X JUNIOR COLLEGE
YEAR 6 PRELIMINARY EXAMINATION
Mock Arrangement
in preparation for Candidates' Examination
Higher 2

# MATHEMATICS 9758/01

Paper 1 **Set I**

**3 hours**

Additional Materials: Answer Paper
Graph paper
List of Formulae and Results (MF27)

**READ THESE INSTRUCTIONS FIRST**

Write your name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

**ABOUT THIS PAPER**

*X Junior College* (XJC) is an unofficial initiative aimed at preparing pre-university and/or junior college students in Singapore for school-level and/or national-level H2 Mathematics examinations through self-prepared mock papers. It has no affiliation with any existing institution in Singapore or worldwide.

This mock paper follows closely the 9758 H2 Mathematics GCE Advanced Level syllabus, most suitable for preparation towards preliminary examinations and A–Levels. The paper intends to explore the unconventional ways and/or applications in which topics within the syllabus can be tested, which may affect the difficulty of this paper to varying degrees. **While it is ideal to attempt this paper under examination constraints, prospective candidates are reminded not to use this potentially non-conforming paper as a definitive gauge for actual performance.**

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**1** When the complex polynomial P($z$) is divided by $(z + \mathrm{i})$, $(z - \mathrm{i})$ and $(z^2 + 1)$, the remainders are $1 + \mathrm{i}$, $1 - \mathrm{i}$ and $Az + B$ respectively. Find $A$ and $B$. [3]

**2** An arithmetic sequence with first term $a$ and common difference $d$ is such that the sum of its first $n$ terms is $m$, and the sum its first $m$ terms is $n$.

**(a)** Find $a$ and $d$ in terms of $m$ and $n$. [4]

**(b)** Hence, show that the sum of the first $(m + n)$ terms of the sequence is $-(m + n)$. [2]

**3** A complex number $z = x + \mathrm{i}y$, where $x$ and $y$ are real, satisfies $|z + 3| = 2\mathrm{Re}(z)$.

**(a)** Show that $x$ and $y$ are related by the equation of a hyperbola. State the equations of its asymptotes. [3]

**(b)** Sketch the part of the hyperbola on which any point $(x, y)$ satisfies the above equation in $z$, including its asymptotes. On your sketch, indicate the values of the axial intercepts of these asymptotes. [2]

**(c)** Deduce the exact range of $\arg(z - 1)$. [2]

**4** A curve $C$ has equation $x^2 + 3xy - y^2 + 4x = 1$.

**(a)** Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ and $y$. [1]

The tangents to $C$ at two distinct points intersect at $(6, -4)$.

**(b)** Show that these points satisfy the equation $2x + 13y = 11$. [3]

**(c)** Hence, find the equation of these tangents, giving your answer in the form $ax + by + c = 0$, where $a$, $b$ and $c$ are integers to be determined. [3]

**5** It is given that $u = r\cos\theta + \mathrm{i}\, r\sin\theta$, for some real values $r$ and $\theta$ such that $r > 0$ and $0 < \theta < \frac{1}{2}\pi$.

**(a)** Express $\dfrac{u - r}{u + r}$ in the form $k\tan\dfrac{1}{2}\theta$, where $k$ is a complex number to be found. [3]

With respect to the origin $O$ on an Argand diagram, the points $Z$ and $W$ represent the complex numbers $(u - r)$ and $(u + r)$ respectively.

**(b)** Explain why triangle $OWZ$ is a right triangle, stating the right angle. [1]

**(c)** Deduce an expression for angle $OWZ$ in terms of $\theta$ and the distance between $Z$ and $W$ in terms of $r$. Hence, express $|u - r|$ and $|u + r|$ in terms of $r$ and $\theta$. [4]

**6** Relative to the origin $O$, the points $A$ and $B$ have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively such that $O$, $A$ and $B$ are not collinear. The point $C$ lies on the line segment $AB$ such that $AC:CB = (1-\lambda):\lambda$ and that $OC$ bisects angle $AOB$.

**(a)** Find $\lambda$ in terms of $|\mathbf{a}|$ and $|\mathbf{b}|$. [4]

**(b)** Another point $D$ lies on the line segment $AB$ such that $AC = BD$. By considering suitable dot products, show that $OD^2 - OC^2 = (|\mathbf{b}| - |\mathbf{a}|)^2$. [4]

**7** **(a)** The functions f and g are defined such that gf exists and

$$\mathrm{f}(x) = \frac{x+1}{2x-1}, \qquad x \in \mathbb{R}, x \neq \frac{1}{2},$$

$$\mathrm{gf}(x) = \frac{2x-7}{x-2}, \qquad x \in \mathbb{R}, x \neq 2.$$

**(i)** Explain why f has an inverse. Find its inverse in a similar form, stating its domain. [3]

**(ii)** Find a simplified expression for $\mathrm{g}(x)$. [2]

**(b)** A function h is given such that

$$\mathrm{h}(x) - 2\mathrm{h}\left(\frac{x-1}{x}\right) + \mathrm{h}\left(\frac{1}{1-x}\right) = \frac{x^3+1}{x^2-x}, \qquad x \in \mathbb{R}, x \neq 0, 1.$$

**(i)** Find the value of $\mathrm{h}(2)$, $\mathrm{h}(0.5)$ and $\mathrm{h}(-1)$. [3]

**(ii)** Deduce an expression for $\mathrm{hhh}(x)$. Hence, given that $\mathrm{hh}(x) = \frac{1}{1-x}$, find $\mathrm{h}(x)$. [2]

**8** The integral $I_n$, where $n = 0, 1, 2, 3, \ldots$, is defined by

$$I_n = \int_0^a x^n \sqrt{ax - x^2}\, \mathrm{d}x,$$

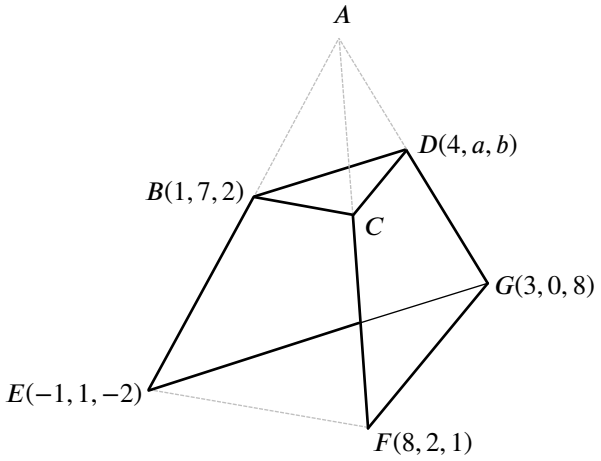
for some constant $a > 0$.

**(a)** By substituting $x = a\sin^2\theta$, find the exact value of $I_0$ in terms of $a$. [4]

**(b)** Show that $I_{n+1} = \left(\frac{2n+3}{2n+6}\right) aI_n$ for $n \geq 0$. [4]

**(c)** Find the exact area of the region enclosed by the curve $y = \sqrt{(2x-x^2)^5}$ and the $x$–axis. [4]

**9**




A cardboard with negligible thickness is cut and folded into the shape following the surface of a truncated tetrahedron *BCDEFG*, which is formed by removing the tetrahedron *ABCD* from the larger tetrahedron *AEFG*. It is known that the faces *BCFE* and *EFG* are hollow, and that the plane *BCD* is parallel to plane *EFG*. The coordinates of *B*, *D*, *E*, *F* and *G* are $(1, 7, 2)$, $(4, a, b)$, $(-1, 1, 2)$, $(8, 2, 1)$ and $(3, 0, 8)$ respectively (see diagram).

**(a)** Show that $a = 6.25$ and find $b$. [2]

**(b)** The outer surface area of the shape can be expressed as $p\left|\overrightarrow{DG} \times \overrightarrow{EG}\right| + q\left|\overrightarrow{DG} \times \overrightarrow{FG}\right| + r\left|\overrightarrow{EG} \times \overrightarrow{FG}\right|$. Find the exact value of $p$, $q$ and $r$ and hence evaluate this area numerically. [6]

[The area of a trapezium is $\frac{1}{2} \times$ height $\times$ sum of parallel sides.]

**(c)** The folded shape is detached along *DG* and flattened such that *BCD* remains attached to *BDGE* at *BD* and to *CDGF* at *CD*. Find the shortest distance between *C* and *E* on this flat surface. [5]

**10** A free-hanging rope is held by two people standing at a distance of $2a$ units from one another. Due to its weight, the rope curves downwards in the shape of a *catenary* defined by parametric equations

$$x = a \ln t, \qquad y = \frac{a}{2}\left(t + \frac{1}{t}\right), \qquad \text{for } t_1 \le t \le t_2,$$

where parameters $t_1$ and $t_2$ correspond to the two ends of the rope at which it is being held.

**(a)** Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $t$. [1]

**(b)** Given that the lowest point on the rope is horizontally in the middle of both people, find $t_1$ and $t_2$. [3]

**(c)** Find, in degrees, the acute angle that the rope ends make with the horizontal as it curves downwards. [2]

The rope is rotated about the line $y = k$ at which height it is being held. The rope remains taut throughout its motion and the shape of the curve is maintained.

**(d)** Show that $y = \dfrac{a}{2}\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right)$. Hence find $k$ exactly in terms of $a$. [2]

**(e)** Determine the exact volume of the space enclosed by the rotating rope, expressing your answer in the form $Aa^3\left(B + C\mathrm{e}^2 + D\mathrm{e}^{-2}\right)$, where $A$, $B$, $C$ and $D$ are exact constants to be determined. [5]

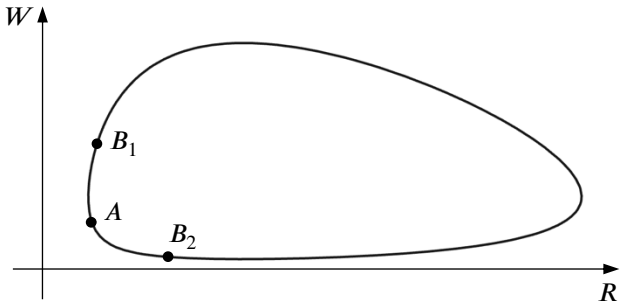
**11** In Ecology, the *Lotka-Volterra* equations are often used to model the population dynamics of a predator-prey ecosystem. Under this model, the population of prey and predators are each defined by a differential equation: the prey population grows at a rate proportional to its own population and declines at a rate proportional to the product of its own population and the predator population, while the predator population declines at a rate proportional to its own population and grows at a rate proportional to the product of its own population and the prey population.

**(a)** An ecosystem initially has 1 000 prey rabbits and 1 000 predator wolves. After $t$ years, the population of rabbits $R$ and wolves $W$, each measured in thousands, are such that the rabbit population grows and declines with a coefficient of 0.6 and 0.4 respectively, while the wolf population grows and declines with a coefficient of 0.2 and 0.8 respectively.

**(i)** Write down the two Lotka-Volterra equations relating $R$, $W$ and $t$. [2]

**(ii)** Hence, show that $2W + R - 3\ln W - 4\ln R = 3$. [5]

**(b)** The graph below shows the population dynamics of the ecosystem described in **(i)**. Point $A$ corresponds to the initial population, while points $B_1$ and $B_2$ correspond to the population in other years.




**(i)** Find, to the nearest hundred, the smallest and largest population size possible for each species. [4]

**(ii)** Which point, $B_1$ or $B_2$, corresponds to the population in the year after the initial? Explain. [2]

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