Paper D Sec 3 Higher Paper Solutions for students

Qn	Solutions
1(ai)	x^4+4y^4 $=x^4+4x^2y^2+4y^4-4x^2y^2$ $=\left(x^2+2y^2\right)^2-4x^2y^2$ $=\left(x^2+2y^2\right)^2-\left(2xy\right)^2$ $=\left(x^2-2xy+2y^2\right)\left(x^2+2xy+2y^2\right)$
1(aii)	5^8+2^6 $=\left(5^2\right)^4+4\left(2\right)^4$ $=\left(25^2-2\times25\times2+2\times2^2\right)\left(25^2+2\times25\times2+2\times2^2\right)$ $=533\times733$ $=13\times41\times733$ Hence, the distinct prime factors are 13, 41 and 733. **2 Marks** if students leave the answer as 533 and 733.
1(bi)	$-1\le\cos x\le1$
1(bii)	$1-\tan^6 x=\sec^6 x$ $1=\sec^6 x+\tan^6 x$ From (i), $0<\cos^6 x\le1 \Rightarrow \sec^6 x=\dfrac{1}{\cos^6 x}\ge1$ $\because \tan^6 x\ge0\Rightarrow\sec^6 x+\tan^6 x\ge1$ Alternatively, $\Rightarrow$ It can only be possible that $\sec^6 x=1$ and $\tan^6 x=0$. i.e. $\cos^6 x=1$ and $\tan^6 x=0$ $\cos x=\pm1$ and $\tan x=0$ $x=-360°,-180°,0°,180°,360°$ $1-\tan^6 x=\sec^6 x$ Let $y=\tan^2 x$. $1-y^3=\left(1+y\right)^3$ (apply $\tan^2 x+1=\sec^2 x$) $\Rightarrow 1-y^3=y^3+3y^2+3y+1$ $2y^3+3y^2+3y=0$ $3y\left(2y^2+3y+3\right)=0$ $y=0$ or $2y^2+3y+3=0$ (no solution) $\tan^2 x=0$ $x=-360°,-180°,0°,180°,360°$

2(a)	$\sqrt{24}\left(\sqrt{8+4\sqrt{3}}+\sqrt{8-4\sqrt{3}}\right)$ $=2\sqrt{6}\left(\sqrt{8+2\sqrt{12}}+\sqrt{8-2\sqrt{12}}\right)$ $=2\sqrt{6}\left(\sqrt{\left(\sqrt{6}+\sqrt{2}\right)^2}+\sqrt{\left(\sqrt{6}-\sqrt{2}\right)^2}\right)$ $=2\sqrt{6}\left(\sqrt{6}+\sqrt{2}+\sqrt{6}-\sqrt{2}\right)$ $=24$ Alternatively, students may try squaring the expression first, $24\left(\sqrt{8+4\sqrt{3}}+\sqrt{8-4\sqrt{3}}\right)^2$ $=24\left(8+4\sqrt{3}+2\sqrt{8+4\sqrt{3}}\sqrt{8-4\sqrt{3}}+8-4\sqrt{3}\right)$ $=24\left(16+2\sqrt{64-16(3)}\right)$ $=24(16+8)$ $=24^2$ $\Rightarrow\sqrt{24}\left(\sqrt{8+4\sqrt{3}}+\sqrt{8-4\sqrt{3}}\right)=24$
2(bi)	Transformation I: scaling of the graph along the *x*-axis by factor of 2 (stretching). Transformation II: scaling of the graph along the *y*-axis by factor of $\frac{1}{3}$ (compression). Transformation III: translation of the graph to the negative direction of *x*-axis by 1 unit. <u>Alternatively,</u> Transformation I: translation of the graph to the negative direction of *x*-axis by $\frac{1}{2}$ unit. Transformation II: scaling of the graph along the *y*-axis by factor of $\frac{1}{3}$ (compression). Transformation III: scaling/stretching of the graph along the *x*-axis by factor of 2. <u>Alternatively,</u> Transformation I: translation of the graph to the negative direction of *x*-axis by 1 unit. Transformation II: translation of the graph to the positive direction of *y*-axis by 3 unit. Transformation III: scaling of the graph along the *y*-axis by factor of $\frac{1}{12}$ (compression).
2(bii)	The translation of the graph does not change the area enclosed by the graph and *x*-axis (1 mark) but the scaling does. The area will increase and decrease

	proportionately with stretching and compression respectively.(1 mark) Hence, the area enclosed = $\frac{4}{3}\times 2\times\frac{1}{3}=\frac{8}{9}$ units^2
3(a)	$y^2-4x-2y+5=0\Rightarrow(y-1)^2=4(x-1)$ The parabola has vertex (1, 1). Since the *x*-coordinate of the point is always equal to its distance from the fixed point (focus point) *Q*, we can infer that the directrix is $x=0$. Hence, the focus point *Q* has coordinates (2, 1).
3(bi)	$\frac{2x}{(x+1)(x-2)}\ge -1 \qquad (x\ne -1 \text{ or } 2)$ $\frac{2x}{(x+1)(x-2)}+1\ge 0$ $\frac{2x+(x+1)(x-2)}{(x+1)(x-2)}\ge 0$ $\frac{x^2+x-2}{(x+1)(x-2)}\ge 0$ $\frac{(x-1)(x+2)}{(x+1)(x-2)}\ge 0$ Multiply both sides with $(x+1)^2(x-2)^2$, $(x-1)(x+2)(x+1)(x-2)\ge 0$ $\Rightarrow x\le -2$ or $-1<x\le 1$ or $x>2$
3(bii)	$\frac{2\mathrm{e}^{-x}}{(\mathrm{e}^{-x}+1)(1-2\mathrm{e}^{-x})}\ge -1$ Multiply both numerator and denominator by e^{2x}, $\frac{2\mathrm{e}^{x}}{(1+e^{x})(\mathrm{e}^{x}-2)}\ge -1$ Let $y=\mathrm{e}^x$, we have $\frac{2y}{(1+y)(y-2)}\ge -1$ From (i), $y\le -2$ or $-1<y\le 1$ or $y>2$ i.e. $\mathrm{e}^x\le -2$ or $-1<\mathrm{e}^x\le 1$ or $\mathrm{e}^x>2$ Since e^x is always positive, $\Rightarrow 0<\mathrm{e}^x\le 1$ or $\mathrm{e}^x>2$ $\therefore x\le 0$ or $x>\ln 2$

4(a)	$\sqrt{\frac{2x+4}{x-2}}-3\sqrt{\frac{x-2}{2x+4}}=-2$ Let $y=\sqrt{\frac{2x+4}{x-2}}$. $y-\frac{3}{y}=-2$ $y^2+2y-3=0$ $(y-1)(y+3)=0$ $y=1$ or $y=-3$ (N.A. since $\sqrt{\frac{2x+4}{x-2}}\geq 0$) $\sqrt{\frac{2x+4}{x-2}}=1$ $\frac{2x+4}{x-2}=1$ $2x+4=x-2$ $x=-6$ Note: Minus 1 mark if student did not reject $\frac{22}{7}$ while using other methods such as squaring the equation.
4(b)	$\frac{\sin x+\cos x}{\tan x}+\frac{\sin x+\cos x}{\cot x}$ $=\frac{\cos x(\sin x+\cos x)}{\sin x}+\frac{\sin x(\sin x+\cos x)}{\cos x}$ $=\frac{\cos^2 x(\cos x+\sin x)+\sin^2 x(\sin x+\cos x)}{\sin x\cos x}$ $=\frac{(\sin x+\cos x)(\sin^2 x+\cos^2 x)}{\sin x\cos x}$ $=\frac{\sin x+\cos x}{\sin x\cos x}$ $=\frac{\sin x}{\sin x\cos x}+\frac{\cos x}{\sin x\cos x}$ $=\frac{1}{\sin x}+\frac{1}{\cos x}$ $=$ RHS Alternatively,

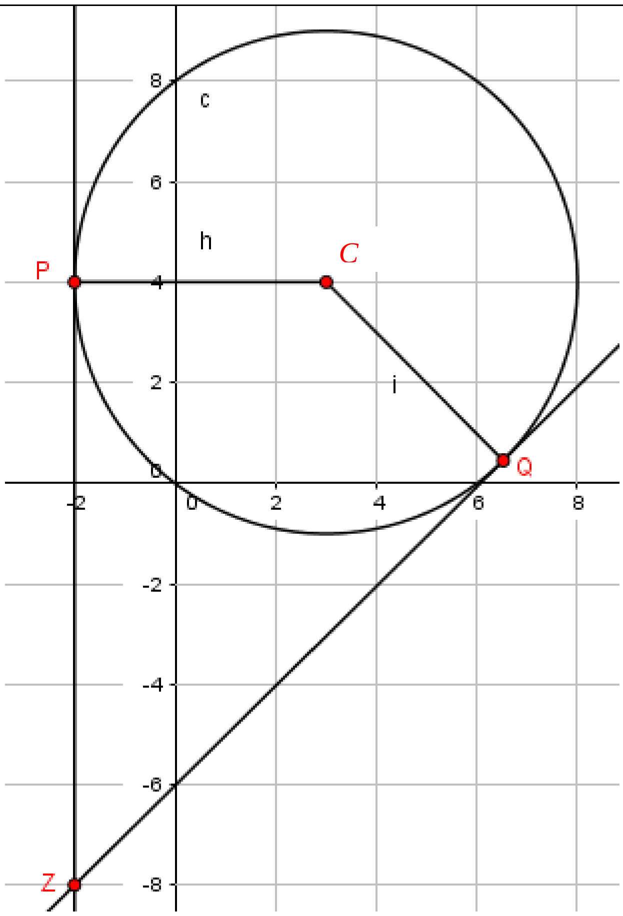
	$\dfrac{\sin x+\cos x}{\tan x}+\dfrac{\sin x+\cos x}{\cot x}$ $=(\sin x+\cos x)\cot x+(\sin x+\cos x)\tan x$ $=\cos x+\dfrac{\cos^2 x}{\sin x}+\dfrac{\sin^2 x}{\cos x}+\sin x$ $=\left(\cos x+\dfrac{\sin^2 x}{\cos x}\right)+\left(\dfrac{\cos^2 x}{\sin x}+\sin x\right)$ $=\dfrac{\cos^2 x+\sin^2 x}{\sin x}+\dfrac{\sin^2 x+\cos^2 x}{\cos x}$ $=\dfrac{1}{\sin x}+\dfrac{1}{\cos x}$ $=\text{RHS}$ Accept any other equivalent methods.
5(i)	Let $f(x)=P(x)-x^2-2x+6$. Since $x^2-x-2=(x-2)(x+1)$ and $x^2+x-2=(x-1)(x+2)$, from given conditions, by remainder theorem, $P(2)=3(2)-4=2$ $P(-1)=3(-1)-4=-7$ $P(1)=(1)-4=-3$ $P(-2)=(-2)-4=-6$ $\Rightarrow f(2)=P(2)-(2)^2-2(2)+6=0$ $f(-1)=P(-1)-(-1)^2-2(-1)+6=0$ $f(1)=P(1)-(1)^2-2(1)+6=0$ $P(-2)=P(-2)-(-2)^2-2(-2)+6=0$ By factor theorem, $(x-2),(x+1),(x-1)$ and $(x+2)$ are all factors of $f(x)$. Hence $P(x)-x^2-2x+6$ has both factors x^2-x-2 and x^2+x-2. <u>Alternatively,</u>

	Let $P(x)=(x^2-x-2)Q_1(x)+3x-4$. $\Rightarrow P(x)-x^2-2x+6$ $=(x^2-x-2)Q_1(x)+3x-4-x^2-2x+6$ $=(x^2-x-2)Q_1(x)-x^2+x+2$ $=(x^2-x-2)[Q_1(x)-1]$ $\therefore P(x)-x^2-2x+6$ has factor x^2-x-2. Similarly, let $P(x)=(x^2+x-2)Q_2(x)+x-4$. $\Rightarrow P(x)-x^2-2x+6$ $=(x^2+x-2)Q_2(x)+x-4-x^2-2x+6$ $=(x^2+x-2)Q_2(x)-x^2-x+2$ $=(x^2+x-2)[Q_2(x)-1]$ $\therefore P(x)-x^2-2x+6$ has factor x^2+x-2. $\Rightarrow P(x)-x^2-2x+6$ has both factors x^2-x-2 and x^2+x-2
5(ii)	From (i), since $P(x)$ is a degree 4 polynomial and each of x^2-x-2 and x^2-x-2 has degree 2, and the leading term of $P(x)-x^2-2x+6$ is x^4, $P(x)-x^2-2x+6=(x^2-x-2)(x^2+x-2)$ $P(x)=(x^2-x-2)(x^2+x-2)+x^2+2x-6$ $\quad=(x^2-2)^2-x^2+x^2+2x-6$ $\quad=x^4-4x^2+2x-2$ Note: Minus 1 mark if students never explain using the degree of polynomial/leading term.
6(a)	$45^a=75^b=15^c$ $\Rightarrow 45=15^{\frac{c}{a}}$ and $75=15^{\frac{c}{b}}$ $15^{\frac{c}{a}}\times 15^{\frac{c}{b}}=45\times 75=15^3$ $15^{\frac{c}{a}+\frac{c}{b}}=15^3$ $\therefore \frac{c}{a}+\frac{c}{b}=3$ Alternatively,

	Let $45^a = 75^b = 15^c = k > 0$ $\Rightarrow \ln 45^a = \ln 75^b = \ln 15^c$ i.e. $a\ln 45 = b\ln 75 = c\ln 15$ $\dfrac{c}{a} = \dfrac{\ln 45}{\ln 15},\ \dfrac{c}{b} = \dfrac{\ln 75}{\ln 15}$ $\dfrac{c}{a} + \dfrac{c}{b}$ $= \dfrac{\ln 45}{\ln 15} + \dfrac{\ln 75}{\ln 15}$ $= \dfrac{\ln(45\times 75)}{\ln 15}$ $= 3$
6(bi)	$\mathrm{fg}(x) = 2\left(\sqrt[3]{\dfrac{x+1}{2}}\right)^3 - 1 = x$ $\therefore$ f and g are inverse functions.
6(bii)	$\underbrace{\mathrm{fg}\ldots\mathrm{fgfg}}_{2017\ \text{'fg'}}(x) = \mathrm{g}^{-1}(x)$ $x = \mathrm{g}^{-1}(x)$ $x = \mathrm{f}(x)$ $x = 2x^3 - 1$ $2x^3 - x - 1 = 0$ When $x = 1,\ 2(1)^3 - 1 - 1 = 0$. By factor theorem, $(x-1)$ is a factor of $2x^3 - x - 1$. $(x-1)(2x^2 + 2x + 1) = 0$ $\Rightarrow x = 1$ or $2x^2 + 2x + 1 = 0$ $\Delta = 2^2 - 4(2)(1) < 0$ $\Rightarrow$ No solution $\therefore x = 1$
7(i)	Gradient of OX = $\dfrac{0-1}{0-(-1)} = -1$ Mid-point of OX: $\left(-\dfrac{1}{2},\ \dfrac{1}{2}\right)$ Equation of perpendicular bisector of OX: $y - \dfrac{1}{2} = \dfrac{-1}{-1}\left(x + \dfrac{1}{2}\right)$ $y = x + 1$ Gradient of OY = $\dfrac{0-7}{0-7} = 1$

	Mid-point of *OY*: $\left(\frac{7}{2}, \frac{7}{2}\right)$ Equation of perpendicular bisector of *OY*: $y-\frac{7}{2}=\frac{-1}{1}\left(x-\frac{7}{2}\right)$ $y=-x+7$ Since the centre of the circle is the intersection of the two perpendicular bisectors of the chords, we have $y=x+1$ $y=-x+7$ Solving the simultaneous linear equations, $x=3$ and $y=4$ $\Rightarrow$ Centre of the circle: $(3,4)$ Radius of the circle = $\sqrt{3^2+4^2}=5$ $\therefore$ equation of the circle: $(x-3)^2+(y-4)^2=25$ or $x^2+y^2-6x-8y=0$
7(ii)	Greatest distance occurs when (2, 6), centre of the circle and the point on the circumference lie on a straight line. Hence, Greatest distance = $5+\sqrt{(2-3)^2+(6-4)^2}=5+\sqrt{5}$ Note: Minus 1 mark if answer is left as $\sqrt{30+10\sqrt{5}}$.

7(iii)



From the sketch, one of the tangent lines ZP is parallel to y-axis. (only needed when student uses tangent later)

$$CZ = \sqrt{(-2-3)^2 + (-8-4)^2} = 13$$

$$\sin \angle CZP = \frac{5}{13} \quad \text{or } \tan \angle CZP = \frac{5}{4-(-8)} = \frac{5}{12}$$

$$\angle CZP = 22.62°$$

$$\Rightarrow \angle CZQ = 22.62° \times 2 = 45.2° \text{ (1 d.p.)}$$