2026 J2 H2 FM

FM Statistics 1 - Discrete and Continuous Random Variable – Solutions

1* The piano shop opens at 9 am every day. The waiting time (in hours) between successive arrivals of customers at the shop has an exponential distribution with mean $\frac{1}{\lambda}$. Records show that on average, the shop sees no customers in the first 5 minutes on 20% of the days.

(a) Show that the mean number of customers in one hour is $12\ln 5$. [2]

(b) Given that the first customer has not arrived by 9.05 am, find the probability that the shop sees its first customer by 9.15 am. [2]

(c) Find the probability that the 15th customer arrives after 10.30 am. [2]

Records show that, on average, 1 in 10 of its customers bought a piano.
Let W be the random variable denoting the number of customers served **before** the customer who bought the second piano.

(d) Find $\mathrm{P}(W = w)$. [2]

(e) By considering the mean and variance of an appropriate geometric distribution, find $\mathrm{E}(W)$. [3]

1	**DHS_HCI_RI_TMJC Prelim 9649/2020/02/Q8**	
(a)	**Method 1: Use Exponential Distribution** Let X be the waiting time (in hours) between successive customers. $X \sim \mathrm{Exp}(\lambda)$ $\mathrm{P}\left(X > \frac{5}{60}\right) = 0.2$ $\mathrm{e}^{-\frac{\lambda}{12}} = 0.2$ $\lambda = 12\ln 5$ Mean number of customers in 1 hour $= 12\ln 5$ **Method 2: Use Poisson Distribution** Let be A be number of customers arriving in 5 min. $A \sim \mathrm{Po}\left(\frac{5\lambda}{60}\right)$ $\mathrm{P}(A = 0) = 0.2$ $\mathrm{e}^{-\frac{\lambda}{12}} = 0.2$ $\lambda = 12\ln 5$ Mean number of customers in 1 hour $= 12\ln 5$	**M1: Distribution + Probability Statement** **AG1: Apply formula to show answer** **M1: Distribution + Probability Statement** **AG1: Apply formula to show answer**
(b)	**Method 1: Use Exponential Distribution** $\mathrm{P}\left(X \le \frac{15}{60} \middle\vert X > \frac{5}{60}\right) = 1 - \mathrm{P}\left(X > \frac{1}{4} \middle\vert X > \frac{1}{12}\right)$ $= 1 - \mathrm{P}\left(X > \frac{1}{6}\right)$	**M1**

	$= \mathrm{P}\left(X \le \frac{1}{6}\right) = 1-\mathrm{e}^{-\frac{12\ln 5}{6}} = 0.96$	**A1**
	Method 2: Use Poisson Distribution $\mathrm{P}(\text{First customer arrives by 9.15 am} \mid A=0)$ $= \mathrm{P}(B \ge 1)$ where B is the number of customers who arrive at the shop in 10 mins $B \sim \mathrm{Po}(2\ln 5)$	**M1**
	$= 1-\mathrm{e}^{-\frac{12\ln 5}{6}} = 0.96$	**A1**
	Method 3: Use Conditional Probability $\mathrm{P}\left(X \le \frac{15}{60} \,\middle\vert\, X > \frac{5}{60}\right) = \dfrac{\mathrm{P}\left(\frac{1}{12} < X \le \frac{1}{4}\right)}{\mathrm{P}\left(X > \frac{1}{12}\right)}$ $= \dfrac{\mathrm{P}\left(X \le \frac{1}{4}\right) - \mathrm{P}\left(X \le \frac{1}{12}\right)}{\mathrm{P}\left(X > \frac{1}{12}\right)}$ $= \dfrac{1-\mathrm{e}^{-\lambda/4} - \left(1-\mathrm{e}^{-\lambda/12}\right)}{\mathrm{e}^{-\lambda/12}}$ $= 0.96$	
(iii)	Let C be the number of customers arriving in 1.5 hours. $C \sim \mathrm{Po}(18\ln 5)$	**B1**
	$\mathrm{P}(C \le 14) = 0.00162$	**B1**
(b)	Let W be the number of customers served before the customer who bought the second piano. For $w = 1, 2, 3, \ldots$ $\mathrm{P}(W=w) = \underbrace{\binom{w}{1}(0.1)(0.9)^{w-1}}(0.1) = w(0.9)^{w-1}(0.1)^2$ Exactly 1 out of w customers bought a piano. Need to ensure that the $(w+1)$th customer bought a piano.	**M1 A1**
	Method 2: Let Y be the number of customers served up to and including the one who bought a piano. $Y \sim \mathrm{Geo}(0.1)$	

	$\mathrm{P}(W=w)=\mathrm{P}(Y_1+Y_2-1=w)$ $=\mathrm{P}(Y_1+Y_2=w+1)$ $=\sum_{r=1}^{w}\mathrm{P}(Y_1=r)\mathrm{P}(Y_2=w+1-r)$ since Y_1,Y_2 are independent $=\sum_{r=1}^{w}(0.9)^{r-1}(0.1)(0.9)^{w-r}(0.1)$ $=\sum_{r=1}^{w}\underbrace{(0.9)^{w-1}(0.1)^2}_{\text{independent of } r}$ $=w(0.9)^{w-1}(0.1)^2$	
(c)	$\mathrm{E}(W)=\sum_{y=1}^{\infty}w\cdot w(0.9^{w-1})(0.1)^2$ $=0.1\sum_{y=1}^{\infty}w^2(0.9^{w-1})(0.1)$ Consider $Y\sim\mathrm{Geo}(0.1)$: $\mathrm{E}(Y)=\sum_{y=1}^{\infty}y(0.9^{y-1})(0.1)=\frac{1}{0.1}=10$ $\mathrm{Var}(Y)=\mathrm{E}(Y^2)-[\mathrm{E}(Y)]^2$ $\mathrm{E}(Y^2)=\mathrm{Var}(Y)+[\mathrm{E}(Y)]^2$ $\sum_{y=1}^{\infty}y^2(0.9^{y-1})(0.1)=\frac{1-0.1}{0.1^2}+10^2=190$ Thus $\mathrm{E}(W)=0.1(190)=19$ **Alternatively, a better method** (Note that question specified the use of mean and variance of geometric distribution, hence we should not do like this) Let $W=Y_1+Y_2-1$, where $Y_1,Y_2\sim\mathrm{Geo}(0.1)$ $\mathrm{E}(W)=\mathrm{E}(Y_1+Y_2-1)=2\mathrm{E}(Y)-1$ $=2\left(\frac{1}{0.1}\right)-1$ $=19$	 **M1** **M1** **A1**

2* The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} \frac{2}{25}(x+1), & -1 \le x \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

The random variable Y is defined by $Y = X^2$.

(a) Show that $P(Y \le y) = \frac{4}{25}\sqrt{y}$ for $0 \le y \le 1$. [3]

(b) Find the cumulative distribution function of Y. [4]

(c) Find the probability density function of Y. [2]

2	**EJC JC2 Prelim 9649/2019/02/Q7**	
(a)	Given $Y = X^2$. For $0 \le y \le 1$, $P(Y \le y) = P(X^2 \le y)$ $= P(-\sqrt{y} \le X \le \sqrt{y})$ $= \frac{2}{25}\int_{-\sqrt{y}}^{\sqrt{y}} (x+1)\,dx$ Note that for $0 \le y \le 1$, $-1 \le x \le 1$ $= \frac{2}{25}\left[\frac{x^2}{2} + x\right]_{-\sqrt{y}}^{\sqrt{y}}$ $= \frac{2}{25}\left[\frac{y}{2} + \sqrt{y} - (\frac{y}{2} - \sqrt{y})\right]$ $= \frac{4}{25}\sqrt{y}$ (shown)	**B1** **M1** **A1**
(b)	Note, for (a) we considered $0 \le y \le 1$ which corresponds to $-1 \le x \le 1$, Now, $1 \le x \le 4 \Rightarrow 1 \le y \le 16$, For $1 \le y \le 16$, $P(Y \le y) = P(Y \le 1) + P(1 \le Y \le y)$ $= P(Y \le 1) + P(1 \le X \le \sqrt{y})$ $= \frac{4}{25}\sqrt{1} + \frac{2}{25}\int_{1}^{\sqrt{y}} (x+1)\,dx$ $(\because 1 \le \sqrt{y} \le 4)$ $= \frac{4}{25} + \frac{2}{25}\left[\frac{x^2}{2} + x\right]_{1}^{\sqrt{y}}$ $= \frac{4}{25} + \frac{2}{25}\left[\frac{y}{2} + \sqrt{y} - (\frac{1}{2} + 1)\right]$ $= \frac{2}{25}\left(\frac{y}{2} + \sqrt{y} + \frac{1}{2}\right)$ $\left[\text{OR } \frac{1}{25}\left(y + 2\sqrt{y} + 1\right)\right]$	**M1** **M1** **A1**

	$G(y)=\begin{cases} 0, & y<0 \\ \frac{4}{25}\sqrt{y}, & 0\le y\le 1 \\ \frac{1}{25}\left(y+2\sqrt{y}+1\right), & 1\le y\le 16 \\ 1, & y>16 \end{cases}$	**A1**
(c)	$g(y)=G'(y)=\begin{cases} \frac{2}{25\sqrt{y}}, & 0<y\le 1 \\ \frac{1}{25}\left(1+\frac{1}{\sqrt{y}}\right), & 1\le y\le 16 \\ 0 & \text{otherwise} \end{cases}$	**M1 -** differentiate **A1**

3 The number of distinct uranium deposits in a given area is a Poisson random variable with parameter 10. Independently, the number of distinct plutonium deposits in this area is a Poisson random variable with parameter 5. In a fixed period of time, each deposit is discovered independently with probability $\frac{1}{50}$.

(a) Find the probability that no less than 2 deposits are discovered during this period. [2]

(b) Given that no more than 3 deposits are found in this period of time, find the probability that there are at least 1 uranium deposit. [4]

(a)	Let U and V be the number of uranium deposits and plutonium deposits discovered during this period of time. $U \sim Po(0.2)$ and $V \sim Po(0.1)$. Let $T=U+V$. $T \sim Po(0.3)$. $\text{Prob. Req'd} = P(T\ge 2)$ $=1-P(T\le 1)$ $=0.0369$	M1 A1
(b)	$P(U\ge 1 \mid T\le 3)$ $=\frac{P(U\ge 1 \text{ and } T\le 3)}{P(T\le 3)}$ $=\frac{P(1\le U\le 3 \text{ and } V=0)+P(1\le U\le 2 \text{ and } V=1)+P(U=1 \text{ and } V=2)}{P(T\le 3)}$ $=\frac{0.1810065}{0.9997342}=0.181$	M1 M1 M1 A1

4 A particle moves along a horizontal line, relative to a fixed origin, so that its position at a given time t is $x = x_0 \sin t$, where x_0 is a positive constant.
Let X be the continuous random variable denoting the particle's position.

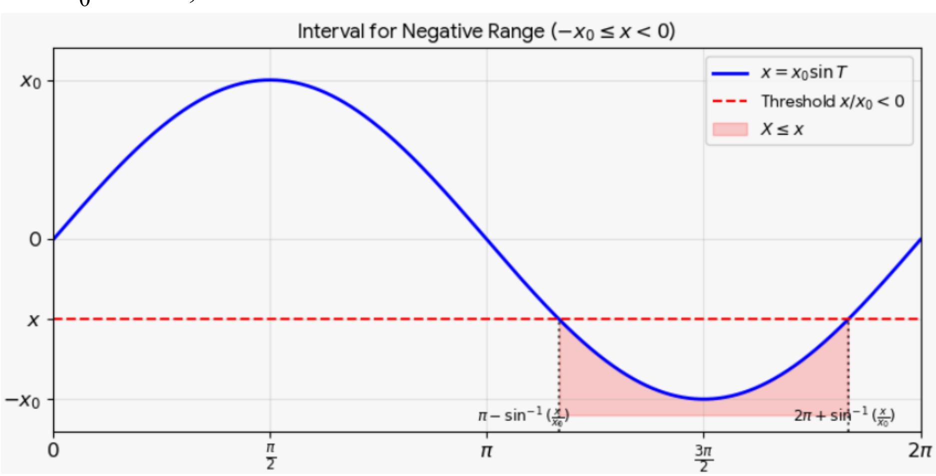
(a) Show that the cumulative density function of X is given by

$$\mathrm{F}(x) = \begin{cases} 1 & x > x_0, \\ \dfrac{1}{2} + \dfrac{1}{\pi}\sin^{-1}\left(\dfrac{x}{x_0}\right) & -x_0 \le x \le x_0, \\ 0 & x < -x_0, \end{cases}$$

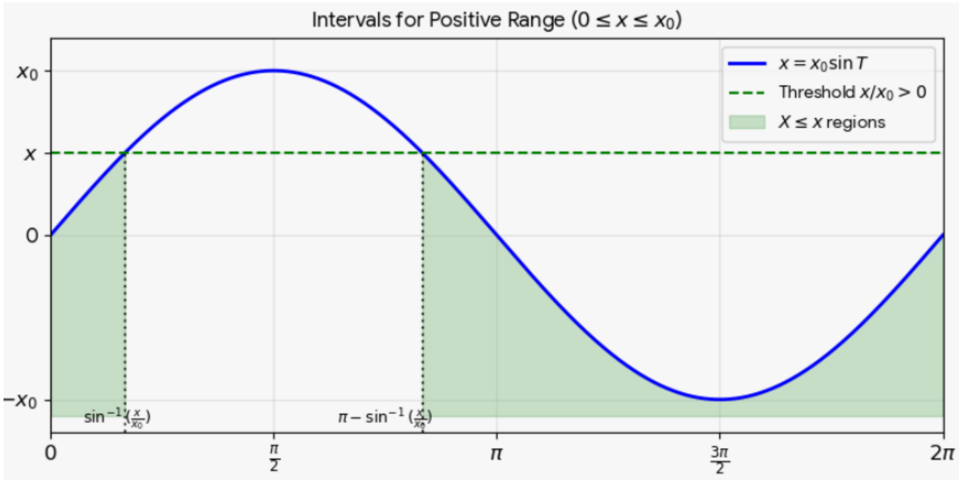
and hence, find the probability density function of X. [5]

(b) Find the exact probability that the magnitude of X is at most $\frac{1}{2}x_0$. [2]

(c) Show that the expectation of X is 0, and find the exact variance of X using the substitution $x = x_0 \sin\theta$. [7]

4	**ACJC JC2 Prelim 9649/2019/02/Q10**	
(a)	$x = x_0 \sin t \Rightarrow t = \sin^{-1}\dfrac{x}{x_0}$ Consider one period. Thus $T \sim \mathrm{U}(0, 2\pi)$, $\mathrm{f}(t) = \dfrac{1}{2\pi}$. If $x < -x_0$, $\mathrm{P}(X \le x) = 0$ If $-x_0 \le x < 0$,  $\mathrm{P}(X \le x) = \mathrm{P}\left(\pi - \underbrace{\sin^{-1}\left(\frac{x}{x_0}\right)}_{<0} < T \le 2\pi + \underbrace{\sin^{-1}\left(\frac{x}{x_0}\right)}_{<0}\right)$ $= \frac{1}{2\pi}\left(\pi + 2\underbrace{\sin^{-1}\left(\frac{x}{x_0}\right)}_{<0}\right)$ by symmetry $= \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\left(\frac{x}{x_0}\right)$	

If $0 \le x \le x_0$,



$$\begin{aligned}
\mathrm{P}(X \le x) &= \mathrm{P}(X \le 0) + \mathrm{P}(0 < X \le x) \\
&= \mathrm{P}(\pi \le T \le 2\pi) + \mathrm{P}\left(0 < T \le \sin^{-1}\left(\tfrac{x}{x_0}\right) \text{ or } \pi - \sin^{-1}\left(\tfrac{x}{x_0}\right) \le T < \pi\right) \\
&= \frac{1}{2\pi}(\pi) + 2 \times \frac{1}{2\pi}\left(\sin^{-1}\left(\frac{x}{x_0}\right)\right) \quad \text{by symmetry} \\
&= \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\left(\frac{x}{x_0}\right)
\end{aligned}$$

Hence, $$\mathrm{F}(x) = \begin{cases} 1 & x > x_0, \\ \dfrac{1}{2} + \dfrac{1}{\pi}\sin^{-1}\left(\dfrac{x}{x_0}\right) & -x_0 \le x \le x_0, \\ 0 & x < -x_0. \end{cases}$$

$$\begin{aligned}
\mathrm{f}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x) \\
&= \frac{1}{\pi\sqrt{x_0^2 - x^2}} \\
&= \begin{cases} \dfrac{1}{\pi\sqrt{x_0^2 - x^2}} & -x_0 < x < x_0, \\ 0 & x < -x_0 \text{ or } x > x_0 \end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{P}\left(|X| \le \tfrac{1}{2}x_0\right) &= \mathrm{P}\left(-\tfrac{1}{2}x_0 \le X \le \tfrac{1}{2}x_0\right) \\
&= \mathrm{F}\left(\tfrac{1}{2}x_0\right) - \mathrm{F}\left(-\tfrac{1}{2}x_0\right) && \textbf{M1} \\
&= \frac{1}{\pi}\left(\sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}\left(-\tfrac{1}{2}\right)\right) \\
&= \tfrac{1}{3} && \textbf{A1}
\end{aligned}$$

(c)	$\mathrm{E}(X)=\int_{-\infty}^{\infty} x\,\mathrm{f}(x)\,\mathrm{d}x$ $=\int_{-x_0}^{x_0} \frac{x}{\pi\sqrt{x_0^2-x^2}}\,\mathrm{d}x$ $=\frac{1}{2\pi}\left[2\sqrt{x_0^2-x^2}\right]_{-x_0}^{x_0}$ $=0$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=\int_{-\infty}^{\infty} x^2\,\mathrm{f}(x)\,\mathrm{d}x-0$ $=\int_{-x_0}^{x_0} \frac{x^2}{\pi\sqrt{x_0^2-x^2}}\,\mathrm{d}x$ $=\frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{x_0^2\sin^2\theta}{x_0\cos\theta}x_0\cos\theta\,\mathrm{d}\theta$ $=\frac{x_0^2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin^2\theta\,\mathrm{d}\theta=\frac{x_0^2}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta$ $=\frac{x_0^2}{\pi}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}=\frac{x_0^2}{2}$	**M1** **A1** **B1** **M1 –** substitution **M1** **M1, A1**

5	Three fair six-sided die are thrown. The random variable X represents the highest of the three scores on the dice. **(a)** Show that $P(X=6)=\frac{91}{216}$. [2] **(b)** By finding the probability distribution of X, show that E(X) = $\frac{119}{24}$ and find the exact value of Var (X). [5]

Solution:

(a)	$\mathrm{P}(X=6)=1-\mathrm{P}(\text{no '6' is obtained})=1-\left(\frac{5}{6}\right)^3=\frac{91}{216}$ Alternatively, $\mathrm{P}(X=6)=\mathrm{P}(\{6,6,6\}\text{ or }\{6,6,*\}\text{ or }\{6,*,*\}\text{ or}\{6,*,!\})$ $=\frac{1}{6}(\frac{1}{6})(\frac{1}{6})+\frac{1}{6}(\frac{1}{6})(\frac{5}{6})(3)+\frac{1}{6}(\frac{5}{6})(\frac{1}{6})(3)+\frac{1\times{}^5C_2\times 3!}{6^3}$ $=\frac{91}{216}$ $\mathrm{P}(X=5)=\underbrace{\left(\frac{5}{6}\right)^3}_{\text{Each of the 3 die gives 1,2,3,4 or 5}}-\underbrace{\left(\frac{4}{6}\right)^3}_{\text{Each of the 3 die gives 1,2,3 or 4}}=\frac{61}{216}$ $\mathrm{P}(X=4)=\left(\frac{4}{6}\right)^3-\left(\frac{3}{6}\right)^3=\frac{37}{216}$	M1: use complement A1 B1: 3 of the 6 cases correct

	$P(X=3)=\left(\frac{3}{6}\right)^3-\left(\frac{2}{6}\right)^3=\frac{19}{216}$ $P(X=2)=\left(\frac{2}{6}\right)^3-\left(\frac{1}{6}\right)^3=\frac{7}{216}$ $P(X=1)=\left(\frac{1}{6}\right)^3=\frac{1}{216}$ <table><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>$P(X=x)$</td><td>$\frac{1}{216}$</td><td>$\frac{7}{216}$</td><td>$\frac{19}{216}$</td><td>$\frac{37}{216}$</td><td>$\frac{61}{216}$</td><td>$\frac{91}{216}$</td></tr></table>$E(X)=\frac{1}{216}+2\left(\frac{7}{216}\right)+3\left(\frac{19}{216}\right)+4\left(\frac{37}{216}\right)+5\left(\frac{61}{216}\right)+6\left(\frac{91}{216}\right)=\frac{119}{24}$ $\mathrm{Var}(X)$ $=\left[\frac{1}{216}+2^2\left(\frac{7}{216}\right)+3^2\left(\frac{19}{216}\right)+4^2\left(\frac{37}{216}\right)+5^2\left(\frac{61}{216}\right)+6^2\left(\frac{91}{216}\right)\right]-\left(\frac{119}{24}\right)^2$ $=\frac{2261}{1728}$	B1: All 6 correct. B1:Use of correct formulae for E(X) B1: Var(X)