2024 NYJC-TJC-VJC H3 Prelim

Inequalities

1 Let $x_1, x_2, \ldots, x_n$ be real numbers and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive real numbers.

(a) Prove that

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 \leq \frac{1}{n}\sum_{i=1}^{n} x_i^{\,2}.$$ [2]

(b) Prove that

$$\left(\sum_{i=1}^{n} \lambda_i x_i\right)^2 \leq \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \lambda_i x_i^{\,2}\right)$$

and find a necessary and sufficient condition for the equality to hold. [3]

(c) Let $y_1, y_2, \ldots, y_n$ be real numbers, not all zero, and $t \geq 2$. Prove that

$$\left(\sum_{i=1}^{n} \frac{1}{t^i} y_i\right)^2 < \sum_{i=1}^{n} \frac{y_i^2}{t^i}.$$ [3]

Functions

2 For any real number s, the greatest integer less than or equal to s is denoted by $\lfloor s \rfloor$. For example, $\lfloor 3.7 \rfloor = 3$ and $\lfloor 5 \rfloor = 5$.

(a) The functions f and g are defined by

$$\mathrm{f}: x \mapsto \lfloor x \rfloor - x \qquad \text{for } x \in \mathbb{R},$$

$$\mathrm{g}: x \mapsto \frac{15}{4}x^2 - \frac{41}{4}x + \frac{11}{2} \qquad \text{for } x \geq 1.$$

(i) Prove that f is a periodic function with the period of the function as 1. [3]

(ii) On the same diagram, sketch the graph of $y = \mathrm{f}(x)$ for $-1.5 \leq x < 3$ and the graph of $y = \mathrm{g}(x)$, labelling the coordinates of all endpoints and the points of intersection of the two graphs clearly. [3]

(iii) Find the area bounded by the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{g}(x)$ and the line $x = 1$. [3]

(b) **(i)** Solve the equation $\lfloor \log_3 k \rfloor = 2$, where $k \in \mathbb{Z}^+$. [2]

(ii) A sequence $\{x_n\}$ is given by the recurrence relation $x_n = x_{\lfloor \frac{n}{3} \rfloor} + 1$, $x_0 = 2$.

Write down the values of $x_1, x_2, \ldots, x_9$. Hence, write down x_n in terms of n, for $n = 1, 2, 3, \ldots$. [3]

Calculus

3 **In this question, you may swap the summation and integration (differentiation) symbols without justification, i.e.** $\sum_{n=0}^{\infty}\left(\int_0^a f_n(t)\,dt\right)=\int_0^a\left[\sum_{n=0}^{\infty} f_n(t)\right]dt$ **and** $\sum_{n=0}^{\infty}\left(\frac{d}{dt}f_n(t)\right)=\frac{d}{dt}\sum_{n=0}^{\infty} f_n(t)$**.**

(a) Show that $\frac{(n!)^2}{(2n)!}=\binom{2n}{n}^{-1}$. [1]

(b) Use induction and integration by parts to prove that for all $n\in\mathbb{Z}_0^+$,

$$\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\,dx=\frac{4^n}{2n+1}\binom{2n}{n}^{-1}.$$ [7]

(c) Use the result in part **(b)** to prove that for $|x|<1$,

$$\sum_{n=0}^{\infty}\frac{4^n}{2n+1}\binom{2n}{n}^{-1}x^{2n+1}=\int_0^{\frac{\pi}{2}}\left[\frac{x\sin t}{1-x^2\left(1-\cos^2 t\right)}\right]dt.$$ [3]

(d) Using the result in part **(c)**, and a suitable substitution, show that for $|x|<1$,

$$\sum_{n=0}^{\infty}\frac{4^n}{2n+1}\binom{2n}{n}^{-1}x^{2n+1}=\frac{\sin^{-1}x}{\sqrt{1-x^2}}.$$ [3]

(e) Deduce the exact value of

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}.$$ [4]

Numbers

4 Let p be a prime number. It is given that x, y and z are positive integers such that $0<x<y<z<p$, and $x^3\equiv y^3\equiv z^3 \pmod{p}$.

(a) Show that p divides x^2+y^2+xy. [3]

(b) Hence, or otherwise, show that p divides $x+y+z$. [2]

(c) Prove that $x+y+z$ divides $x^2+y^2+z^2$. [6]

Functions

5 The function $\mathrm{f}:\mathbb{R}^+ \to \mathbb{R}^+$ is such that

$$\mathrm{f}(x)\mathrm{f}(y\mathrm{f}(x)) = \mathrm{f}(x+y) \text{ for all } x, y \in \mathbb{R}^+.$$

(a) Prove by contradiction, or otherwise, that $\mathrm{f}(x) \le 1$ for all $x \in \mathbb{R}^+$. [3]

(b) Show that if $\mathrm{f}(x_0) = 1$ for some $x_0 \in \mathbb{R}^+$, then $\mathrm{f}(x) = 1$ for all $x \in \mathbb{R}^+$. [3]

(c) Suppose now that $\mathrm{f}(x) < 1$ for all $x \in \mathbb{R}^+$. By replacing x with 1 and y with $x + \dfrac{1}{\mathrm{f}(x)} - 1$ in the original equation $\mathrm{f}(x)\mathrm{f}(y\mathrm{f}(x)) = \mathrm{f}(x+y)$, show that

$$\mathrm{f}\left(\left(x + \frac{1}{\mathrm{f}(x)} - 1\right)\mathrm{f}(1)\right) = \mathrm{f}(x).$$ [2]

(d) Given that $\mathrm{f}(1) = \dfrac{1}{2}$, show that f is strictly decreasing on $\mathbb{R}^+$ and find $\mathrm{f}(x)$. [4]

Numbers

6 **(a)** Suppose that a sequence of prime numbers a_1, a_2, a_3, a_4, a_5 forms an arithmetic progression with positive common difference k and $a_1 > 5$.

(i) Show that $k \equiv 0 \pmod{10}$. [3]

(ii) State one such sequence of a_1, a_2, a_3, a_4, a_5. [1]

(b) Let $b_1, b_2, b_3, ..., b_p$ be an arithmetic progression of prime numbers with positive common difference d, such that p is a prime number and $b_1 > p$.

(i) Prove that p divides d. [4]

(ii) Using a suitable example, show that if $b_1 \le p$, then p may not divide d. [2]

(iii) Show that if $b_1, b_2, b_3, ..., b_{18}$ is an arithmetic progression of prime numbers with positive common difference m, and $b_1 > 18$, then m is at least 510510. [2]

Combinatorics

7 **(a)** A set S containing four distinct positive integers is called "connected" if for every $x \in S$ at least one of the numbers $x-1$ and $x+1$ belongs to S. Let C_n be the number of connected subsets of the set $\{1, 2, \ldots n\}$.

(i) Evaluate C_7. [3]

(ii) Find a general formula for C_n. [3]

(b) A permutation of a set is called a derangement if none of the objects appear in their "natural" (i.e. ordered) place. For example, the only derangements of the set $\{1, 2, 3\}$ are $\{2, 3, 1\}$ and $\{3, 1, 2\}$.

Find the number of derangements of the set $\{1, 2, 3, \ldots, n\}$, showing your workings clearly. Hence show that as n gets large, the number of derangements is approximately $\dfrac{n!}{\mathrm{e}}$. [5]

Numbers

8 **(a)** Prove that if p is a prime and a is an integer, then

$$p \mid a^n \Rightarrow p \mid a$$

where n is a positive integer. [3]

(b) It is given that the real number x satisfies the polynomial equation

$$x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0,$$

for some integers $c_0, c_1, \ldots, c_{n-1}$.

By writing x in the form $\dfrac{a}{b}$ for some $a, b \in \mathbb{Z}, b \neq 0$ and stating any assumption you need to make, prove that x is either an integer or an irrational number. [7]

(c) By expressing $\left(x^2 - 11\right)^2$ as a polynomial in x where $x = \sqrt{6} - \sqrt{3} - \sqrt{2}$, prove that x is irrational. [4]

—— END OF PAPER ——