



EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2026

General Certificate of Education Advanced Level

Higher 2

FURTHER MATHEMATICS

Paper 1

9649/01

27 August 2026

3 hours

Additional Materials: Printed Answer Booklet
 List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.

Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.

You must show all necessary working clearly.

The number of marks is given in brackets [] at the end of each question or part question.

This document consists of **5** printed pages and **3** blank pages.

- 1 The terms in the sequence $u_0, u_1, u_2, \dots$ satisfy the recurrence relation

$$u_r = 4u_{r-1} - u_{r-2} \text{ for } r \geq 2.$$

- (a) Find the general solution of this recurrence relation. [2]
- (b) Find, in an exact form, an expression for u_r in terms of r in the case that $u_0 = 0$ and $u_1 = 2$. [2]
- (c) Find the exact value of u_0 in the case that $u_2 = 1 - \frac{4}{7}\sqrt{3}$ and u_r tends to zero as r tends to infinity. [2]

- 2 Consider the differential equation

$$\frac{dy}{dx} = \frac{e^{10x} - y}{2y - 1},$$

where $y = 1$ when $x = 0$.

- (a) Use two steps of both the Euler method and the improved Euler method to calculate approximations to the value of y when $x = 1$, giving each answer correct to 2 decimal places. [5]
- (b) Explain the large discrepancy between these two answers. [2]

- 3 The growth of a mold colony on a circular petri dish is the subject of a biology experiment. A student proposes modelling the area of the mold using a recurrence relation:

$$A_{n+1} = p\sqrt{A_n}(1 - qA_n),$$

where p and q are positive constants and A_n represents the area of the mold in mm^2 , n days after the start of the experiment.

- (a) It is given that $p = 20$ and $A_0 = 100$.
- (i) Find the exact value of q for which this model predicts that the area of the mold remains constant. [2]
- (ii) Determine the range of values of q for which this model predicts that A_1 is less than zero. [1]
- (b) The values of p and q are now chosen so that for $n \geq 1$,

$$A_n = \begin{cases} 400, & n \text{ is odd,} \\ 100, & n \text{ is even.} \end{cases}$$

- (i) Determine the exact values of p and q . [3]
- (ii) Investigate what the model, with these values of p and q , predicts about the mold area when $A_0 = 99$ and when $A_0 = 101$. Hence comment on the proposed model. [3]

4 Let $I = \int_1^5 0.5x + \sin(e^{0.5x}) dx$.

(a) Use the trapezium rule with four strips to obtain an approximation to I , correct to 3 decimal places. [3]

(b) Use Simpson's rule with five ordinates to obtain an approximation to I , correct to 3 decimal places. [3]

It is given that the approximation to I , correct to 3 decimal places, from the graphing calculator is 6.158.

(c) State which of the approximations in (a) or (b) is more accurate and explain, with aid of a diagram, why the less accurate method has such a large discrepancy. [3]

5 The curve C has equation $y = x + \sin(x^2)$. The area bounded by the portion of C over the interval $[0, \sqrt{3\pi}]$, the positive x -axis and the line $x = \sqrt{3\pi}$ is rotated 2π radians about the y -axis.

(a) Find the exact volume of the solid of revolution. [4]

(b) Find, correct to 3 decimal places, the area of the surface of revolution that is generated by the portion of C . [3]

The same portion of C is now rotated 2π radians about the x -axis instead.

(c) Find the area of the new surface of revolution that is generated. [2]

6 The linear transformation T maps points (x, y, z) to image points (x', y', z') such that

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ -4 & -4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(a) Given that T has an invariant plane, Π , and an invariant line, l , find vector equations for Π and l . [6]

(b) (i) Given that a , b and c are constants, state the values of t for which the system of equations

$$\begin{aligned} (3-t)x + y + 2z &= a \\ x + (3-t)y + 2z &= b \\ -4x - 4y + (-6-t)z &= c \end{aligned}$$

has no unique solutions. [1]

(ii) For each of these values of t , find the relationship(s) between a , b and c such that the system of equations is consistent. [3]

7 Consider the function $f(x) = \frac{\cos(e^{1.1x})}{x}$. Solutions are required for the equation $f(x) = 0$ for $x > 0$.

(a) Evaluate $f(1)$ and $f(2)$, and hence comment on the possible number of roots of $f(x) = 0$ in the interval $(1, 2)$. [2]

(b) Name the feature that must exist on the graph of $y = f(x)$ for the number of roots of $f(x) = 0$ in the interval $(1, 2)$ to be non-zero. [1]

It is given that α , the 2nd smallest positive root of $f(x) = 0$, lies in the interval $(1, 2)$.

(c) (i) Use the Newton-Raphson method, with initial approximation $x_0 = 1$, to determine the values of x_1 , x_2 and x_3 . Give each answer correct to 3 decimal places. [3]

(ii) Explain why the values of x_3 and x_4 are the same (correct to 3 decimal places). [2]

(iii) Explain why the use of $x_0 = 1$ in the Newton-Raphson method does not give convergence to α . [1]

(iv) Find the value of α correct to 3 decimal places. [1]

(v) By considering the graph of $y = f(x)$, for $x > 0$, suggest a reason why the Newton-Raphson method should not be used to find the roots of $f(x) = 0$ for large values of x . [1]

8 (a) The matrices

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -1 & -1 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 8 & -4 & -5 \\ 4 & -1 & -4 \\ 5 & -4 & -2 \end{pmatrix}$$

have characteristic equations equivalent to $(3 - \lambda)^2(1 + \lambda) = 0$.

(i) For each matrix, find a basis for the set of eigenvectors corresponding to the eigenvalue 3. [3]

(ii) For each matrix, determine whether it is diagonalisable. [1]

(b) The vector space V is $\mathbb{R}^4$. The set S consists of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ in $\mathbb{R}^4$ given by $\mathbf{a} = (-1, 0, 1, 2)$, $\mathbf{b} = (0, 1, 2, -1)$, $\mathbf{c} = (1, 2, -1, 0)$ and $\mathbf{d} = (2, -1, 0, 1)$.

(i) Show that the set S spans V . [3]

(ii) Express the vector $\mathbf{u}_1 = (1, 0, 0, 0)$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. [2]

(iii) Write down similar expressions for each of the vectors

$$\mathbf{u}_2 = (0, 1, 0, 0), \mathbf{u}_3 = (0, 0, 1, 0) \text{ and } \mathbf{u}_4 = (0, 0, 0, 1)$$

in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. [3]

- 9 A curve C has polar equation

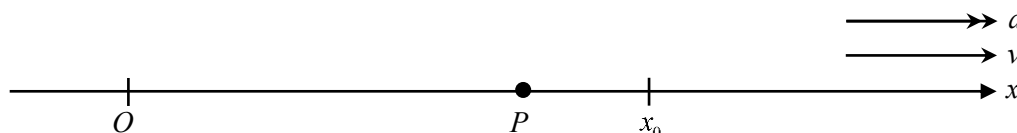
$$r = (1 + \sin 3\theta)(2 + \cos \theta) \text{ for } 0 \leq \theta < 2\pi.$$

- (a) Find, in an exact form, the equation of the tangents at the pole. [2]
- (b) State the number of loops that C has. [1]
- (c) For each loop, find the values of θ and r for which C is furthest from the pole. [3]
- (d) Sketch C , labelling clearly the features found in all previous parts above. [3]
- (e) Find the area enclosed by the smallest loop of C . [2]
- (f) Find the arc length of the largest loop of C . [2]

- 10 [In this question, all variables are in standard SI units.]

In an experiment to investigate the motion of objects submerged in viscous fluid, an object P of fixed mass m is attached to the end of a metal spring and submerged in a pool of the viscous fluid.

The origin O is the equilibrium point for the system, where there will not be any motion unless an external force is applied. P is displaced from O by a displacement of x_0 and released from rest. P moves along the x -axis with a displacement from O of x , an acceleration of a and a velocity of v at time t from its release.



Newton's Second Law states that the sum of all forces that act on P is equal to the product ma , where the sign of each force represents its direction of application. All quantities, x , v , a and forces are positive when they are in the direction of the positive x -axis.

In this experiment, there are only two forces acting on P ,

- (1) the spring force S with magnitude $k|x|$, for a positive constant k , which acts in the direction from P towards O , and
 - (2) the drag force D with magnitude $b|v|$, for a positive constant b , which directly opposes the motion of P .
- (a) By writing down expressions for the forces, show that the motion of P is described by the differential equation

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0. \quad [2]$$

- (b) State, in terms of m , b and k , the possible forms of the solution to the differential equation and the corresponding conditions satisfied by the positive constants m , b and k . [4]
- (c) Determine, with justification, the eventual motion of P for each of the possible forms in part (b). [3]

It is now known that $m = 0.1$, $b = 0.5$, $k = 1.5$, and $x_0 = 0.1$.

- (d) Find an expression for x in terms of t . Leave all coefficients in exact form. [4]
- (e) The motion is considered to have “stopped” after the last time at which the magnitude of the displacement is 0.005. Find, correct to 3 decimal places, the time it takes for the oscillations to stop. [1]

1	Solution
(a)	Rewriting the given recurrence relation, $u_r - 4u_{r-1} + u_{r-2} = 0$ The characteristic equation is given by $m^2 - 4m + 1 = 0$ $m = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$ Hence, the general solution of the recurrence relation is $u_r = A(2 + \sqrt{3})^r + B(2 - \sqrt{3})^r, \text{ where } r = 0, 1, 2, \dots$
(b)	From $u_0 = 0$, we have $A + B = 0$ From $u_1 = 2$, we have $A(2 + \sqrt{3}) + B(2 - \sqrt{3}) = 2$ Solving the two equations simultaneously, $A = \frac{1}{\sqrt{3}} \text{ and } B = -\frac{1}{\sqrt{3}}$ Therefore, $u_r = \frac{1}{\sqrt{3}}(2 + \sqrt{3})^r - \frac{1}{\sqrt{3}}(2 - \sqrt{3})^r$, where $r = 0, 1, 2, \dots$
(c)	From u_r tends to zero as r tends to infinity, we know $A = 0$. From $u_2 = 1 - \frac{4}{7}\sqrt{3}$, we have $B(2 - \sqrt{3})^2 = 1 - \frac{4}{7}\sqrt{3}$ $B = \frac{1 - \frac{4}{7}\sqrt{3}}{7 - 4\sqrt{3}} = \frac{1}{7}$ $u_0 = \frac{1}{7}$

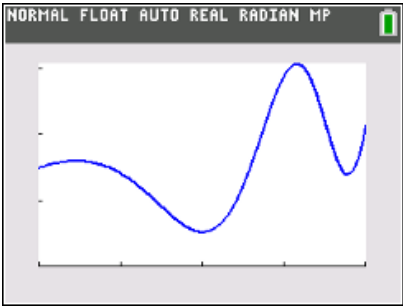
2	Solution																							
(a)	Using the Euler method,																							
	<table><tr><td>x</td><td>y</td><td>dy/dx</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>0.5</td><td>1</td><td>147.41316</td></tr><tr><td>1</td><td>74.70658</td><td></td></tr></table>	x	y	dy/dx	0	1	0	0.5	1	147.41316	1	74.70658												
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<table><tr><td>x</td><td>y</td><td>$dy/dx1$</td><td>u</td><td>$dy/dx2$</td><td>dy/dx</td></tr><tr><td>0</td><td>1</td><td>0</td><td>1</td><td>147.41316</td><td>73.70658</td></tr><tr><td>0.5</td><td>37.85329</td><td>1.4799214</td><td>38.5932505</td><td>288.60588</td><td>145.0429</td></tr><tr><td>1</td><td>110.37474</td><td></td><td></td><td></td><td></td></tr></table>	x	y	$dy/dx1$	u	$dy/dx2$	dy/dx	0	1	0	1	147.41316	73.70658	0.5	37.85329	1.4799214	38.5932505	288.60588	145.0429	1	110.37474				
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When $x = 1$, $y \approx 110.37$ (2 d.p.)																								
(b)	As seen from the values of the subsequent gradients, the value of the gradients increases exponentially when x increases, but the gradient is zero at the initial point. So the 1 st step of the Euler method proposes no change to the y -value, while the improved Euler method takes the large increase of the gradients into consideration by including the gradient from an intermediate estimate, leading to the large discrepancy.																							
	Additional Note: <table><tr><td>Step Size</td><td>Euler</td><td>Improved Euler</td><td>Difference</td><td>% Diff</td></tr><tr><td>0.5</td><td>74.71</td><td>110.37</td><td>35.66</td><td>32.31%</td></tr><tr><td>0.25</td><td>34.89</td><td>70.19</td><td>35.30</td><td>50.29%</td></tr><tr><td>0.2</td><td>36.21</td><td>60.23</td><td>24.02</td><td>39.88%</td></tr></table> From the table of comparison, reducing the step size does not have much of an effect on the absolute difference between the estimates and in fact increases the percentage difference (this is calculated with the values from improved Euler as the base since it is the more accurate method).	Step Size	Euler	Improved Euler	Difference	% Diff	0.5	74.71	110.37	35.66	32.31%	0.25	34.89	70.19	35.30	50.29%	0.2	36.21	60.23	24.02	39.88%			
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(a) (ii)	$A_1 = 20\sqrt{100}(1 - q100) < 0$ $q > \frac{1}{100}$																																																
(b) (i)	When n is even, $400 = p\sqrt{100}(1 - 100q)$ $40 = p(1 - 100q) \dots\dots\dots (1)$ When n is odd, $100 = p\sqrt{400}(1 - 400q)$ $5 = p(1 - 400q) \dots\dots\dots (2)$ (1) / (2), $8 = \frac{1 - 100q}{1 - 400q} \Rightarrow q = \frac{7}{3100}$ Sub in (1), $40 = p\left(1 - 100\left(\frac{7}{3100}\right)\right) \Rightarrow p = \frac{155}{3}$																																																
(b) (ii)	$A_{n+1} = \frac{155}{3}\sqrt{A_n}\left(1 - \frac{7}{3100}A_n\right)$ With $A_0 = 100$, the sequence cycles between 100 and 400. NORMAL FLOAT AUTO REAL RADIAN MP Plot1 Plot2 Plot3 TYPE: SEQ(n) SEQ(n+1) SEQ(n+2) nMin=0 $\frac{155}{3}\sqrt{u(n)}\left(1 - \frac{7}{3100}u(n)\right)$ u(0)=100 u(1)= v(n+1)= v(0)= NORMAL FLOAT AUTO REAL RADIAN MP PRESS + FOR ΔTb1 <table border="1"> <thead> <tr> <th>n</th> <th>u</th> </tr> </thead> <tbody> <tr><td>0</td><td>100</td></tr> <tr><td>1</td><td>400</td></tr> <tr><td>2</td><td>100</td></tr> <tr><td>3</td><td>400</td></tr> <tr><td>4</td><td>100</td></tr> <tr><td>5</td><td>400</td></tr> <tr><td>6</td><td>100</td></tr> <tr><td>7</td><td>400</td></tr> <tr><td>8</td><td>100</td></tr> <tr><td>9</td><td>400</td></tr> <tr><td>10</td><td>100</td></tr> </tbody> </table> n=10 When $A_0 = 99$, the sequence drifts away from the 100/400 cycle. NORMAL FLOAT AUTO REAL RADIAN MP SECOND CONDITION IF NEEDED Plot1 Plot2 Plot3 TYPE: SEQ(n) SEQ(n+1) SEQ(n+2) nMin=0 $\frac{155}{3}\sqrt{u(n)}\left(1 - \frac{7}{3100}u(n)\right)$ u(0)=99 u(1)= v(n+1)= v(0)= v(1)= NORMAL FLOAT AUTO REAL RADIAN MP PRESS + FOR ΔTb1 <table border="1"> <thead> <tr> <th>n</th> <th>u</th> </tr> </thead> <tbody> <tr><td>0</td><td>99</td></tr> <tr><td>1</td><td>399.16</td></tr> <tr><td>2</td><td>101.86</td></tr> <tr><td>3</td><td>401.51</td></tr> <tr><td>4</td><td>96.649</td></tr> <tr><td>5</td><td>397.08</td></tr> <tr><td>6</td><td>106.41</td></tr> <tr><td>7</td><td>404.91</td></tr> <tr><td>8</td><td>89.087</td></tr> <tr><td>9</td><td>389.56</td></tr> <tr><td>10</td><td>122.73</td></tr> </tbody> </table> n=10 When $A_0 = 101$, the sequence drifts away from the 100/400 cycle.	n	u	0	100	1	400	2	100	3	400	4	100	5	400	6	100	7	400	8	100	9	400	10	100	n	u	0	99	1	399.16	2	101.86	3	401.51	4	96.649	5	397.08	6	106.41	7	404.91	8	89.087	9	389.56	10	122.73
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Plot1	Plot2	Plot3	
TYPE: SEQ(n)	SEQ(n+1)	SEQ(n+2)	
nMin=0			
$u(n+1) = \frac{155}{3} \sqrt{u(n)} \left(1 - \frac{7}{3100} u(n) \right)$			
u(0)=101			
u(1)=			
v(n+1)=			
v(0)=			
v(1)=			

NORMAL FLOAT AUTO REAL RADIAN MP PRESS + FOR Δ Tb1			
n	u		
0	101		
1	400.82		
2	98.182		
3	398.45		
4	103.42		
5	402.72		
6	93.962		
7	394.56		
8	111.91		
9	408.45		
10	81.119		
n=10			

The proposed model is highly sensitive to initial conditions. Even a minor deviation from the expected cycle values of 100 (or 400) causes the sequence to diverge rapidly, indicating that it is mathematically unstable and therefore a poor predictive model for biological growth.

4	Solution
(a)	Using the trapezium rule, $I \approx \left(\frac{1}{2}\right) \left[0.5 + \sin(e^{0.5}) + 2(1 + \sin(e^1)) + 2(1.5 + \sin(e^{1.5})) + 2(2 + \sin(e^2)) + 2.5 + \sin(e^{2.5}) \right]$ $= 6.642353454 = 6.642 \text{ (3 d.p.)}$
(b)	Using the Simpson's rule, $I \approx \left(\frac{2}{6}\right) \left[0.5 + \sin(e^{0.5}) + 4(1 + \sin(e^1)) + 2(1.5 + \sin(e^{1.5})) + 4(2 + \sin(e^2)) + 2.5 + \sin(e^{2.5}) \right]$ $= 7.297993133 = 7.298 \text{ (3 d.p.)}$
(c)	The trapezium rule in (a) is more accurate.  It can be seen from the sketch of $y = 0.5x + \sin(e^{0.5x})$ that there are 3 turning points in the interval $[3, 5]$. A parabola fitted to the 3 points where $x = 3, 4, 5$ will have a maximum point near $x = 4$ and mostly lie above the graph over the interval, resulting in a gross overestimate of the area.

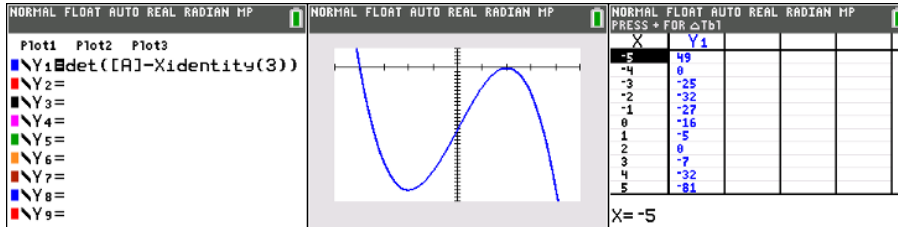
5	Solution
(a)	Using the shell method, $\begin{aligned}\text{Volume} &= \int_0^{\sqrt{3\pi}} 2\pi x \left[x + \sin(x^2) \right] dx \\ &= \pi \int_0^{\sqrt{3\pi}} 2x^2 + 2x \sin(x^2) dx \\ &= \pi \left[\frac{2x^3}{3} - \cos(x^2) \right]_0^{\sqrt{3\pi}} \\ &= \pi \left[\frac{2(\sqrt{3\pi})^3}{3} - \cos(3\pi) - 0 + \cos(0) \right] \\ &= 2\pi(\pi\sqrt{3\pi} + 1)\end{aligned}$
(b)	[Note that the function is not 1-to-1, so the surface area formula is modified.] $\begin{aligned}\text{Surface Area} &= \int_0^{\sqrt{3\pi}} 2\pi x \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\ &= \int_0^{\sqrt{3\pi}} 2\pi x \sqrt{1 + [1 + 2x \cos(x^2)]^2} dx \\ &= 87.13478314 = 87.135 \text{ (3 d.p.)}\end{aligned}$
(c)	$\begin{aligned}\text{New Surface Area} &= \int_0^{\sqrt{3\pi}} 2\pi y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\ &= \int_0^{\sqrt{3\pi}} 2\pi [x + \sin(x^2)] \sqrt{1 + [1 + 2x \cos(x^2)]^2} dx \\ &= 97.39467591 = 97.4 \text{ (3 s.f.)}\end{aligned}$

6 Solution**(a)**

Let $\mathbf{M} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ -4 & -4 & -6 \end{pmatrix}$ be the transformation matrix.

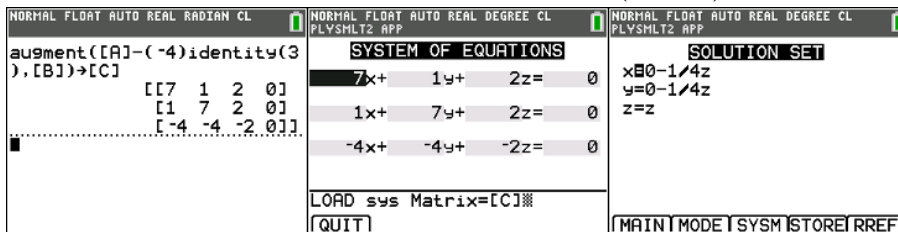
The invariant line or plane corresponds to the eigenvectors of $\mathbf{M}$.

Consider $\det(\mathbf{M} - \lambda \mathbf{I}) = 0$, we have $\det \begin{pmatrix} 3-\lambda & 1 & 2 \\ 1 & 3-\lambda & 2 \\ -4 & -4 & -6-\lambda \end{pmatrix} = 0$



From GC, the eigenvalues are $\lambda = -4$ or $\lambda = 2$

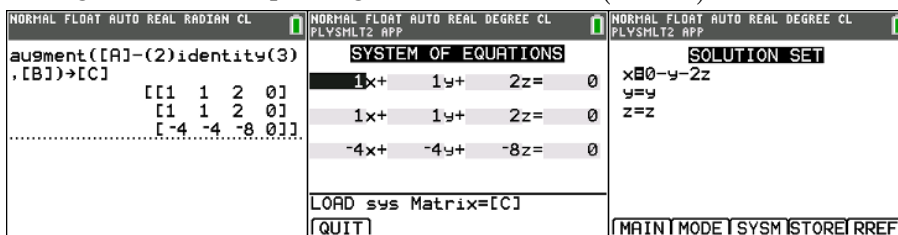
For eigenvector corresponding to $\lambda = -4$, consider $(\mathbf{M} - 4\mathbf{I})\mathbf{v} = \mathbf{0}$,



From GC, the eigenvector is $z \begin{pmatrix} 1 \\ -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$

So invariant line has vector equation $\mathbf{r} = \lambda \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$

For eigenvector corresponding to $\lambda = 2$, consider $(\mathbf{M} - 2\mathbf{I})\mathbf{v} = \mathbf{0}$,



From GC, $x + y + 2z = 0$,

So the invariant plane has vector equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$.

OR

From GC, the eigenvectors are $y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

So invariant plane has vector equation $\mathbf{r} = \mu \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \mu, \gamma \in \mathbb{R}$

(b)
(i) The system of equations is equivalent to $(\mathbf{M} - t\mathbf{I}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

It has no unique solution when $\det(\mathbf{M} - t\mathbf{I}) = 0$. So $t = -4$ or $t = 2$.

(b)
(ii) Since the eigenvectors are linearly independent,

$\mathbf{x} = \lambda \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \lambda, \mu, \gamma \in \mathbb{R}$ represents all possible vectors in $\mathbb{R}^3$.

When $t = -4$,

$$[\mathbf{M} + 4\mathbf{I}]\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow (-4+4)\lambda \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} + (2+4)\mu \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + (2+4)\gamma \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow 6\mu \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 6\gamma \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

For consistency, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ lies on the invariant plane $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$,

and so $a + b + 2c = 0$.

When $t = 2$,

$$[\mathbf{M} - 2\mathbf{I}]\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow (-4-2)\lambda \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} + (2-2)\mu \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + (2-2)\gamma \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow (-6)\lambda \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

For consistency, $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ lies on the invariant line $\mathbf{r} = \lambda \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$,

and so $-a = -b = \frac{c}{4}$ or $4a = 4b = -c$.

OR

For $t = -4$, we have

$$\left(\begin{array}{ccc|c} 7 & 1 & 2 & a \\ 1 & 7 & 2 & b \\ -4 & -4 & -2 & c \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 7 & 2 & b \\ 7 & 1 & 2 & a \\ -4 & -4 & -2 & c \end{array} \right)$$

$$\xrightarrow{\begin{matrix} R_2 \leftarrow R_2 - 7R_1 \\ R_3 \leftarrow R_3 + 4R_1 \end{matrix}} \left(\begin{array}{ccc|c} 1 & 7 & 2 & b \\ 0 & -48 & -12 & a-7b \\ 0 & 24 & 6 & c+4b \end{array} \right) \xrightarrow{\begin{matrix} R_2 \leftarrow -\frac{1}{12}R_2 \\ R_3 \leftarrow \frac{1}{6}R_3 \end{matrix}} \left(\begin{array}{ccc|c} 1 & 7 & 2 & b \\ 0 & 4 & 1 & (a-7b)/(-12) \\ 0 & 4 & 1 & (c+4b)/6 \end{array} \right)$$

For equations to be consistent, we need

$$\frac{a-7b}{-12} = \frac{c+4b}{6}$$

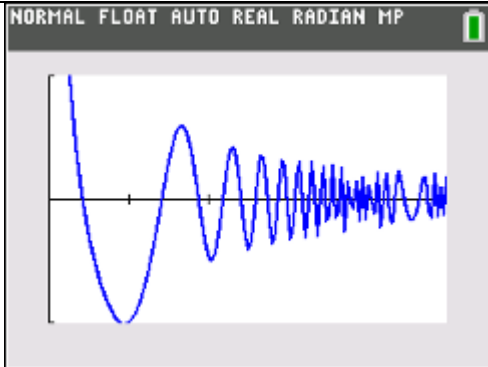
$$a+b+2c=0$$

For $t=2$, we have

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 1 & 2 & b \\ -4 & -4 & -8 & c \end{array} \right) \xrightarrow{R_3 \leftarrow \frac{1}{-4}R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 1 & 2 & b \\ 1 & 1 & 2 & c/(-4) \end{array} \right)$$

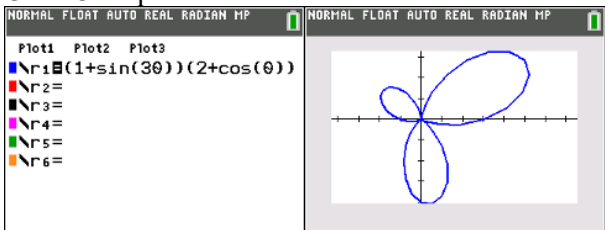
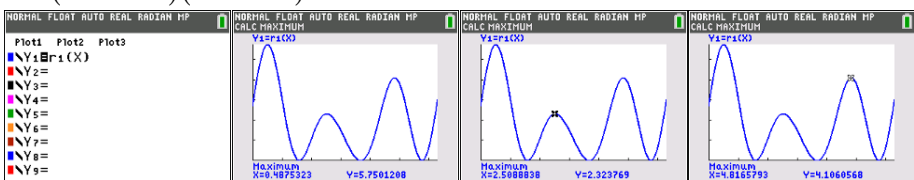
For equations to be consistent, we need

$$a=b=\frac{c}{-4}$$

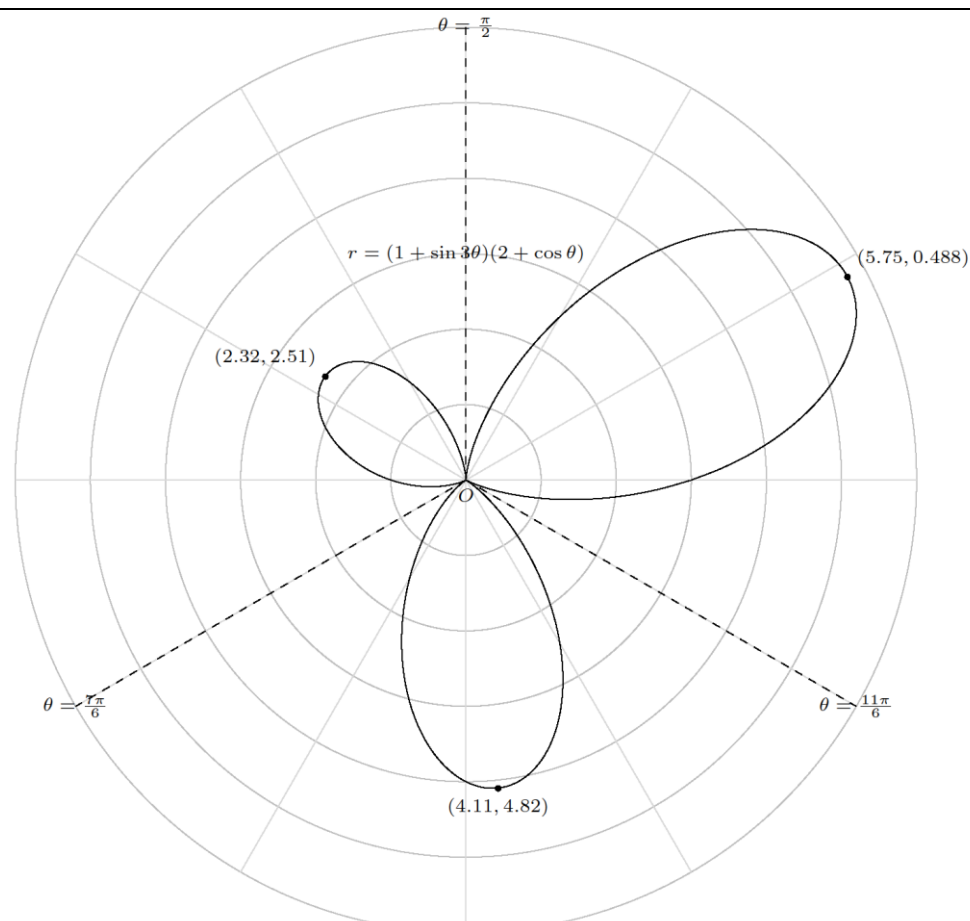
7	Solution						
(a)	$f(1) = -0.99057, \quad f(2) = -0.46058.$ Since the trigo function in the numerator is bounded by the interval $[-1, 1]$, the turning points lie near the graphs of $y = \pm \frac{1}{x}$ and hence there can only be an even number of roots in the interval $(1, 2)$.						
(b)	There must be a (local) maximum point. [Note: the maximum points lie near $y = \frac{1}{x}$ which will be above the x -axis]						
(c)	The Newton-Raphson equation is						
(i)	$x_{n+1} = x_n - \left(\frac{\cos(e^{1.1x_n})}{x_n} \right) \div \left(\frac{-1.1x_n e^{1.1x_n} \sin(e^{1.1x_n}) - \cos(e^{1.1x_n})}{x_n^2} \right)$ $= x_n + \frac{x_n \cos(e^{1.1x_n})}{1.1x_n e^{1.1x_n} \sin(e^{1.1x_n}) + \cos(e^{1.1x_n})}$ From GC, <table><tr><td>x_1</td><td>x_2</td><td>x_3</td></tr><tr><td>2.842</td><td>2.881</td><td>2.872</td></tr></table>	x_1	x_2	x_3	2.842	2.881	2.872
x_1	x_2	x_3					
2.842	2.881	2.872					
(c)	Rewriting the Newton-Raphson equation,						
(ii)	$x_{n+1} - x_n = \frac{x_n \cos(e^{1.1x_n})}{1.1x_n e^{1.1x_n} \sin(e^{1.1x_n}) + \cos(e^{1.1x_n})} = \frac{1}{1.1e^{1.1x_n} \tan(e^{1.1x_n}) + (x_n)^{-1}}$ At $n = 3$, $x_4 - x_3 \approx \frac{1}{1.1e^{1.1(2.8722681)} \tan(e^{1.1(2.8722681)}) + (2.8722681)^{-1}}$ $\approx \frac{1}{7970.993} \approx 1.2545 \times 10^{-4}$ This value is too small to affect the value of the iterations at 3 decimal places.						
(c)	The initial approximation of $x_0 = 1$ is very close to the turning point at $x = 0.93921$, so the tangent to the curve will have a very gentle slope, leading to an intercept that is very far away and hence the iterations converges to another root instead of α .						
(c)	From GC, $\alpha \approx 1.409$						
(iv)							
(c)							
(v)	 The graph reveals that as x gets larger, the turning points and the roots of f are packed closely together so it is difficult to choose an initial estimate that will let further estimates converge to the intended root.						

8	Solution
(a) (i)	Solving $(\mathbf{A} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$, we have $\begin{pmatrix} -1 & -1 & -2 \\ -1 & -1 & -2 \\ -1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ From GC, $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} z$ So a basis of eigenvectors corresponding to $\lambda = 3$ is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$. Solving $(\mathbf{B} - 3\mathbf{I})\mathbf{v} = \mathbf{0}$, we have $\begin{pmatrix} 5 & -4 & -5 \\ 4 & -4 & -4 \\ 5 & -4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ From GC, $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} z$ So a basis of eigenvectors corresponding to $\lambda = 3$ is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.
(a) (ii)	$\mathbf{A}$ is diagonalisable since there are 3 linearly independent eigenvectors, and $\mathbf{B}$ is not diagonalisable since there are only 2 linearly independent eigenvectors.
(b) (i)	Consider $\mathbf{M} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$ where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are the columns. From GC, $\det(\mathbf{M}) = 32 \neq 0$. So the column vectors are linearly independent in $\mathbb{R}^4 = V$. Hence S spans V. (shown)
	<u>Alternatively:</u> From GC, $\mathbf{M} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$ reduces to $\begin{pmatrix} 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ So the row vectors are linearly independent in $\mathbb{R}^4 = V$. Hence S spans V. (shown)
(b) (ii)	Augmenting $\mathbf{M}$ with $\mathbf{u}_1$, we have $\begin{pmatrix} -1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & -1 & 0 \\ 1 & 2 & -1 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 \end{pmatrix}$. From GC, we have $\mathbf{u}_1 = -\frac{1}{8}\mathbf{a} + \frac{1}{8}\mathbf{b} + \frac{1}{8}\mathbf{c} + \frac{3}{8}\mathbf{d}$
(b) (iii)	By observing that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{u}_1$, $\mathbf{u}_4$, $\mathbf{u}_3$, $\mathbf{u}_2$ have the same cyclic nature, we can deduce that

	$\mathbf{u}_4 = -\frac{1}{8}\mathbf{b} + \frac{1}{8}\mathbf{c} + \frac{1}{8}\mathbf{d} + \frac{3}{8}\mathbf{a} = \frac{3}{8}\mathbf{a} - \frac{1}{8}\mathbf{b} + \frac{1}{8}\mathbf{c} + \frac{1}{8}\mathbf{d}$ $\mathbf{u}_3 = -\frac{1}{8}\mathbf{c} + \frac{1}{8}\mathbf{d} + \frac{1}{8}\mathbf{a} + \frac{3}{8}\mathbf{b} = \frac{1}{8}\mathbf{a} + \frac{3}{8}\mathbf{b} - \frac{1}{8}\mathbf{c} + \frac{1}{8}\mathbf{d}$ $\mathbf{u}_2 = -\frac{1}{8}\mathbf{d} + \frac{1}{8}\mathbf{a} + \frac{1}{8}\mathbf{b} + \frac{3}{8}\mathbf{c} = \frac{1}{8}\mathbf{a} + \frac{1}{8}\mathbf{b} + \frac{3}{8}\mathbf{c} - \frac{1}{8}\mathbf{d}$
	Note: M is an example of a circulant matrix .

9	Solution
(a)	For tangents at the pole, $r = 0$ So $(1 + \sin 3\theta)(2 + \cos \theta) = 0$ Since $2 + \cos \theta > 0$, $1 + \sin 3\theta = 0$ $3\theta = \frac{3}{2}\pi, \frac{7}{2}\pi, \frac{11}{2}\pi.$ $\theta = \frac{1}{2}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi.$ So equations of tangents at the pole are $\theta = \frac{\pi}{2}$, $\theta = \frac{7\pi}{6}$ and $\theta = \frac{11\pi}{6}$.
(b)	C has 3 loops. 
(c)	$r = (1 + \sin 3\theta)(2 + \cos \theta)$  From GC, the distances are 5.75 at $\theta = 0.488$, 2.32 at $\theta = 2.51$ and 4.11 at $\theta = 4.82$.

(d)



NORMAL FLOAT AUTO REAL RADIAN MP					NORMAL FLOAT AUTO REAL RADIAN MP				
PRESS + FOR Δ Tb1					PRESS + FOR Δ Tb1				
θ	r				θ	r			
0	3				1.0472	2.5			
0.5236	5.7321				1.5708	0			
1.0472	2.5				2.0944	1.5			
1.5708	0				2.618	2.2679			
2.0944	1.5				3.1416	1			
2.618	2.2679				3.6652	0			
3.1416	1				4.1888	1.5			
3.6652	0				4.7124	4			
4.1888	1.5				5.236	2.5			
4.7124	4				5.7596	0			
5.236	2.5				6.2832	3			
$\theta=5.235987755983$					$\theta=6.28318530718$				

(e)

$$\text{Area enclosed} = \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{7\pi}{6}} r^2 d\theta = 2.19$$

NORMAL FLOAT AUTO REAL RADIAN MP				
$\frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{7\pi}{6}} (r_1^2) d\theta$				
				2.190259634

(f)

$$\text{Arc length} = \int_0^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta + \int_{\frac{11\pi}{6}}^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = 13.8$$

NORMAL FLOAT AUTO REAL RADIAN MP					NORMAL FLOAT AUTO REAL RADIAN MP				
$\int_0^{\pi/2} \left(\sqrt{r_1^2 + \left(\frac{d}{d\theta}(r_1)\right)^2} \right) d\theta$					$\int_{11\pi/6}^{2\pi} \left(\sqrt{r_1^2 + \left(\frac{d}{d\theta}(r_1)\right)^2} \right) d\theta$				
				10.71783441					3.062752544
$\int_{11\pi/6}^{2\pi} \left(\sqrt{r_1^2 + \left(\frac{d}{d\theta}(r_1)\right)^2} \right) d\theta$					$\int_0^{\pi/2} \left(\sqrt{r_1^2 + \left(\frac{d}{d\theta}(r_1)\right)^2} \right) d\theta + \int_{11\pi/6}^{2\pi}$				
				3.062752544					13.78058696

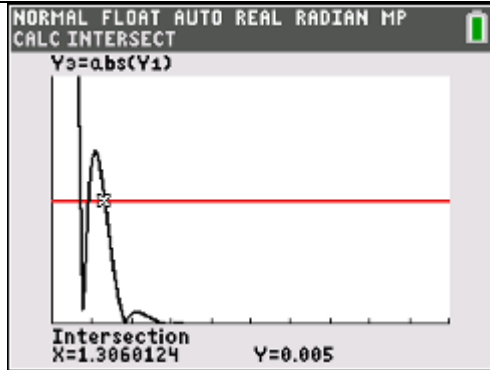
10	Solution
(a)	Since S is in the opposite direction of the displacement x, $S = -kx$, and D is in the opposite direction of the velocity v, $D = -bv$. $ma = S + D = -bv - kx$ $\Rightarrow ma + bv + kx = 0$ $\Rightarrow m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$
(b)	The characteristic equation is $m\lambda^2 + b\lambda + k = 0$ with roots $\lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$. For distinct real roots, $b^2 - 4mk > 0$ and $x = Ae^{\left(\frac{-b + \sqrt{b^2 - 4mk}}{2m}\right)t} + Be^{\left(\frac{-b - \sqrt{b^2 - 4mk}}{2m}\right)t}$. For repeated real roots, $b^2 - 4mk = 0$ and $x = (A + Bt)e^{\left(\frac{-b}{2m}\right)t}$. For complex conjugate roots $\lambda = \alpha \pm i\beta = \frac{-b}{2m} \pm i \frac{\sqrt{4mk - b^2}}{2m}$, $b^2 - 4mk < 0$ and $x = e^{\left(\frac{-b}{2m}\right)t} \left[A \sin\left(\frac{\sqrt{4mk - b^2}}{2m}t\right) + B \cos\left(\frac{\sqrt{4mk - b^2}}{2m}t\right) \right]$.
(c)	For $b^2 - 4mk > 0$, note that $\sqrt{b^2 - 4mk} < \sqrt{b^2} = b \Rightarrow -b + \sqrt{b^2 - 4mk} < 0$ So the coefficients of t in $x = Ae^{\left(\frac{-b + \sqrt{b^2 - 4mk}}{2m}\right)t} + Be^{\left(\frac{-b - \sqrt{b^2 - 4mk}}{2m}\right)t}$ are both negative and the displacement decreases slowly to zero (since $\frac{-b}{2m} < \frac{-b + \sqrt{b^2 - 4mk}}{2m} < 0$) without oscillating. For $b^2 - 4mk = 0$, the coefficient of t in the exponential expression is negative in $x = (A + Bt)e^{\left(\frac{-b}{2m}\right)t}$ and the displacement decreases quickly to zero without oscillating. For $b^2 - 4mk < 0$, the coefficient of t in the exponential expression is negative in $x = e^{\left(\frac{-b}{2m}\right)t} \left[A \sin\left(\frac{\sqrt{4mk - b^2}}{2m}t\right) + B \cos\left(\frac{\sqrt{4mk - b^2}}{2m}t\right) \right]$ and the displacement oscillates with a decreasing amplitude before coming to rest.
(d)	Since $b^2 - 4mk = 0.5^2 - 4(0.1)(1.5) = -0.35 < 0$, $x = e^{-\frac{5}{2}t} \left[A \sin\left(\frac{\sqrt{35}}{2}t\right) + B \cos\left(\frac{\sqrt{35}}{2}t\right) \right] \text{ and}$ $\frac{dx}{dt} = e^{-\frac{5}{2}t} \left[A \left(\frac{\sqrt{35}}{2}\right) \cos\left(\frac{\sqrt{35}}{2}t\right) - B \left(\frac{\sqrt{35}}{2}\right) \sin\left(\frac{\sqrt{35}}{2}t\right) \right]$ $+ \left(-\frac{5}{2}\right) e^{-\frac{5}{2}t} \left[A \sin\left(\frac{\sqrt{35}}{2}t\right) + B \cos\left(\frac{\sqrt{35}}{2}t\right) \right]$ $= e^{-\frac{5}{2}t} \left[\left(\frac{\sqrt{35}}{2}A - \frac{5}{2}B\right) \cos\left(\frac{\sqrt{35}}{2}t\right) - \left(\frac{\sqrt{35}}{2}B + \frac{5}{2}A\right) \sin\left(\frac{\sqrt{35}}{2}t\right) \right]$

Given at $t=0$, $x=0.1$ and $\frac{dx}{dt}=0$,

$$\frac{1}{10} = B \text{ and } 0 = \frac{\sqrt{35}}{2} A - \frac{5}{2} \left(\frac{1}{10} \right) \Rightarrow A = \frac{1}{2\sqrt{35}}$$

$$\text{Hence } x = e^{-2.5t} \left[\frac{1}{2\sqrt{35}} \sin \left(\frac{\sqrt{35}}{2} t \right) + \frac{1}{10} \cos \left(\frac{\sqrt{35}}{2} t \right) \right]$$

(e)



From the GC, time taken = 1.306 seconds (3 d.p.)