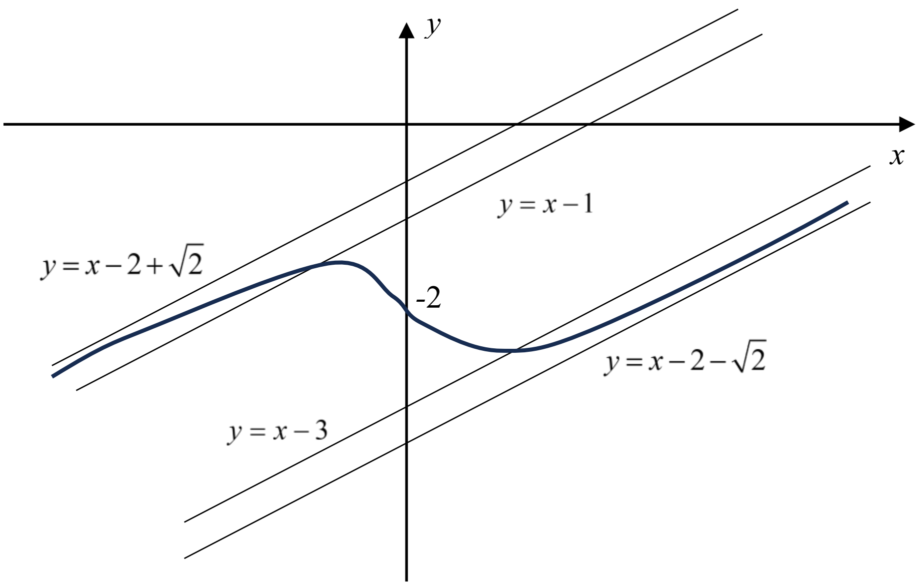
| Q1 | Suggested Answers |
|---|---|
| **(a)** | $y = mx + c$<br>$\dfrac{dy}{dx} = m$<br>Substitute into the DE:<br>$m = x^2 + (mx+c)^2 - 2x(mx+c) - 4x + 4(mx+c) + 3$<br>Comparing coefficients:<br>$x^2$ : $\quad 0 = 1 + m^2 - 2m \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$<br>Constant: $m = c^2 + 4c + 3$<br>$c^2 + 4c + 2 = 0 \Rightarrow c = \dfrac{-4 \pm \sqrt{16-8}}{2} = -2 \pm \sqrt{2}$<br>Thus the solutions are $y = x - 2 + \sqrt{2}$ and $y = x - 2 - \sqrt{2}$ |
| **(b)** | $\dfrac{dy}{dx} = x^2 + y^2 - 2xy - 4x + 4y + 3 = (y-x)^2 + 4(y-x) + 3 = 0$<br>Thus $(y-x+3)(y-x+1) = 0$<br>The equations of the two lines are $y = x - 3$ and $y = x - 1$ |
| **(c)** | $\dfrac{dy}{dx} = x^2 + y^2 - 2xy - 4x + 4y + 3$<br>$\dfrac{d^2y}{dx^2} = 2x + 2y\dfrac{dy}{dx} - 2\left(y + x\dfrac{dy}{dx}\right) - 4 + 4\dfrac{dy}{dx} = 2x - 2y - 4$ since $\dfrac{dy}{dx} = 0$ at stationary points<br>If the stationary point lies on $y = x - 3$, then $x - y - 3 = 0$<br>Then $\dfrac{d^2y}{dx^2} = 2x - 2y - 4 = 2(x - y - 3 + 1) = 2 > 0$<br>Thus the stationary point is a minimum point.<br><br>If the stationary point lies on $y = x - 1$, then $x - y - 1 = 0$<br>Then $\dfrac{d^2y}{dx^2} = 2x - 2y - 4 = 2(x - y - 1 - 1) = -2 < 0$<br>Thus the stationary point is a maximum point. |
| **(d)** |  |

| Q2 | Suggested Answers |
|---|---|
| (a) | $\mathrm{f}(n)=(n+3)^2+2$<br>$\mathrm{g}(n)=(n-2)^2+2$<br>$\mathrm{h}(n)=(n-1)^2+4$<br>Note that $\mathrm{f}(n)=\mathrm{g}(n+5)$ for every $n\in\mathbb{Z}$. i.e. $n \mapsto n+5$ is a bijection on $\mathbb{Z}$.<br>Hence the functions have the same range. |
| | To find common values in the range of f and h, consider<br>$(n+3)^2+2=(s-1)^2+4$<br>$(n+3)^2-(s-1)^2=2$<br>Since there are no perfect squares that differ by 2, there are no integer solutions.<br>There are no common values in the ranges of f and h. |
| (b)(i) | $a^2+ab+b^2=\left(a+\frac{1}{2}b\right)^2+\frac{3}{4}b^2\geq 0$ |
| (b)(ii) | $(a-1-b)\left[(a-1)^2+(a-1)b+b^2\right]$<br>$=(a-1)^3-b(a-1)^2+(a-1)^2b-(a-1)b^2+(a-1)b^2-b^3$<br>$=(a-1)^3-b^3$ |
| (c) | $\mathrm{p}(n)=n^3-3n^2+7n=(n-1)^3+4n+1$<br>To find common integers in the range of p and q, consider<br>$(n-1)^3+4n+1=s^3+4s-6$<br>$(n-1)^3-s^3=4s-6-4n-1$<br>$(n-1-s)\left[(n-1)^2+(n-1)s+s^2\right]=4(s-n+1)-11$<br>$(n-1-s)\left[(n-1)^2+(n-1)s+s^2\right]+4(n-1-s)=-11$<br>$(n-1-s)\underbrace{\left[(n-1)^2+(n-1)s+s^2+4\right]}_{\geq 4}=-11$<br>Thus $(n-1-s)=-1$ and $(n-1)^2+(n-1)s+s^2+4=11$<br>Thus $n=s$.<br>$(s-1)^2+(s-1)s+s^2+4=11$<br>$3s^2-3s-6=0$<br>$(s+1)(s-2)=0$<br>$s=-1$ or $2$<br><br>Thus the values common to both ranges are $\mathrm{q}(-1)=-11$ and $\mathrm{q}(2)=10$. |

| Q3 | Suggested Solutions |
|---|---|
| a(i) | $\int_0^a \mathrm{f}(x)\,\mathrm{d}x=\int_a^0 \mathrm{f}(a-u)(-1)\,\mathrm{d}u \qquad (\text{with } x=a-u)$<br>$=\int_0^a \mathrm{f}(a-u)\,\mathrm{d}u$<br>$=\int_0^a \mathrm{f}(a-x)\,\mathrm{d}x$ |

| | |
|---|---|
| **a(ii)** | Let $I=\int_0^{\frac{\pi}{2}}\frac{\cos^n x}{\sin^n x+\cos^n x}\ \mathrm{d}x$. <br> $I=\int_0^{\frac{\pi}{2}}\frac{\cos^n\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)+\cos^n\left(\frac{\pi}{2}-x\right)}\ \mathrm{d}x \qquad \big(\text{by part (a)(i)}\big)$ <br> $=\int_0^{\frac{\pi}{2}}\frac{\sin^n x}{\cos^n x+\sin^n x}\ \mathrm{d}x \qquad \left(\begin{array}{l}\because \sin\left(\frac{\pi}{2}-x\right)=\cos x\\ \& \ \cos\left(\frac{\pi}{2}-x\right)=\sin x\end{array}\right)$ <br> $2I=\int_0^{\frac{\pi}{2}}\frac{\cos^n x}{\sin^n x+\cos^n x}\ \mathrm{d}x+\int_0^{\frac{\pi}{2}}\frac{\sin^n x}{\cos^n x+\sin^n x}\ \mathrm{d}x$ <br> $=\int_0^{\frac{\pi}{2}}\frac{\cos^n x+\sin^n x}{\sin^n x+\cos^n x}\ \mathrm{d}x$ <br> $=\frac{\pi}{2}$ <br> $I=\frac{\pi}{4}$ |
| **b** | $\int_0^a x\,\mathrm{f}(x)\ \mathrm{d}x=\int_0^a (a-x)\,\mathrm{f}(a-x)\ \mathrm{d}x \qquad \big(\text{by part (a)(i)}\big)$ <br> $=\int_0^a (a-x)\mathrm{f}(x)\ \mathrm{d}x \qquad \because \mathrm{f}(x)=\mathrm{f}(a-x)$ <br> $=a\int_0^a \mathrm{f}(x)\ \mathrm{d}x-\int_0^a x\,\mathrm{f}(x)\ \mathrm{d}x$ <br> $2\int_0^a x\,\mathrm{f}(x)\ \mathrm{d}x=a\int_0^a \mathrm{f}(x)\ \mathrm{d}x$ <br> $\int_0^a x\,\mathrm{f}(x)\ \mathrm{d}x=\frac{a}{2}\int_0^a \mathrm{f}(x)\ \mathrm{d}x$ |
| **c(i)** | $I_n=\int_0^{\frac{\pi}{2}}\cos^n(2x)\ \mathrm{d}x$ <br> $=\int_0^{\frac{\pi}{2}}\cos(2x)\cos^{n-1}(2x)\ \mathrm{d}x$ <br> $=\left[\frac{1}{2}\sin(2x)\cos^{n-1}(2x)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}-(n-1)\sin^2(2x)\cos^{n-2}(2x)\ \mathrm{d}x$ <br> $=(n-1)\int_0^{\frac{\pi}{2}}\left[1-\cos^2(2x)\right]\cos^{n-2}(2x)\ \mathrm{d}x$ <br> $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}(2x)-\cos^n(2x)\ \mathrm{d}x$ <br> $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}(2x)\ \mathrm{d}x-(n-1)\int_0^{\frac{\pi}{2}}\cos^n(2x)\ \mathrm{d}x$ |

| | |
|---|---|
| | $(1+n-1)I_n=(n-1)I_{n-2}$<br>$nI_n=(n-1)I_{n-2}$ |
| **c(ii)** | $\int_0^{\pi} x\cos^8(2x)\ \mathrm{d}x$<br>$=\frac{\pi}{2}\int_0^{\pi}\cos^8(2x)\ \mathrm{d}x \qquad \begin{pmatrix}\text{by part (b)}\\ \because \cos^8\left(2(\pi-x)\right)=\cos^8(2x)\end{pmatrix}$<br>$=\frac{\pi}{2}\left(2\int_0^{\frac{\pi}{2}}\cos^8(2x)\ \mathrm{d}x\right) \qquad \left(\because \cos^8(2x)\text{ symmetrical along } x=\frac{\pi}{2}\right)$<br>$=\pi\int_0^{\frac{\pi}{2}}\cos^8(2x)\ \mathrm{d}x$<br>$=\pi I_8$<br>$=\pi\left(\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times I_0\right)$<br>$=\frac{35\pi}{128}\int_0^{\frac{\pi}{2}}\cos^0(2x)\ \mathrm{d}x$<br>$=\frac{35\pi}{128}\left(\frac{\pi}{2}\right)$<br>$=\frac{35}{256}\pi^2$ |

| **Q4** | |
|---|---|
| **(a)** | $\lvert z_1+z_2\rvert^2=\left(z_1+z_2\right)\left(z_1^*+z_2^*\right)$<br>$=\lvert z_1\rvert^2+\lvert z_2\rvert^2+z_1z_2^*+z_1^*z_2$<br>$=\lvert z_1\rvert^2+\lvert z_2\rvert^2+2\,\mathrm{Re}(z_1z_2^*)$<br>$\le\lvert z_1\rvert^2+\lvert z_2\rvert^2+2\lvert z_1z_2^*\rvert$<br>$=\lvert z_1\rvert^2+\lvert z_2\rvert^2+2\lvert z_1\rvert\lvert z_2\rvert$<br>$=\left(\lvert z_1\rvert+\lvert z_2\rvert\right)^2$<br>Hence $\lvert z_1+z_2\rvert\le\lvert z_1\rvert+\lvert z_2\rvert$. |
| | Let $\mathrm{P}_n$ be the proposition that $\lvert z_1+z_2+\ldots+z_n\rvert\le\lvert z_1\rvert+\lvert z_2\rvert+\ldots+\lvert z_n\rvert$ for all $n\in\mathbb{Z}^+$.<br>Clearly, $\mathrm{P}_1$ and $\mathrm{P}_2$ is true.<br>Assume $\mathrm{P}_k$ is true for some *k*.<br>Need to prove $\mathrm{P}_{k+1}$ is true. i.e. $\lvert z_1+z_2+\ldots+z_{k+1}\rvert\le\lvert z_1\rvert+\lvert z_2\rvert+\ldots+\lvert z_{k+1}\rvert$<br>$\lvert z_1+z_2+\ldots+z_{k+1}\rvert\le\lvert z_1+z_2+\ldots+z_k\rvert+\lvert z_{k+1}\rvert$ |

| | |
|---|---|
| | $\leq \lvert z_1 \rvert + \lvert z_2 \rvert + \ldots + \lvert z_{k+1} \rvert$<br>$\mathrm{P}_k$ is true $\Rightarrow \mathrm{P}_{k+1}$ is true.<br>Since $\mathrm{P}_1$ and $\mathrm{P}_2$ is true and $\mathrm{P}_k$ is true $\Rightarrow \mathrm{P}_{k+1}$ is true, hence by mathematical induction, $\mathrm{P}_n$ is true for all $n \in \mathbb{Z}^+$. |
| **(b)** | Given $u_r - u_{r+1}^2 > 0$ and $u_{2023} - u_1^2 > 0$,<br>$u_1 - u_2^2$, $u_2 - u_3^2$, …, $u_{2022} - u_{2023}^2$, $u_{2023} - u_1^2$ is a sequence of positive numbers<br>Using AM-GM,<br>$\sqrt[2023]{\left[\prod_{r=1}^{2022}(u_r - u_{r+1}^2)\right](u_{2023} - u_1^2)} \leq \frac{1}{2023}\left[\sum_{r=1}^{2022}(u_r - u_{r+1}^2) + (u_{2023} - u_1^2)\right]$<br>Now,<br>$\frac{1}{2023}\left[\sum_{r=1}^{2022}(u_r - u_{r+1}^2) + (u_{2023} - u_1^2)\right]$<br>$= \frac{1}{2023}\left[(u_1 - u_2^2) + (u_2 - u_3^2) + \ldots + (u_{2022} - u_{2023}^2) + (u_{2023} - u_1^2)\right]$<br>$= \frac{1}{2023}\left[(u_1 - u_1^2) + (u_2 - u_2^2) + \ldots + (u_{2022} - u_{2022}^2) + (u_{2023} - u_{2023}^2)\right]$<br>$= \frac{1}{2023}\sum_{r=1}^{2023}(u_r - u_r^2)$<br>$\leq \frac{1}{2023}\left(\frac{1}{4}\right)(2023)$ since $(u_r - u_r^2) \leq \frac{1}{4}$<br>$= \frac{1}{4}$<br>So $\sqrt[2023]{\left[\prod_{r=1}^{2022}(u_r - u_{r+1}^2)\right](u_{2023} - u_1^2)} \leq \frac{1}{4}$<br>$\Rightarrow \quad W = \left[\prod_{r=1}^{2022}(u_r - u_{r+1}^2)\right](u_{2023} - u_1^2) \leq \left(\frac{1}{4}\right)^{2023} \quad \Rightarrow \quad 4^{2025} W \leq 4^{2025}\left(\frac{1}{4}\right)^{2023} = 16$ |

| Qn. | Suggested Solutions |
|---|---|

**5a**

Let $\mathrm{A}_1,\ldots,\mathrm{A}_m$ represent the different surnames and $\mathrm{B}_1,\ldots,\mathrm{B}_n$ represent the different birth months. For each boy (rows 1 to 18 in table below), put a tick under column $\mathrm{A}_i$ and $\mathrm{B}_j$ if his surname is $\mathrm{A}_i$ and birth month is $\mathrm{B}_j$.

| Boy | $\mathrm{A}_1$ | … | $\mathrm{A}_m$ | $\mathrm{B}_1$ | … | $\mathrm{B}_n$ |
|---|---|---|---|---|---|---|
| 1 | ✓ | | | | | ✓ |
| 2 | | | ✓ | ✓ | | |
| ⋮ | | | | | | |
| 18 | | | ✓ | | | ✓ |

Let $a_1,\ldots,a_m,b_1,\ldots,b_n$ be the number of ticks in the columns $\mathrm{A}_1,\ldots,\mathrm{A}_m,\mathrm{B}_1,\ldots,\mathrm{B}_n$ respectively.

Since each row has 2 ticks, the total number of ticks is

$a_1+\ldots+a_m+b_1+\ldots+b_n=18\times 2=36$.

Also, since all integers from 0 to 7 were seen in the boys' responses,

$a_1+\ldots+a_m+b_1+\ldots+b_n\geq 1+2+\ldots+8=36$.

Hence, $a_1,\ldots,a_m,b_1,\ldots,b_n$ is an arrangement of the integers 1, 2, …, 8, which implies that $m+n=8$ and consequently $m,n\leq 7$.

WLOG, suppose $a_1=8$, i.e. 8 boys have the same surname.

Since the number of birth months, $n$, is at most 7, by pigeonhole principle, there are at least 2 of these 8 boys with the same surname, that have the same birth month.

**5b (i)**

The 18 boys can split themselves up according to $6+4+4+4$ or $5+5+4+4$.

Number of ways

$$=\binom{18}{6}\binom{12}{4}\binom{8}{4}\binom{4}{4}\times{}^4\mathrm{C}_1+\binom{18}{5}\binom{13}{5}\binom{8}{4}\binom{4}{4}\times{}^4\mathrm{C}_2=7204317120$$

**5b (ii)**

Each of the 6 players takes 5 dice, leaving 50 dice to be distributed.

Number of ways

$$=\binom{50+6-1}{6-1}=3478761$$

**5c**

Let $A$, $B$ and $C$ be the sets of questions that Alex, Ben and Charlie answered correctly respectively.

Let $a$, $b$, $c$ be the difficult questions that Alex, Ben and Charlie answered correctly respectively.

| | |
|---|---|
| | $a+b+c$<br>$=\lvert A\rvert-\lvert A\cap B\rvert-\lvert A\cap C\rvert+\lvert A\cap B\cap C\rvert$<br>$\quad+\lvert B\rvert-\lvert A\cap B\rvert-\lvert B\cap C\rvert+\lvert A\cap B\cap C\rvert$<br>$\quad+\lvert C\rvert-\lvert A\cap C\rvert-\lvert B\cap C\rvert+\lvert A\cap B\cap C\rvert$<br>$=\lvert A\rvert+\lvert B\rvert+\lvert C\rvert-2\left(\lvert A\cap B\rvert+\lvert A\cap C\rvert+\lvert B\cap C\rvert\right)+3\lvert A\cap B\cap C\rvert$<br>$=2\left[\lvert A\rvert+\lvert B\rvert+\lvert C\rvert-\left(\lvert A\cap B\rvert+\lvert A\cap C\rvert+\lvert B\cap C\rvert\right)+\lvert A\cap B\cap C\rvert\right]$<br>$\quad-\left(\lvert A\rvert+\lvert B\rvert+\lvert C\rvert\right)+\lvert A\cap B\cap C\rvert$<br>$=2\lvert A\cup B\cup C\rvert-\left(\lvert A\rvert+\lvert B\rvert+\lvert C\rvert\right)+\lvert A\cap B\cap C\rvert$<br>$=2(50)-(40+30+20)+\lvert A\cap B\cap C\rvert$<br>$=10+\lvert A\cap B\cap C\rvert$<br>Hence, there were $a+b+c-\lvert A\cap B\cap C\rvert=10$ more difficult questions than easy questions. |
| **5d** | Let $F_i$ be the event where Alex met the $i^{\text{th}}$ friend, $i=1,2,...,5$.<br>Number of times Alex participated<br>$=\left\lvert\bigcup_{i=1}^{5}F_i\right\rvert+3$<br>$=\sum_{i}\lvert F_i\rvert-\sum_{i<j}\lvert F_i\cap F_j\rvert+\sum_{i<j<k}\lvert F_i\cap F_j\cap F_k\rvert$<br>$\quad-\sum_{i<j<k<l}\lvert F_i\cap F_j\cap F_k\cap F_l\rvert+\left\lvert\bigcap_{i=1}^{5}F_i\right\rvert+3$<br>$=\binom{5}{1}(11)-\binom{5}{2}(5)+\binom{5}{3}(3)-\binom{5}{4}(2)+\binom{5}{5}(1)+3$<br>$=29$<br><br><u>Alternative</u><br>Let $n_i$ be the number of competitions where Alex meets exactly *i* friends, $i=0,1,2,...,5$.<br><br>Alex meets his friends a total number of times given by<br>$\sum_{i=1}^{5}in_i=n_1+2n_2+3n_3+4n_4+5n_5=\binom{5}{1}\times 11=55\quad---(1)$.<br>And we also have the following:<br>$\binom{2}{2}n_2+\binom{3}{2}n_3+\binom{4}{2}n_4+\binom{5}{2}n_5=\binom{5}{2}\times 5=50$<br>$\Rightarrow n_2+3n_3+6n_4+10n_5=50\quad----(2)$<br>$\binom{3}{3}n_3+\binom{4}{3}n_4+\binom{5}{3}n_5=\binom{5}{3}\times 3=30$<br>$\Rightarrow n_3+4n_4+10n_5=30\qquad---(3)$ |

$$\binom{4}{4}n_4+\binom{5}{4}n_5=\binom{5}{4}\times 2=10$$

$$\Rightarrow n_4+5n_5=10 \qquad ---(4)$$

$$\binom{5}{5}n_5=\binom{5}{5}\times 1=1$$

$$\Rightarrow n_5=1 \qquad ---(5)$$

Solving Equations (1), (2), (3), (4) and (5) gives
$n_1=10, n_2=10, n_3=0, n_4=5, n_5=1$.

Number of times Alex participated
$=n_0+n_1+...+n_5=3+10+10+0+5+1=29$

**Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.**

**Continued Fractions**

A continued fraction for a real number $x$ is an expression of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \ ,$$

or simply expressed as $x = [a_0; a_1, a_2, \ldots]$, where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{R}$ for each $i \geq 1$.

A continued fraction with $a_i \in \mathbb{Z}^+$ for each $i \geq 1$ is known as a simple continued fraction.

It is known that the simple continued fraction for $x$ is finite if and only if $x$ is rational.
For example, the number

$$\frac{9}{7} = 1 + \cfrac{1}{3 + \cfrac{1}{2}} = [1;3,2] \text{ is rational}$$

while the number

$$\ln 3 = 1 + \cfrac{1}{10 + \cfrac{1}{7 + \cfrac{1}{9 + \ddots}}} = [1;10,7,9,\ldots] \text{ is irrational.}$$

Let $a_0, a_1, a_2, \ldots$ be a sequence of integers, with all the terms positive except perhaps $a_0$. We define two sequences of integers as follows:

$$p_{-1} = 1,\ p_0 = a_0,\ p_n = a_n p_{n-1} + p_{n-2},\ n \geq 1,$$
$$q_{-1} = 0,\ q_0 = 1,\ q_n = a_n q_{n-1} + q_{n-2},\ n \geq 1. \qquad (*)$$

For $x = [a_0; a_1, a_2, \ldots]$, and for each $n \in \mathbb{Z}^+$, we define the $n$th convergent of $x$ as

$$r_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} = [a_0; a_1, a_2, \ldots, a_n] \ .$$

**6** **(a)** **(i)** Show that $\sqrt{2}=1+\dfrac{1}{1+\sqrt{2}}$. Hence, write $\sqrt{2}$ as a simple continued fraction to show that it is irrational. [2]

**(ii)** The number $\pi = 3.14159265\ldots$ can be expressed as a simple continued fraction

$$\pi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \ddots}}}.$$

Find the values of $b_0$, $b_1$, $b_2$, and $b_3$. [2]

**(iii)** Let $x = [1; a_1, a_2, a_3, \ldots]$, where for each $n \in \mathbb{Z}^+$, $a_{2n-1} = 3$ and $a_{2n} = 4$. Find the value of $x$, giving your answer in the form $a + b\sqrt{c}$, where $a$ and $b$ are rational numbers and $c$ is a positive integer. [3]

**(b)** Let $a_0, a_1, a_2, \ldots$ be a sequence of integers, with all the terms positive except perhaps $a_0$. Let $p_n$ and $q_n$ be defined by the recurrence relations in (*) as stated in the mathematical text.

**(i)** Prove that for each $n \in \mathbb{Z}^+$ and any positive real number $y$,

$$[a_0; a_1, a_2, \ldots, a_{n-1}, y] = \frac{y p_{n-1} + p_{n-2}}{y q_{n-1} + q_{n-2}}.$$ [5]

**(ii)** Hence, show that $r_n$, the $n$th convergent of $[a_0; a_1, a_2, \ldots]$, is given by $r_n = \dfrac{p_n}{q_n}$. [1]

**(iii)** Prove that for each $n \in \mathbb{Z}^+$,

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}, \text{ and}$$

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$ [4]

**(iv)** Explain why the $n$th convergent, $r_n = \dfrac{p_n}{q_n}$, is always in its lowest terms. [1]

**(v)** Prove that $r_2 < r_4 < r_6 < \ldots < r_5 < r_3 < r_1$. [2]

| **Qn.** | **Suggested Solutions** |
|---|---|
| **6a (i)** | $\dfrac{1}{1+\sqrt{2}} = \dfrac{1}{1+\sqrt{2}} \times \dfrac{1-\sqrt{2}}{1-\sqrt{2}} = \dfrac{1-\sqrt{2}}{1-2} = \sqrt{2}-1 \Rightarrow \sqrt{2} = 1 + \dfrac{1}{1+\sqrt{2}}$ |

| | |
|---|---|
| | $\sqrt{2}=1+\dfrac{1}{1+\sqrt{2}}$ <br> $=1+\dfrac{1}{1+\left(1+\dfrac{1}{1+\sqrt{2}}\right)}$ <br> $=1+\dfrac{1}{2+\dfrac{1}{1+\left(1+\dfrac{1}{1+\sqrt{2}}\right)}}$ <br> $=1+\dfrac{1}{2+\dfrac{1}{2+\ddots}}=[1;2,2,2,\ldots]$ <br> Hence, $\sqrt{2}$ is an infinite continued fraction which shows that it is irrational. |
| **6a (ii)** | $b_0=\lfloor \pi \rfloor=3$ <br> $b_1=\left\lfloor \dfrac{1}{\pi-3} \right\rfloor=7$ <br> $b_2=\left\lfloor \dfrac{1}{\dfrac{1}{\pi-3}-7} \right\rfloor=15$ <br> $b_4=\left\lfloor \dfrac{1}{\dfrac{1}{\dfrac{1}{\pi-3}-7}-15} \right\rfloor=1$ |
| **6a (iii)** | Given $x=[1;3,4,3,4,\ldots]$, let $y=[0;3,4,3,4,\ldots]$. <br> $y=\dfrac{1}{3+\dfrac{1}{4+\ddots}}=\dfrac{1}{3+\dfrac{1}{4+y}}=\dfrac{4+y}{13+3y}$ <br> $13y+3y^2=4+y$ <br> $3y^2+12y-4=0$ <br> $y=\dfrac{-12+\sqrt{192}}{6}$ $\qquad (\because y=[0;3,4,3,4,...]>0)$ <br> $y=-2+\dfrac{4}{3}\sqrt{3}$ <br> $\therefore x=1+y=-1+\dfrac{4}{3}\sqrt{3}$ |
| **6b (i)** | Let $T_n$ be the statement $[a_0;a_1,a_2,\ldots,a_{n-1},y]=\dfrac{yp_{n-1}+p_{n-2}}{yq_{n-1}+q_{n-2}}$ for each $n\in\mathbb{Z}^+$. <br> Given $p_{-1}=1$, $p_0=a_0$, $p_n=a_np_{n-1}+p_{n-2}$ and $q_{-1}=0$, $q_0=1$, $q_n=a_nq_{n-1}+q_{n-2}$. |

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| | Since $\dfrac{yp_0+p_{-1}}{yq_0+q_{-1}}=\dfrac{y(a_0)+1}{y(1)+0}=a_0+\dfrac{1}{y}=[a_0;y]$, $T_1$ is true.<br><br>Assume that $T_k$ is true for some $k\in\mathbb{Z}^+$, i.e. $[a_0;a_1,a_2,\ldots,a_{k-1},y]=\dfrac{yp_{k-1}+p_{k-2}}{yq_{k-1}+q_{k-2}}$ for any $y\in\mathbb{R}^+$.<br><br>$[a_0;a_1,a_2,\ldots,a_{k-1},a_k,y]$<br>$=\left[a_0;a_1,a_2,\ldots,a_{k-1},a_k+\dfrac{1}{y}\right]$<br>$=\dfrac{\left(a_k+\frac{1}{y}\right)p_{k-1}+p_{k-2}}{\left(a_k+\frac{1}{y}\right)q_{k-1}+q_{k-2}}\left(\text{induction hypothesis, }\because a_k+\dfrac{1}{y}>0\right)$<br>$=\dfrac{(ya_k)p_{k-1}+p_{k-1}+yp_{k-2}}{(ya_k)q_{k-1}+q_{k-1}+yq_{k-2}}$<br>$=\dfrac{y(a_kp_{k-1}+p_{k-2})+p_{k-1}}{y(a_kq_{k-1}+q_{k-2})+q_{k-1}}$<br>$=\dfrac{yp_k+p_{k-1}}{yq_k+q_{k-1}}$ (by the given recurrence relations)<br>This shows that $T_k$ is true implies that $T_{k+1}$ is true.<br><br>By the principle of mathematical induction, $T_n$ is true for each $n\in\mathbb{Z}^+$. |
| **6b (ii)** | $r_n=[a_0;a_1,\ldots,a_{n-1},a_n]$<br>$=\dfrac{a_np_{n-1}+p_{n-2}}{a_nq_{n-1}+q_{n-2}}$ (by part (b)(i) since $a_n>0$)<br>$=\dfrac{p_n}{q_n}$ (by given recurrence relations) |
| **6b (iii)** | Let $S_n$ be the statement $p_nq_{n-1}-p_{n-1}q_n=(-1)^{n-1}$ for each $n\in\mathbb{Z}^+$.<br><br>$p_1q_0-p_0q_1=(a_1p_0+p_{-1})(1)-a_0(a_1q_0+q_{-1})$<br>$=a_1a_0+1-a_0a_1$<br>$=1=(-1)^{1-1}$<br>Hence, $S_1$ is true.<br><br>Assume $S_k$ is true for some $k\in\mathbb{Z}^+$, i.e. $p_kq_{k-1}-p_{k-1}q_k=(-1)^{k-1}$. |

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| | LHS of $S_{k+1} = p_{k+1}q_k - p_k q_{k+1}$<br>$= \left(a_{k+1}p_k + p_{k-1}\right)q_k - p_k\left(a_{k+1}q_k + q_{k-1}\right)$<br>$= -\left(p_k q_{k-1} - p_{k-1}q_k\right)$<br>$= -(-1)^{k-1}$ $\quad$ (by induction hypothesis)<br>$= (-1)^{(k+1)-1} =$ RHS of $S_{k+1}$<br>This shows that $S_k$ is true implies that $S_{k+1}$ is true.<br><br>By the principle of mathematical induction, $S_n$ is true for each $n \in \mathbb{Z}^+$.<br><br>$p_n q_{n-2} - p_{n-2}q_n$<br>$= \left(a_n p_{n-1} + p_{n-2}\right)q_{n-2} - p_{n-2}\left(a_n q_{n-1} + q_{n-2}\right)$<br>$= a_n\left(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}\right)$<br>$= \begin{cases} a_n\left(p_0 \times 0 - 1 \times 1\right) & \text{if } n = 1 \\ a_n(-1)^{n-1-1} & \text{from previous result if } n \geq 2 \end{cases}$<br>$= (-1)^n a_n$ |
| **6b (iv)** | From part **(b)(iii)**, $p_n q_{n-1} - p_{n-1}q_n = (-1)^{n-1}$ shows that if $d$ divides both $p_n$ and $q_n$, then $d$ divides $(-1)^{n-1}$, and so $d$ divides 1. This implies that $\gcd\left(p_n, q_n\right) = 1$ . |
| **6b (v)** | From part **(b)(iii)**,<br>$p_n q_{n-1} - p_{n-1}q_n = (-1)^{n-1} \Rightarrow r_n - r_{n-1} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}}$ and<br>$p_n q_{n-2} - p_{n-2}q_n = (-1)^n a_n \Rightarrow r_n - r_{n-2} = \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^n a_n}{q_n q_{n-2}}$.<br><br>Since for all $n \in \mathbb{Z}_0^+$, $q_n > 0$, for each $m \in \mathbb{Z}^+$, we have<br>$r_{2m} - r_{2m-2} = \frac{(-1)^{2m} a_{2m}}{q_{2m}q_{2m-2}} > 0$, $r_{2m+1} - r_{2m-1} = \frac{(-1)^{2m+1} a_{2m+1}}{q_{2m+1}q_{2m-1}} < 0$, and<br>$r_{2m} - r_{2m-1} = \frac{(-1)^{2m-1}}{q_{2m}q_{2m-1}} < 0$.<br><br>This implies that for each $m \in \mathbb{Z}^+$, $r_{2m-2} < r_{2m}$, $r_{2m+1} < r_{2m-1}$ and $r_{2m} < r_{2m-1}$.<br>Hence, $r_2 < r_4 < r_6 < \ldots < r_5 < r_3 < r_1$. |