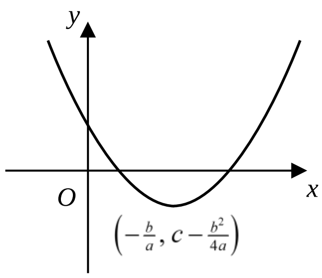
$\mathrm{f} : x \mapsto mx + c, x \in \mathbb{R},$ where m and c are constants.

$R_{\mathrm{f}} = \mathbb{R}$

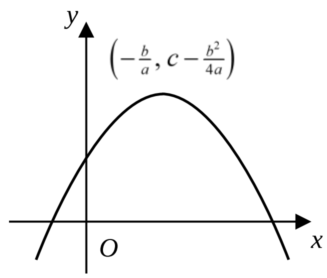


Quadratic Function

$\mathrm{f} : x \mapsto ax^2 + bx + c, x \in \mathbb{R},$ where a, b and c are constants.



$a > 0$



$a < 0$

$$
\begin{aligned}
f(x) &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}
\end{aligned}
$$

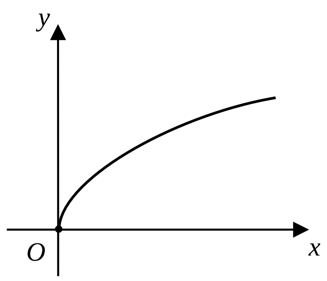
For $a > 0$, $R_f = \left[c - \frac{b^2}{4a}, \infty\right)$

For $a < 0$, $R_f = \left(-\infty, c - \frac{b^2}{4a}\right]$

Root Function

$f : x \mapsto \sqrt{x},\ x \in \mathbb{R},\ x \geq 0$

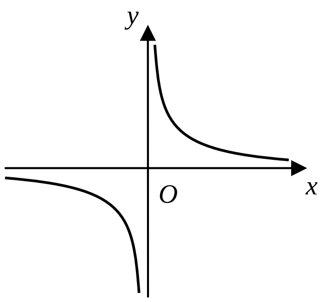
$R_f = \mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$



Reciprocal Function

$f : x \mapsto \frac{1}{x},\ x \in \mathbb{R},\ x \neq 0$

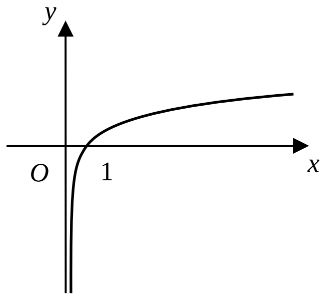
$R_f = \mathbb{R} \setminus \{0\}$
or $(-\infty, 0) \cup (0, \infty)$
or $\{y \in \mathbb{R} : y > 0\} \cup \{y \in \mathbb{R} : y < 0\}$



Logarithmic Function

$f : x \mapsto \ln x,\ x \in \mathbb{R},\ x > 0$

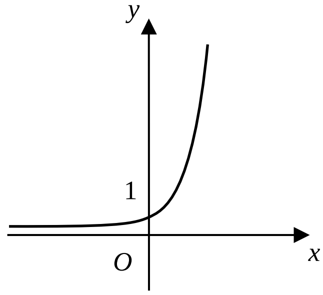
$R_f = \mathbb{R}$



Exponential Function

$\mathrm{f} : x \mapsto \mathrm{e}^x,\ x \in \mathbb{R}$

$R_\mathrm{f} = \mathbb{R}^+$ or $(0, \infty)$



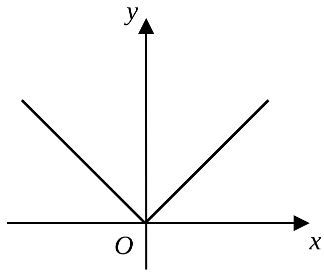
Absolute (or Modulus) Function

$\mathrm{f} : x \mapsto |x|,\ x \in \mathbb{R}$

Alternatively,

$$\mathrm{f}(x) = \begin{cases} x, & x > 0 \\ -x, & x \le 0 \end{cases}$$

$R_\mathrm{f} = \mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$



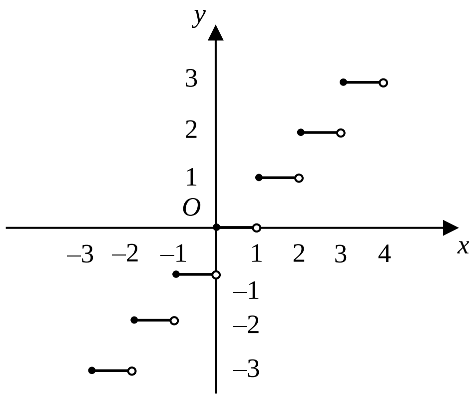
Greatest Integer Function *(Optional)*

$\mathrm{f} : x \mapsto \lfloor x \rfloor,\ x \in \mathbb{R}$

$\lfloor x \rfloor$ is the greatest integer less than or equal to x. In other words,

$$\lfloor x \rfloor = \begin{cases} \vdots & \ldots \\ -2 & -2 \le x < -1 \\ -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & 2 \le x < 3 \\ \vdots & \ldots \end{cases}$$

$R_\mathrm{f} = \mathbb{Z}$

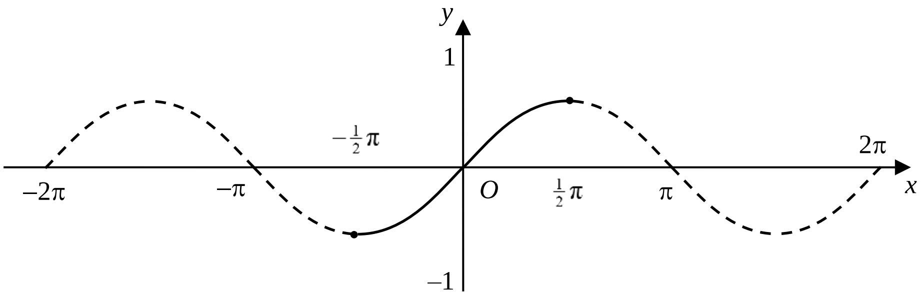


ANNEX B

Graphs of Inverse Trigonometric Functions

Inverse Sine Function

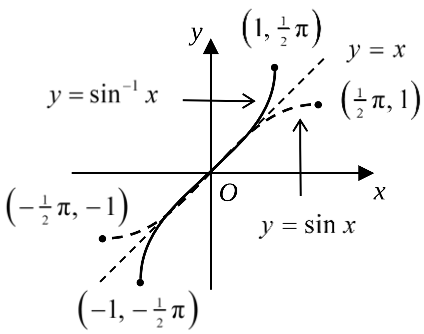
Consider the function $\mathrm{f} : x \mapsto \sin x, -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.



Since f is a 1-1 function for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ (principal range), f^{-1} exists.

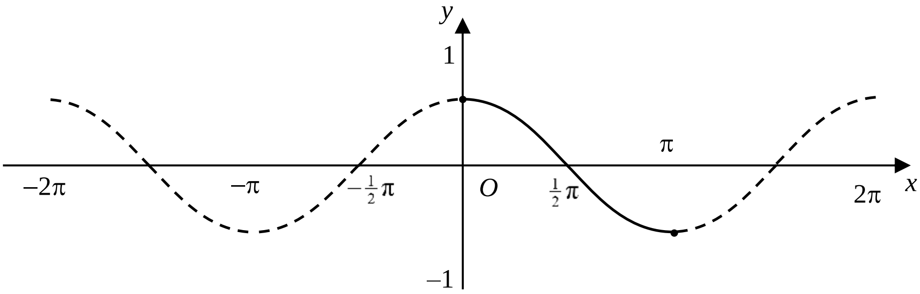
i.e. $\mathrm{f}^{-1} : x \mapsto \sin^{-1} x, -1 \le x \le 1$.

To obtain the graph of $y = \sin^{-1} x$, reflect the graph of $y = \sin x$, $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$, along the line $y = x$ as shown.
Note that $D_{\mathrm{f}^{-1}} = [-1,1] = R_{\mathrm{f}}$ and $R_{\mathrm{f}^{-1}} = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] = D_{\mathrm{f}}$.



Inverse Cosine Function

Consider the function $\mathrm{g}: x \mapsto \cos x, 0 \le x \le \pi$.

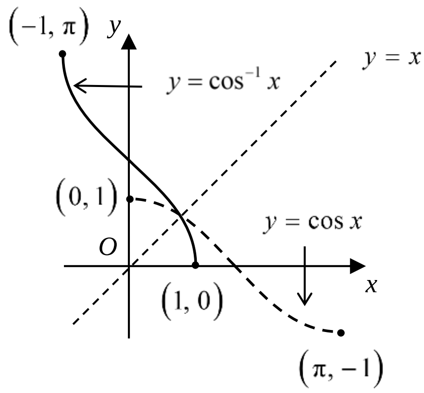


Since g is a 1-1 function for $0 \le x \le \pi$ (principal range), g^{-1} exists.

i.e. $\mathrm{g}^{-1}: x \mapsto \cos^{-1} x, -1 \le x \le 1$.

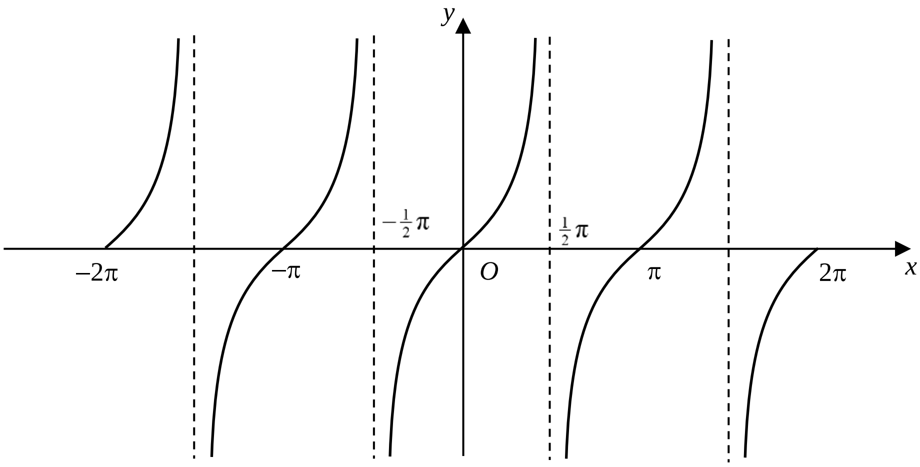
To obtain the graph of $y = \cos^{-1} x$, reflect the graph of $y = \cos x$, $0 \le x \le \pi$, along the line $y = x$ as shown.

Note that $D_{\mathrm{g}^{-1}} = [-1,1] = R_{\mathrm{g}}$ and $R_{\mathrm{g}^{-1}} = [0,\pi] = D_{\mathrm{g}}$.



Inverse Tangent Function

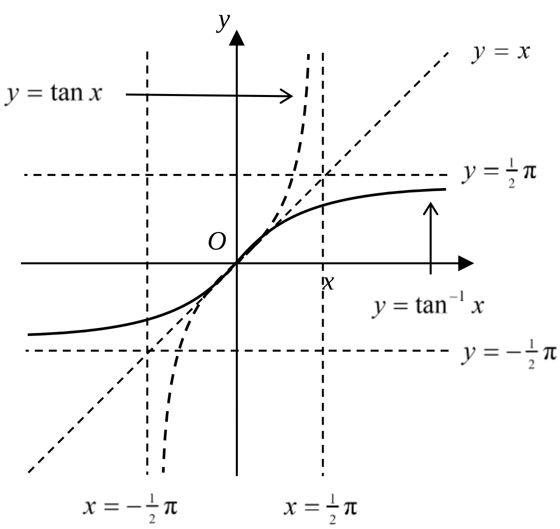
Consider the function $\mathrm{h}: x \mapsto \tan x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.



Since h is a 1-1 function for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ (principal range), h^{-1} exists.

i.e. $\mathrm{h}^{-1}: x \mapsto \tan^{-1} x,\ x \in \mathbb{R}$.

To obtain the graph of $y = \tan^{-1} x$, reflect the graph of $y = \tan x$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, along the line $y = x$ as shown.



Note that $D_{\mathrm{h}^{-1}} = (-\infty, \infty) = R_{\mathrm{h}}$ and $R_{\mathrm{h}^{-1}} = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) = D_{\mathrm{h}}$.

ANNEX C

Practice Questions on Functions

1 For each of the following parts, sketch the graphs of f and g, and state the range. Find also similar expressions for f^{-1} and g^{-1}, stating their domains.

(a) $\mathrm{f}: x \mapsto 10(1-x),\ x \in \mathbb{R}^{+}$ $\qquad$ $\mathrm{g}: x \mapsto \mathrm{e}^{-x},\ x \in \mathbb{R},\ x \geq 0$

(b) $\mathrm{f}: x \mapsto x^{2},\ x \in \mathbb{R}^{-}$ $\qquad$ $\mathrm{g}: x \mapsto \ln(2+x),\ x \in \mathbb{R},\ x \geq -1$

(c) $\mathrm{f}: x \mapsto 4x^{3}+3,\ x \in \mathbb{R}$ $\qquad$ $\mathrm{g}: x \mapsto -2\ln x,\ x \in \mathbb{R},\ 0 < x < 1$

(d) $\mathrm{f}: x \mapsto x^{2}+2x+2,\ x \in \mathbb{R},\ x \geq 1$ $\qquad$ $\mathrm{g}: x \mapsto \dfrac{1}{x},\ x \in \mathbb{R}^{+}$

2 For **1(a)** and **1(c)**, write down an expression for fg, stating its domain and find its range. For **1(b)** and **1(d)**, write down an expression for gf, stating its domain and find its range.

Answers

1 **(a)** $R_{\mathrm{f}} = (-\infty,\ 10)$, $\mathrm{f}^{-1}(x) = 1-0.1x,\ x \in \mathbb{R},\ x < 10$

$R_{\mathrm{g}} = (0,1]$, $\mathrm{g}^{-1}(x) = -\ln x,\ x \in \mathbb{R},\ 0 < x \leq 1$

(b) $R_{\mathrm{f}} = (0,\infty)$, $\mathrm{f}^{-1}(x) = -\sqrt{x},\ x \in \mathbb{R}^{+}$

$R_{\mathrm{g}} = [0,\infty)$, $\mathrm{g}^{-1}(x) = \mathrm{e}^{x}-2,\ x \in \mathbb{R},\ x \geq 0$

(c) $R_{\mathrm{f}} = (-\infty,\infty)$, $\mathrm{f}^{-1}(x) = \sqrt[3]{\dfrac{x-3}{4}}$, $x \in \mathbb{R}$

$R_{\mathrm{g}} = (0,\infty)$, $\mathrm{g}^{-1}(x) = \mathrm{e}^{-\frac{x}{2}},\ x \in \mathbb{R}^{+}$

(d) $R_{\mathrm{f}} = [5,\infty)$, $\mathrm{f}^{-1}(x) = \sqrt{x-1}-1,\ x \in \mathbb{R},\ x \geq 5$

$R_{\mathrm{g}} = (0,\infty)$, $\mathrm{g}^{-1}(x) = \dfrac{1}{x},\ x \in \mathbb{R}^{+}$

2 **(a)** $\mathrm{fg}: x \mapsto 10(1-\mathrm{e}^{-x}),\ x \in \mathbb{R},\ x \geq 0$, $R_{\mathrm{fg}} = [0,10)$

(b) $\mathrm{gf}: x \mapsto \ln(2+x^{2}),\ x \in \mathbb{R}^{-}$, $R_{\mathrm{gf}} = (\ln 2,\infty)$

(c) $\mathrm{fg}: x \mapsto 4(-2\ln x)^{3}+3,\ x \in \mathbb{R},\ 0 < x < 1$, $R_{\mathrm{fg}} = (3,\infty)$

(d) $\mathrm{gf}: x \mapsto \dfrac{1}{x^{2}+2x+2},\ x \in \mathbb{R},\ x \geq 1$, $R_{\mathrm{gf}} = (0,0.2]$