ANDERSON SERANGOON JUNIOR COLLEGE
PHYSICS 9749

Topic 6: Motion in a Circle

Additional Notes:

Watch the human loop the loop on YouTube:

https://youtu.be/GkPudmBQBEY

Content:

- Kinematics of uniform circular motion
- Centripetal acceleration
- Centripetal force

Learning Outcomes:

Candidates should be able to:

Kinematics of uniform circular motion

(a) express angular displacement in radians

(b) show an understanding of and use the concept of angular velocity to solve problems.

(c) recall and use $v = r\omega$ to solve problems.

Centripetal acceleration

(d) describe qualitatively motion in a curved path due to a perpendicular force, and understand the centripetal acceleration in the case of uniform motion in a circle.

(e) recall and use centripetal acceleration $a = r\omega^2$ and $a = \dfrac{v^2}{r}$ to solve problems.

Centripetal force

(f) recall and use centripetal force $F = mr\omega^2$ and $F = \dfrac{mv^2}{r}$ to solve problems.

Demonstrating Science Inquiry Skills

In this chapter, we tackle key questions such as, “What causes an object to move in a circular path?”, “What is a centripetal force?”

Relating Science and Society

An example of an application of the concept of motion in a circle is the determination of the speed limit of vehicles negotiating a circular path. Other examples of motion in a circle include:

- *the rotation of the Earth about its axis,*
- *the motion of the moon about the Earth (approximately circular),*
- *the rotation of a compact disc about its axis,*
- *the motion of a Ferris wheel,*
- *the motion of the second, minute and hour hands of an analogue clock.*

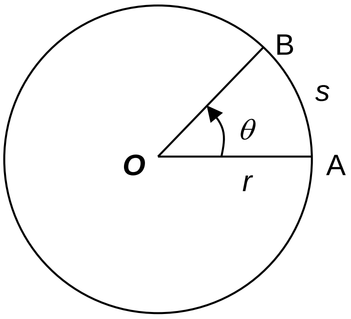
"Additional Notes:"

6A Kinematics of Uniform Circular Motion

6A.1 Angular Displacement θ

Consider an object going round a circular path of radius r from point A to B.

The angular displacement θ is the angle the radius sweeps through in a specific direction from a reference line.



Legend:

s: arc length

r: radius of circle

Recall that
π rad = 180°

Hence,

$1^\circ = \frac{\pi}{180}$ rad

Definition of angular displacement: The angle through which an object turns, usually measured in radians (rad).

$$\boxed{\theta = \frac{s}{r}}$$

Note:

- Since θ is a ratio of a length to another length, it is dimensionless.
- θ is a vector quantity since it has direction (clockwise / anti-clockwise).

Definition of a radian: 1 radian is the angle subtended by an arc of equal length to the radius.

<u>Check Your Understanding 1</u>

How many radians does 1° correspond to?

A 0.035 rad

B 0.0087 rad

C 0.017 rad

Additional Notes:

6A.2 Angular Velocity ω

Definition: Angular velocity ω is defined as the rate of change of angular displacement.

$$\boxed{\omega = \frac{d\theta}{dt}}$$

Note:

- Its unit is rad s^{-1}.
- It is a vector quantity. The direction pointing out of the page along the axis of rotation when the rotation is in the plane of the page anti-clockwise.
- Its magnitude is called angular speed.

Check Your Understanding 2

What is the angular speed of the second hand of a clock?

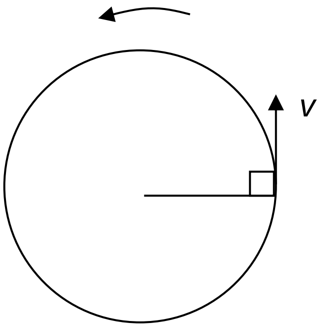
A 0.105 rad s^{-1}

B 9.55 rad s^{-1}

C 0.0523 rad s^{-1}

6A.3 Relationship between Angular Velocity ω and Linear Velocity v

From $\theta = \frac{s}{r}$ and $\omega = \frac{d\theta}{dt}$, we can show that:



$$\omega = \frac{d}{dt}\left(\frac{s}{r}\right) \quad \text{and } r \text{ is a constant}$$

$$= \frac{1}{r}\frac{ds}{dt}$$

$$= \frac{v}{r}$$

Or $v = \frac{2\pi r}{T} = r\frac{2\pi}{T} = r\omega$

i.e. $\boxed{v = r\omega}$

where v is the linear velocity or tangential velocity,

r is the radius of the circular path,

ω is the angular velocity.

Note:

- v has units m s^{-1}
- It is a vector quantity.
- Its magnitude is called linear speed or tangential speed.
- Its direction is tangent to the circle and therefore always perpendicular to the radius.

Check Your Understanding 3

What is the relationship between linear velocity v, radius of rotation r and angular velocity ω?

$$\boxed{} = \frac{\boxed{}}{\boxed{}}$$

6A.4 Uniform Circular Motion

- An object is undergoing <u>uniform circular motion</u> if it moves round a circle with <u>constant speed</u>.
- The time it takes to go one revolution is called the period T.

  For 1 period T, the angular displacement $\Delta\theta = 2\pi$ rad.

$$\boxed{\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T}}$$

- The number of **revolutions per unit time** is called the frequency f, with unit Hertz (Hz). Since it takes time T to make one revolution,

$$\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi}} \quad \text{or} \quad \boxed{\omega = 2\pi f}$$

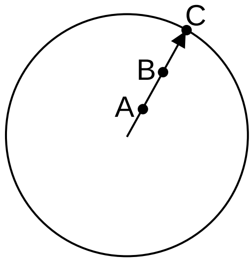
Check Your Understanding 4

An object moving with uniform circular motion has a constant velocity. TRUE / FALSE

Additional Notes:

Example 1

Consider the second hand of a clock. Three points A, B and C lie on the second hand.



(a) State and explain the point which has the largest

(i) angular velocity ω,

(ii) linear speed v.

Key questions on objective

Do the 3 points experience the same change in angular displacement over time?
Yes. The 3 points lie along the same radius as they move. Hence their change in angular displacement per unit time is the same.

Do the 3 points experience the same change in linear displacement over time?
No. C experiences the largest change in linear displacement since it traces a larger circular arc as it moves. Conversely, A experiences the smallest change in displacement.

(b) Point C is 2.0 cm from the centre of the circle, calculate the

(i) angular velocity of point C, [0.105 rad s^{-1}]

(ii) linear velocity of point C. [2.09×10^{-3} m s^{-1}]

Note that r does not refer to the radius of the revolving object. Rather, it refers to the radius of the circular path described by the point considered.

(i) Key question to solve question
What general knowledge do we know about the rotation of the second hand of a clock?
It takes 60 s to go through one round, i.e.

For $t = T = 60$ s, $\Delta\theta = 2\pi$

Example 2

Singapore lies approximately on the Earth's equator. The radius of the Earth is about 6370 km. Calculate the

(i) angular velocity and [7.27×10^{-5} rad s^{-1}]
(ii) linear speed, of Singapore. [463 m s^{-1}]

(i) *Key question to solve question*
What general knowledge do we know about the rotation of the Earth about its axis?
It takes 24 hours for Earth to spin 1 round about its own axis, i.e.

For $t = T = 24$ hours, $\Delta\theta = 2\pi$

Recall that the SI unit of time is the second.

6B Centripetal Acceleration and Centripetal Force

For a <u>uniform</u> circular motion

The speed of an object can be constant, but its velocity is changing, if the object's motion changes direction.

- The speed of the object is constant.
- However, its direction is always changing. Hence its velocity is changing.
- The object thus has an acceleration. By Newton's second law, it must have a resultant force acted on it.

Alternatively,

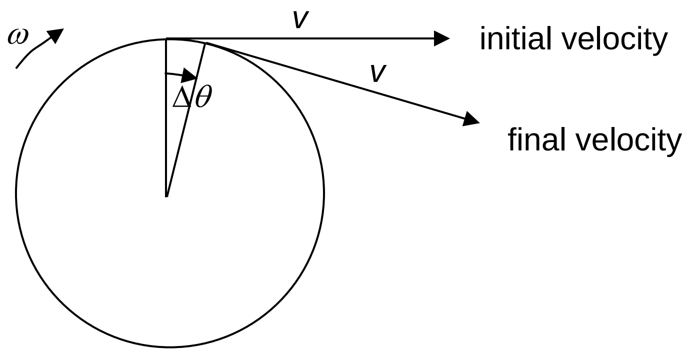
- By Newton's first law, a body moves in a straight line if no resultant force acts on it.
- Hence, there must be a resultant force acting on a body moving in a circular path.

Note:

- Since object moves with constant tangential speed, the tangential acceleration is zero and hence tangential force is zero.
- The resultant force must act perpendicular to the path, such that the force only changes the direction of motion, but not the tangential speed.

6B.1 Centripetal Acceleration

Consider an object undergoing **uniform circular motion with constant linear speed *v***.

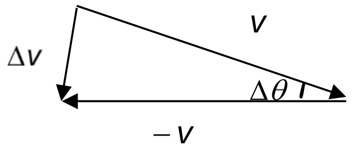


In a small time Δt, it moves through an angular displacement $\Delta\theta$.

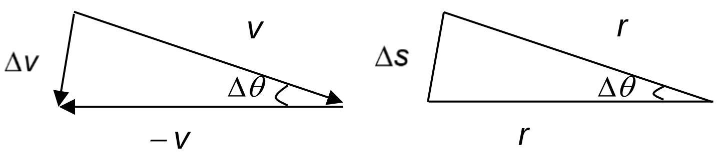
The average acceleration, $a = \dfrac{\text{change in velocity}}{\text{time taken}}$

Change in velocity Δv = final velocity – initial velocity

For small $\Delta\theta$, $a = \dfrac{\Delta v}{\Delta t}$



By similar triangles



$$\frac{\Delta v}{v} = \frac{\Delta s}{r}$$

$$\Delta v = v\Delta\theta$$

Therefore $a = \dfrac{\Delta v}{\Delta t} = \dfrac{v\Delta\theta}{\Delta t} = v\omega$

Since $v = r\omega$ $\qquad$ $a = r\omega^2 = \dfrac{v^2}{r}$

The direction of this acceleration a for small $\Delta\theta$ is towards the centre of the circle.

This acceleration is called the centripetal acceleration.

centripetal acceleration	$a = r\omega^2 = v\omega = \dfrac{v^2}{r}$

Check Your Understanding 5

An object travels at constant speed around a circle of radius 1.0 m in 1.0 s. What is the magnitude of its acceleration?

A 0 m s^{-2}

B 1.0 m s^{-2}

C 6.3 m s^{-2}

D 39 m s^{-2}

6B.2 Centripetal Force

The force that causes a body to go round the circle is called the centripetal force given by

centripetal force	$F_{net} = mr\omega^2 = mv\omega = \dfrac{mv^2}{r}$

- The centripetal force is also directed towards the centre of the circle.
- Centripetal force must NOT be drawn as an additional force in a force diagram. It must be provided by some real force(s).
- Since the centripetal force acts at right angle to the motion, the centripetal force **does no work**. Thus there is no change in kinetic energy.

Recall work is product of force and displacement *in the direction of the force*

Worked Example 3

Explain why a body which is executing uniform circular motion always has acceleration perpendicular to its motion.

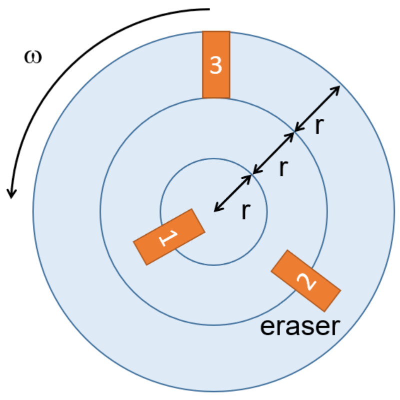
Method 1:
Since its direction is always changing, there must be a force acting on it.
And since its speed is constant, the force cannot have any component acting in or opposite to direction of motion.
Hence, force can only act perpendicular to motion. The acceleration is also in the perpendicular direction to its motion since force and acceleration are always in the same direction.

Method 2
Since kinetic energy is constant, the work done by the object must be zero implying that the force is acting in a perpendicular direction to its motion.

Access SLS and go to the end of section 3 to watch a video to understand why you experience a force when the bus you are riding turns round a bend.

Additional Notes:

6C Uniform horizontal circular motion



3 erasers are on a turntable spinning at with angular speed ω. Friction provides the centripetal force for the eraser to move in circular motion.

As ω increases gradually, which eraser will fly off first?

General Approach to Solving Problems relating to Circular Motion

1) Draw a free body diagram showing all the forces acting on the body.

2) Determine the centre of the circular path.

3) Resolve the forces along two axes:
(i) towards the direction of the centre of the circle
(ii) in a direction perpendicular to the direction in (i)

4) For 3(i), write down a statement for the net force towards the centre of the circular path. **This net force provides the centripetal force**, i.e.

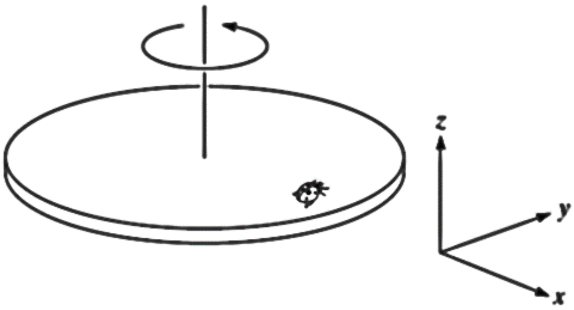
$$F_{net} = mr\omega^2 = mv\omega = \frac{mv^2}{r}$$

5) For 3(ii), the net force is zero.

Check Your Understanding 6

A ladybug sits at the outer merry-go-round that is turning at constant angular speed. What are the respective directions of the **angular velocity** and **acceleration** at the instant as shown in the diagram?

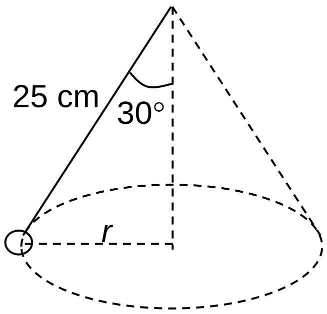
	angular velocity	acceleration
A	+z direction,	+y direction
B	+z direction,	–x direction
C	+y direction,	+x direction
D	+y direction,	–x direction



Additional Notes:

Example 4 (Conical Pendulum)

In a conical pendulum, a 0.50 kg mass is made to go round a horizontal circle at constant speed. The string is of length 25.0 cm and makes an angle of 30° with the vertical. Neglect air resistance.



(a) Draw the forces acting on the mass.

(b) Determine

(i) the tension T in string [5.66 N]

(ii) the linear speed of the mass [0.841 m s^{-1}]

(iii) the period of oscillation of the mass. [0.934 s]

Key question on objective:
Why doesn't the pendulum move higher up or lower down?
There must be an equilibrium of forces in the vertical direction.

Key question to solve the question:
The pendulum's motion is subtended by an angle, is there a reason for it?
With the angle, the tension can be resolved into the vertical and horizontal direction. And since it is in vertical equilibrium, a component of the tension can be equated to weight.

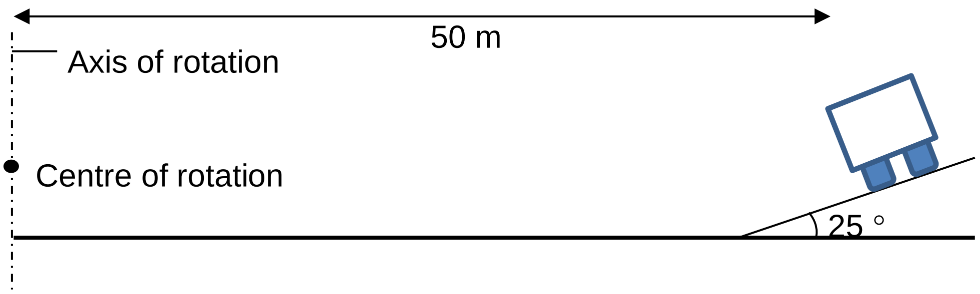
Additional Notes:

Example 5 (combine flat + banked roads + addition of “Wall of Death”)

(a) (i) A 1500 kg car goes round a circular bend of radius 50 m at 12.5 m s^{-1} on a flat road. Determine the magnitude of sideway friction required for the car to go round the bend at this speed.
[4690 N]

(ii) Explain what might happen if the car is driven at a higher speed while maintaining the same radius.

(b) The car is then driven round a road banked at an angle of 25° of radius 50 m as shown.



Additional Notes:

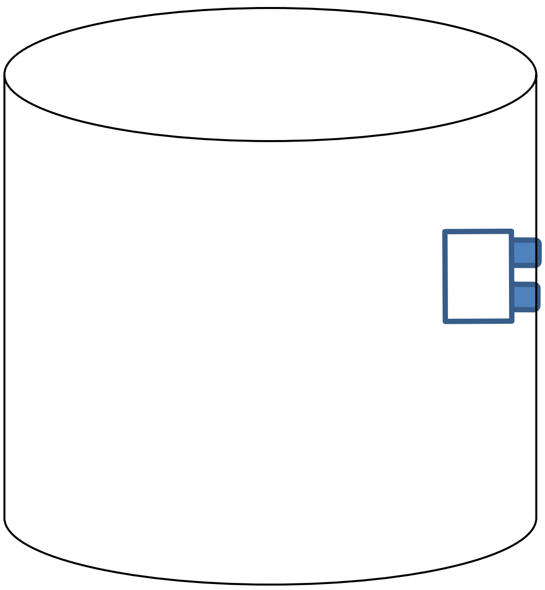
(i) Calculate the maximum speed that the car can travel while maintaining the circular radius by assuming that the sideway friction between the tyres and the road is negligible. [15.1 m s^{-1}]

(ii) In practice, there is significant amount of friction between the tyres and the road. State and explain how the maximum speed of the car will be affected.

Watch the “well of death” on YouTube:

https://youtu.be/QA1wwR9PGxU

(c) The angle of the banked road is increased to 90°, such that the car is moving on the inside of a vertical cylinder. Draw the free body diagram of the car.



What provides the centripetal force?

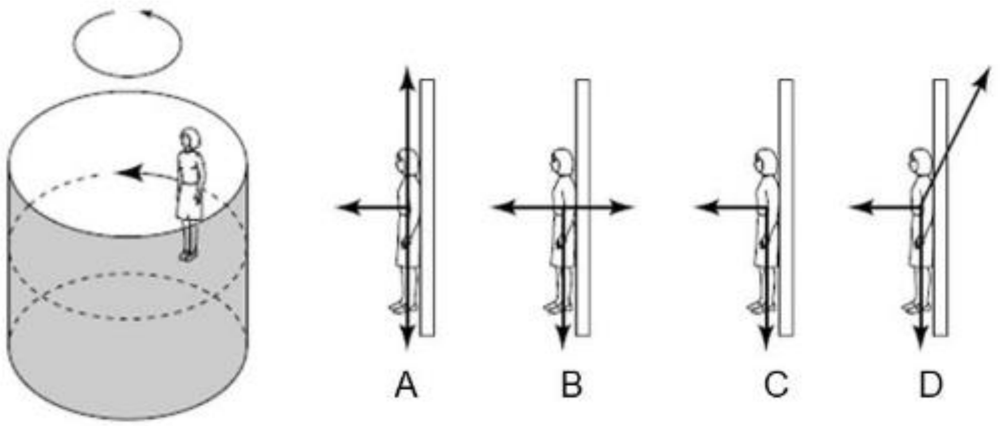
How can the centripetal force increase?

For rotational equilibrium that is not in translational equilibrium, the resultant torque will only be zero about the centre of gravity.

Additional Notes:

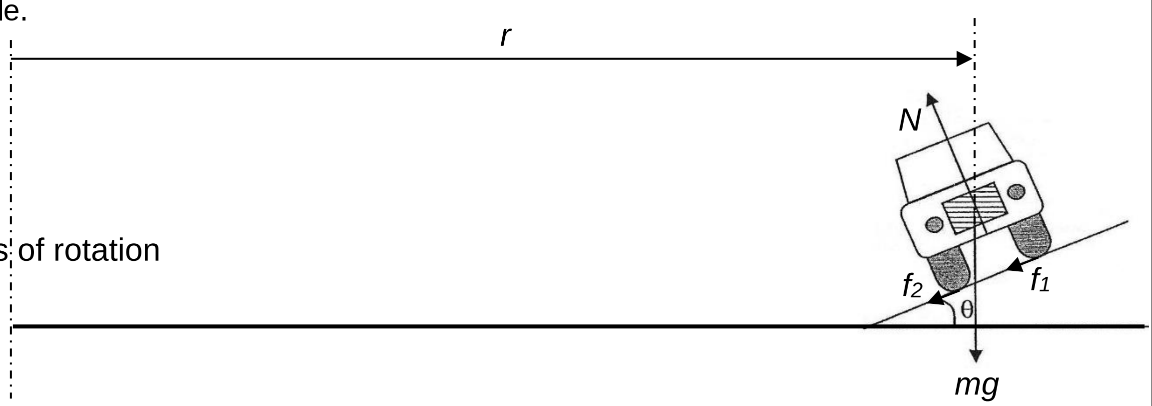
Check Your Understanding 7

A rider in a "barrel of fun" finds herself stuck with her back to the wall. Which diagram correctly shows the forces acting on her?



Additional notes on Banked Roads

In most racing competition (cycling or motor racing), the road is usually banked at an angle.



- In addition to the frictional forces between the road and the tyres, the horizontal component of the normal contact force provides part of the centripetal force.
- The larger the banked angle θ, the smaller the frictional force needed to provide for the centripetal force.
- No frictional force is required if the component of the normal contact force is sufficient to provide the centripetal force required.

Note that the centre of rotation is not on the ground, but in the same plane as the circular path of the car.

Benefits of banked roads:

- drivers could drive at a higher speed with less danger of skidding when turning
- less wear and tear on the tire

Disadvantages of banked roads:

- May encourage driving at high speed when turning
- More expensive to construct roads with banking

6D Non-uniform vertical circular motion

Watch the following YouTube video:

https://youtu.be/-CvhRX59ty0

1) Draw a free body diagram showing all the forces acting on the body.

(Note : The free body diagram will be different at different positions of the circular path, unlike uniform horizontal circular motion)

2) Determine the centre of the circular path.

3) Resolve the forces acting towards the direction of the centre of the circle

4) Write down an expression for the **<u>net force towards the centre of the circular path</u>**. This net force provides the centripetal force, ie.

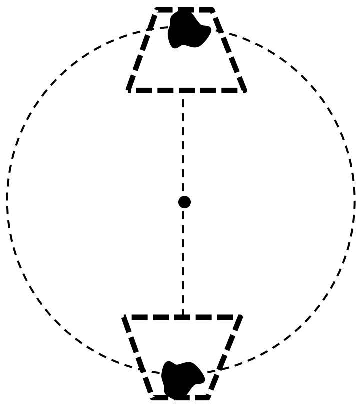
$$F_{net} = ma_c = mr\omega^2 = mv\omega = \frac{mv^2}{r}$$

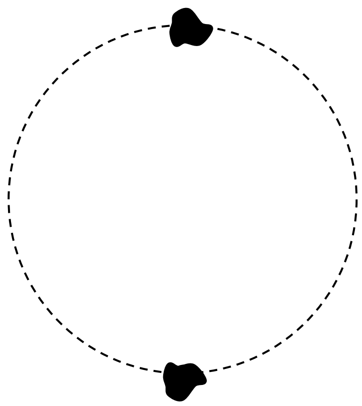
Note that the expression for the net force is different at different positions of the circular path.

5) Conservation of Mechanical Energy can also be applied.

Example 6 (String tied to a pail with a stone in it and spun in a vertical circle)

A string is tied to a pail with a stone in it and spun in a vertical circle as shown on the diagram on the left-hand-side below. The length of the string is 50.0 cm.





(a) By energy consideration, explain why <u>uniform</u> circular motion is not possible in this case.

(b) On the diagram on the right-hand-side above, draw and label the forces acting on the stone at the bottom, and top of the circular path.

(c) Write an equation using Newton's second law for the circular motion at the 2 positions.

Additional Notes:

(d) Calculate the minimum speed of the pail and stone such that the stone will not fall off from the pail. [2.21 m s^{-1}]

Key questions on solving problem:

(d) At which position would the stone fall off? What does it mean when the stone will not fall off?

If the stone does not fall off, i.e. stays in contact with pail, the normal reaction force of the pail on the stone must be larger than or equal to zero. The normal reaction force is smallest at the top. $N = 0$ when the stone does not fall off.

(a) The pail and stone lose GPE and gains KE when descending, thus speed will increase. On ascending, it gains GPE but loses KE, thus speed will decrease. Hence, its speed is not constant and will not maintain uniform circular motion.

(c)

(d)

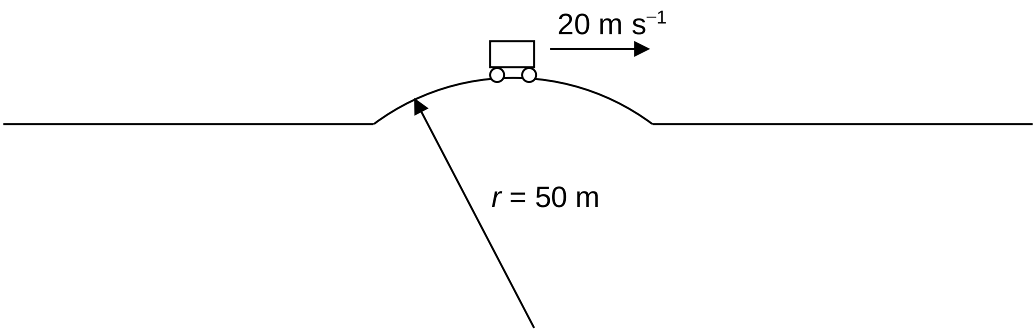
Additional Notes:

Check Your Understanding 8

Which of the following statements is true?

	Uniform Horizontal Circular Motion	**Non-uniform Vertical Circular Motion**
A	Centripetal force = resultant force	Centripetal force = resultant force
B	Centripetal force = resultant force	Centripetal force is not always the resultant force
C	Centripetal force is not always the resultant force	Centripetal force = resultant force
D	Centripetal force is not always the resultant force	Centripetal force is not always the resultant force

Example 7 (Car goes over a hump-back bridge)



At the top of the bridge, the speed of a 1500 kg car is 20 m s^{-1}.

(a) Draw all the forces acting on the car while it is at the top of the bridge. Use symbols to indicate the forces. Name the forces.

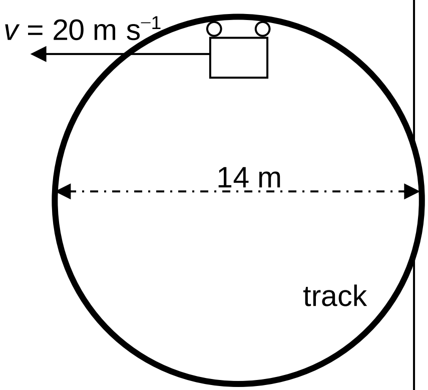
(b) Calculate the force the car exerts on the road while it is at the top of the bridge. [2720 N]

(c) Calculate the maximum speed of the car for it to remain in contact with the top of the hump. [22.1 m s^{-1}]

Additional Notes:

Example 8 (Car experiencing Vertical Circular Motion)

In a particular ride in an amusement park, a car of mass m = 55 kg travels upside down in a carriage at the top of a circular frictionless track of diameter 14 m at a speed of v = 20 m s^{-1}, as shown in the diagram below.



(a) On the diagram of the car below, draw all forces acting on the car. Use symbols to represent the forces. Name the forces.

(b) For the car, write an equation to represent Newton's 2nd law that governs its motion at the top of the circle.

(c) Hence, determine the magnitude of the force exerted by the car on the track. [2600 N]

(d) The engine of the car is switched off. Determine the velocity of the car at the bottom of the circle. [26.0 m s^{-1}]

(d) By drawing a free body diagram, determine the normal reaction force acting on the car at this position. [5850 N]

Key questions on solving problem:

(a) In general, for an object to be in contact with a surface, how should the direction of the normal contact force act?

The normal contact force should be directed from the surface towards the object.

(c) Should the velocity of the car be the same at the top and bottom of the loop?

By conservation of energy, the car loses GPE as it moves to the bottom of the loop. This implies that KE should be gained, i.e. the velocity should increase.

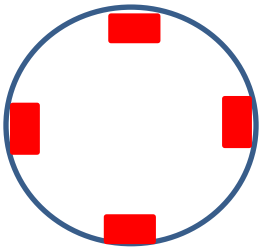
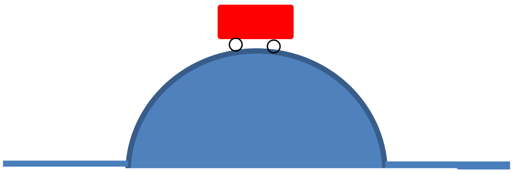
Additional Notes:

Note :

In Example 6(d) & 7(c), the minimum and maximum speeds are found by $v = \sqrt{rg}$.

The reason why $v = \sqrt{rg}$ is minimum for Example 6 and maximum for Example 7 is because Example 6 is a circular motion "inside the circle", whereas Example 7 is a circular motion "outside the circle".

Do not memorize the equation $v = \sqrt{rg}$. Always start from the fundamental of physics.

Inside the circle		**Outside the circle**
	vs	

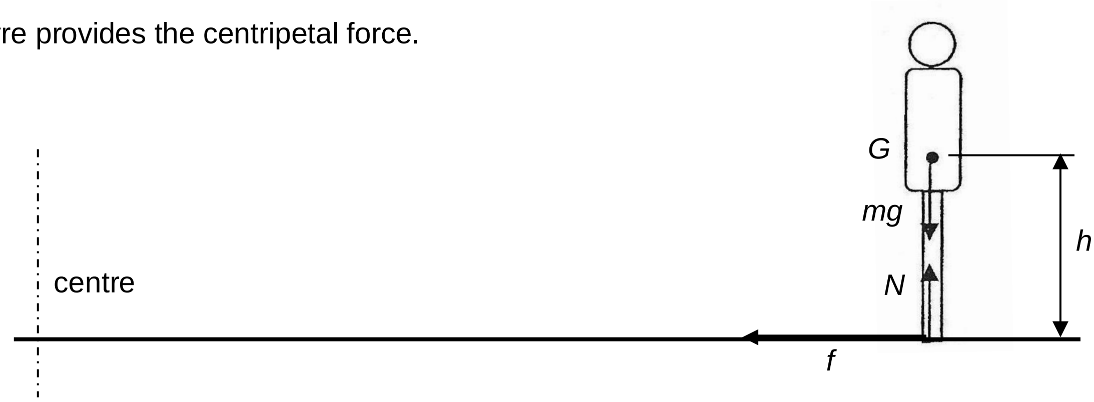
At the top,

$mg + N = mv^2/r$	$mg – N = mv^2/r$
v is min when $N = 0$	v is max when $N = 0$

APPENDIX

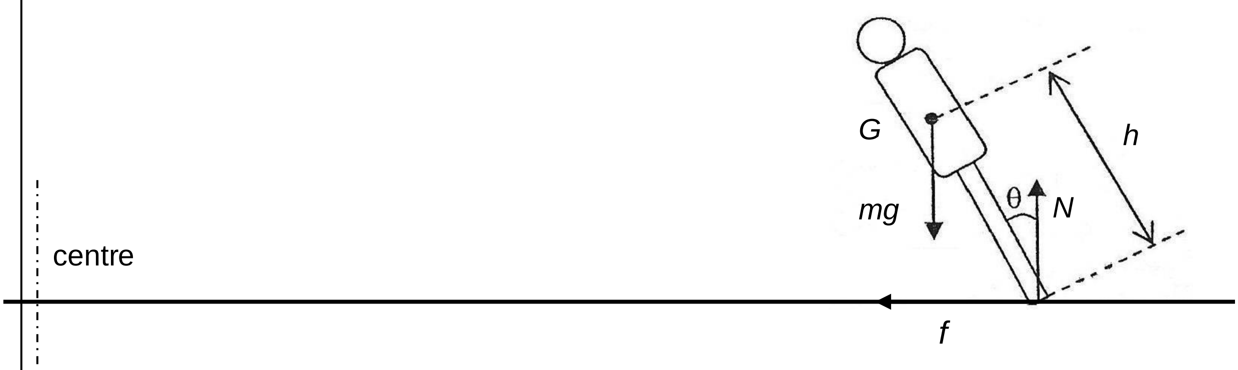
Cyclist going round a corner

Just like a car, when a cyclist goes round a corner, the friction between the road and the tyre provides the centripetal force.



For dynamic equilibrium, the net moment about the centre of mass must be zero. In this case, the cyclist has an unbalanced clockwise moment about the centre of mass *G*. The cyclist will topple outwards.

In order for the cyclist to negotiate a bend, he must tilt his bicycle. In the diagram below, the normal contact force *N* provides the anti-clockwise moment that balances the clockwise moment due to friction *f* about the centre of mass *G*.



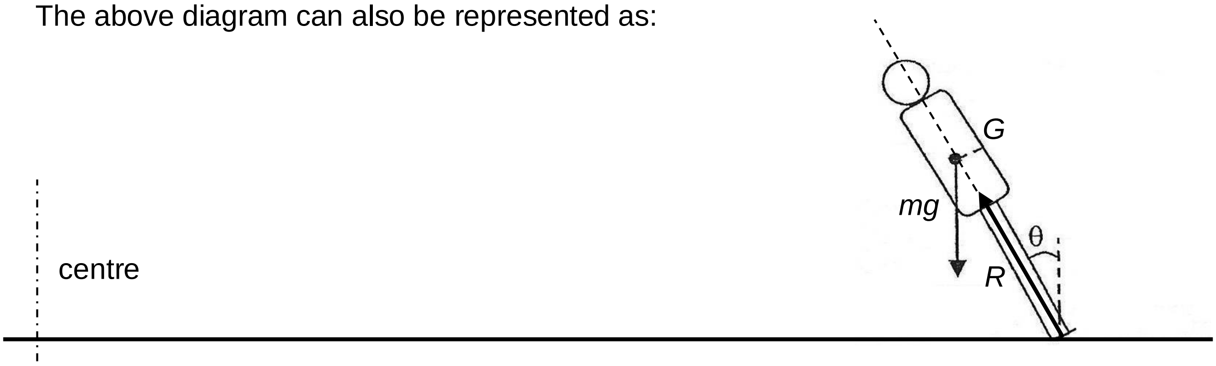
For vertical equilibrium $N = mg$

For horizontal motion, by Newton's 2nd law: $f = m\dfrac{v^2}{r}$

Also, taking moment about *G* $f \times h\cos\theta = N \times h\sin\theta$

i.e. $\tan\theta = \dfrac{f}{N} = \dfrac{v^2}{rg}$

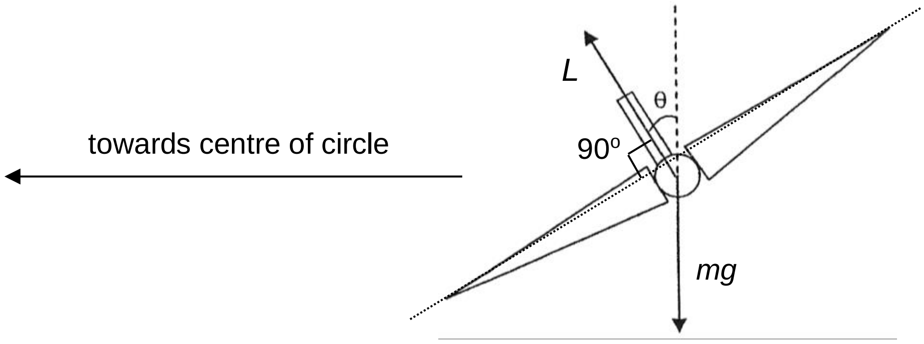
The above diagram can also be represented as:



R is the resultant of *f* and *N* and that this resultant must pass through the centre of mass *G* of the cyclist in order that the net moment about this point is zero.

Airplane making a circular turn

When an airplane makes a turn, it has to tilt the wings at angle to the vertical. This allows a component of the lift force *L* to provide for the centripetal force.



Lift force is 90° to the wings.

If the plane flies in a horizontal circle (i.e. vertical forces are balanced), then

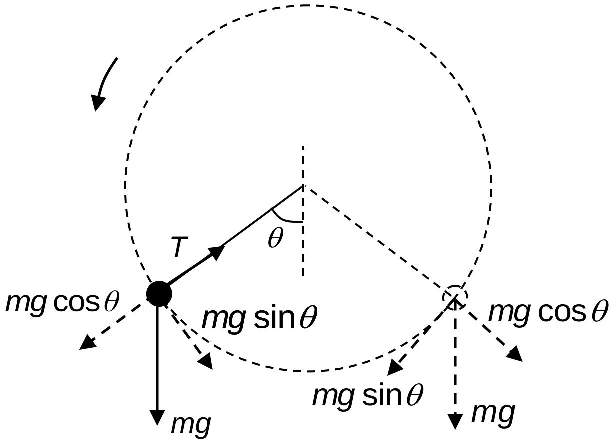
$$L\cos\theta = mg \qquad \text{.... (1)}$$

By Newton's 2nd law: $L\sin\theta = m\dfrac{v^2}{r}$ .... (2)

(2)/(1): $\tan\theta = \dfrac{v^2}{rg}$

Vertical circular motion

An object is fixed to one end of a light rod which rotates in a vertical circle at <u>constant</u> speed.



Consider the free body diagram of the object when the rod makes an angle θ with the vertical.

$$T - mg\cos\theta = \frac{mv^2}{r}$$

$$\Rightarrow T = m\frac{v^2}{r} + mg\cos\theta$$

At the bottom (θ = 0 and 2π rad), the tension is the greatest:

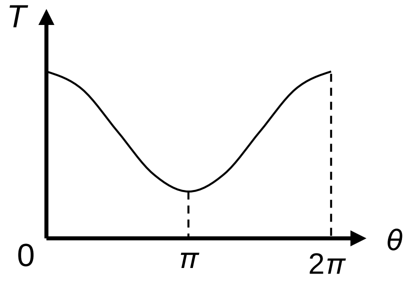
$$T = m\frac{v^2}{r} + mg$$

At the top ($\theta = \pi$ rad), the tension is the smallest:

$$T = m\frac{v^2}{r} - mg$$

At the side, $T = m\dfrac{v^2}{r}$

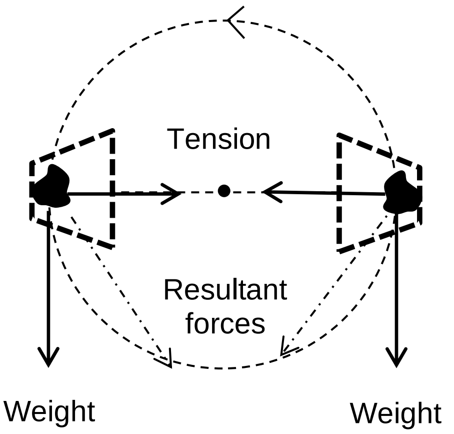
A graph that represents the variation with angle θ of the tension T in the rod:



Additional Notes:

For non-uniform vertical circular motion

The resultant force has a vertical component which changes the speed of the object.



Summary

Concept	Definition	Formula or important concepts
Angular displacement	The angle through which an object turns, usually measured in radians (rad). A more explicit definition is: The angular displacement is defined as the ratio of the arc length to its radius.	$\theta = \frac{s}{r}$
Angular velocity	Angular velocity is defined as the rate of change of angular displacement.	In general, $\omega = \frac{d\theta}{dt}$ For uniform circular motion, $\omega = \frac{2\pi}{T}$
Relationship between tangential velocity and angular velocity		$v = r\omega$
Centripetal acceleration		$a = r\omega^2 = v\omega = \frac{v^2}{r}$
Centripetal force		$F = ma = mr\omega^2 = mv\omega = \frac{mv^2}{r}$