1 A recycling company wants to buy three types of broken machines and process them to recover reusable materials. Each type of broken machine contains different amounts of copper, steel, and plastic for recovery. The amounts of each material recoverable from 1 unit of each type of broken machine are shown in the following table.

	Machine A	Machine B	Machine C
Copper (kg)	2	1	4
Steel (kg)	3	4	3
Plastic (kg)	1	5	8

The unit costs for Machine A, Machine B and Machine C are \$550, \$400 and \$350 respectively. The company recovered exactly 40 kg of copper, 61 kg of steel, and 56 kg of plastic. Find the total amount paid by the recycling company to purchase the machines. [4]

2 **Do not use a calculator in answering this question.**

It is given that one of the roots of the equation $px^4-7x^3+8x^2-3x+10=0$, where . $p\in\mathbb{R}$., is $2-\mathrm{i}$. Find the value of p and the other roots in exact form. [6]

3 **Do not use a calculator in answering this question.**

The points V and W are represented by the complex numbers v and w respectively, where

$$v=\left(3+\sqrt{3}\mathrm{i}\right)\left(1-\sqrt{3}\mathrm{i}\right) \text{ and } w=v\mathrm{i}.$$

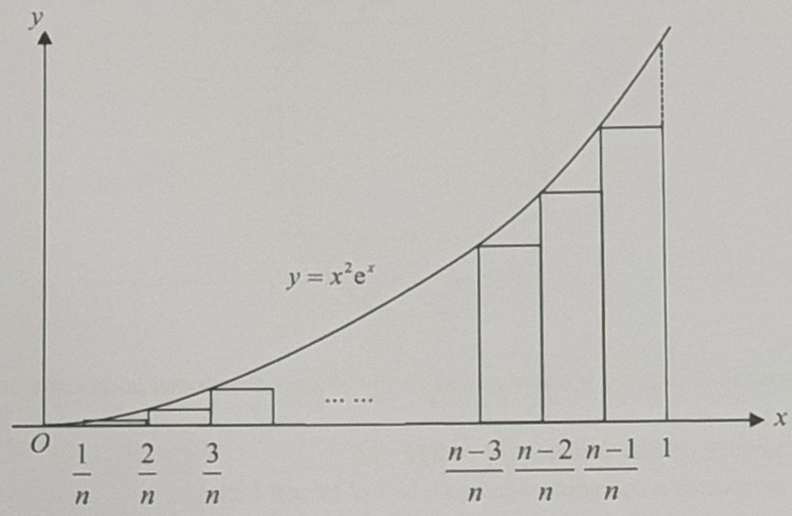
(a) Find the exact values of $|v|$ and $\arg(v)$. [4]

(b) Plot the points V and W on an Argand diagram, showing clearly the geometrical relationship between them. [2]

4 **(a)** Without using a calculator, solve the inequality $\dfrac{11-13x}{4x^2-1}<1$. [4]

(b) Hence solve the inequality $\dfrac{11-13x^2}{4x^4-1}<1$, giving your answer in exact form. [3]

5 The diagram shows a sketch of the curve $y = x^2 e^x$ for $x \geq 0$, with n rectangles of equal width $\frac{1}{n}$, where $n \in \mathbb{Z}^+$, drawn under the curve for $0 \leq x \leq 1$.



Let A_n denote the **total** area of the n rectangles.

(a) Show that $A_n = \frac{1}{n^3} \sum_{r=1}^{n-1} r^2 e^{\frac{r}{n}}$. [3]

(b) Find the exact value of $\lim_{n \to \infty} A_n$. [4]

6 The function f is defined by $f(x) = \sqrt{|2 - x| + 1}, \quad x \in \mathbb{R}, x \geq -1$.

(a) Show that f^{-1} does not exist. [2]

The function f has an inverse if its domain is restricted to $x \geq 2$.

(b) Find an expression for f^{-1} and state its domain. [4]

(c) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram, showing clearly the relationship between the two graphs. [3]

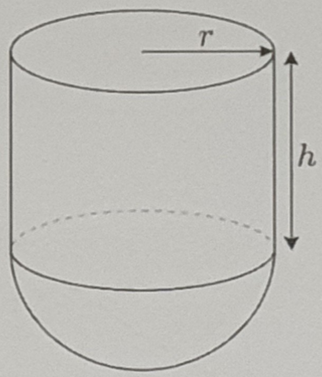
The functions g and h are defined by

$$g(x) = \frac{4}{4x^2 - 1}, \ x \in \mathbb{R}, \ x \neq -\frac{1}{2}, \ x \neq \frac{1}{2},$$

$$h(x) = 1 + e^{-x}, \ x \in \mathbb{R}.$$

(d) Find the exact range of hg. [2]

7 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



An industrial engineer is designing a storage tank in the shape of a closed composite solid which consists of three parts:

- the curved surface of a hemisphere of radius r m.
- the curved surface of a cylinder of radius r m and height h m.
- the circular top of radius r m.

The total volume of the tank is fixed at k m^3, where k is a positive constant. You can assume that the wall of the tank is of negligible thickness.

During the preliminary design phase, the engineer uses a simulation software to study the tank's proportions. The software models r increasing at a constant rate of 1 ms^{-1}, while the volume remains at k m^3 and the composite structure of the tank is kept intact.

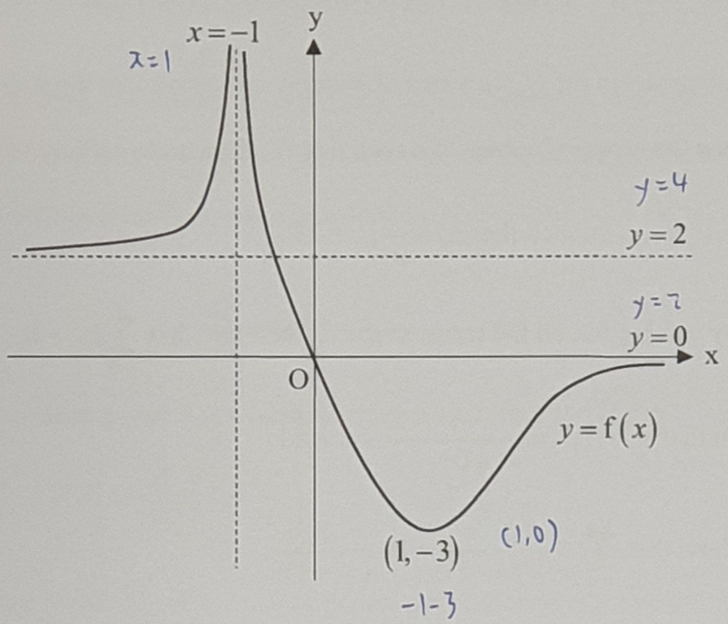
(a) Find $\frac{\mathrm{d}h}{\mathrm{d}t}$ at the instant when $r = \sqrt[3]{\frac{6k}{7\pi}}$. [4]

The total external surface area is given as S m^2.

(b) Show that $S = \frac{5}{3}\pi r^2 + \frac{2k}{r}$. $\frac{5}{3}\pi r^2 + \frac{7}{3}\pi r^2$ [3]

(c) Find the value of r, in terms of k, for which S is a minimum value. [4]

8 **(a)** The diagram shows the graph of $y=\mathrm{f}(x)$. It has a turning point at $(1,-3)$, asymptotes $x=-1$, $y=0$ and $y=2$, and crosses the origin.



On the axes provided, sketch the graphs of 1,-1

(i) $y=\mathrm{f}(-x)$, [2]

(ii) $y=\mathrm{f}(|x|)+2$, [3]

(iii) $y=\mathrm{f}'(x)$, [3]

labelling clearly the equation(s) of any asymptote(s), coordinates of any axial intercept(s) and turning point(s) where applicable.

(b) State a sequence of transformations that will transform the curve with equation $y=\ln 8x^3-1$ on to the curve with equation $y=\ln x$. [3]

9 **(a)** Let $U_1, U_2, U_3, \ldots$ denote the terms of a sequence and, for each n, $W_n = U_1 + \ldots + U_n, \quad n \geq 1$.

It is given that $W_n = \dfrac{k^n}{3^{n-1}} - 3$, where k is a non-zero constant, $k \neq 3$.

(i) Find an expression for U_n in terms of k and n, and prove that the sequence is geometric. [4]

(ii) Determine the range of values of k such that W, the sum to infinity of the series W_n, exists. [1]

(b) A sequence $u_1, u_2, u_3, \ldots$ is such that $u_r = \dfrac{1}{r!}, \quad r \geq 1$.

(i) Show, by writing out all the terms of the summation, that $\sum_{r=1}^{4}(u_r - u_{r+1}) = u_1 - u_5$. [1]

It can be shown that $u_r - u_{r+1} = \dfrac{1}{r! + (r-1)!}$ for $r \geq 1$.

(ii) Hence show that $\sum_{r=1}^{N} \dfrac{1}{r! + (r-1)!} = 1 - \dfrac{1}{(N+1)!}$. [2]

(iii) Hence show that $\dfrac{1}{2!+1!} + \dfrac{1}{3!+2!} + \dfrac{1}{4!+3!} + \ldots = \dfrac{1}{2}$. [3]

10 The plane p has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, where λ, μ are parameters.

(a) Use a vector product to show that $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ is normal to p. Hence find the cartesian equation of p. [3]

The line l has equation $\mathbf{r} = \begin{pmatrix} t \\ 1 \\ -2 \end{pmatrix} + s \begin{pmatrix} -t \\ t^2 \\ 2t+4 \end{pmatrix}$, where s is a parameter and t is an unknown constant.

7

(b) Show that, for all values of t, l is not perpendicular to p. [3]

(c) Given that l does not intersect p, show that $t = -2$. [4]

(d) For $t = -2$, find the shortest distance between l and p. [3]

11 At a local prawn farming kelong, the prawn population, P, at the time t months after the opening of the farm, is measured by Mr Xia, the prawn farmer.

(a) Mr Xia proposes that the rate of change of P is proportional to P. Write down a differential equation for this model. [1]

When the farm first opened, there were 2000 prawns. After 12 months, there are 3000 prawns.

(b) Solve the differential equation in part **(a)**. Comment on the suitability of this model. [5]

Mr Xia realises that he needs to take into consideration the size of his kelong, and proposes an alternative model

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{N}\right) \quad ---(*),$$

where r is a positive constant, and N is a positive integer such that $0 < P < N$.

(c) Show that $\dfrac{d^2P}{dt^2} = r^2P\left(1 - \dfrac{P}{N}\right)\left(1 - \dfrac{2P}{N}\right)$. [2]

$rP(1-\frac{P}{N}) \times r(1-\frac{2P}{N})$

$\frac{dP}{dt} \times \frac{d^2P}{dt^2}$

(d) Hence show that if the rate of growth of prawn population is at a maximum, then $P = \dfrac{N}{2}$. [You do not need to verify that this value of P gives a maximum.]

Furthermore, find the corresponding rate of growth, leaving your answer in terms of N. [3]

After some time, Mr Xia wants to sell some of the prawns every month at a constant rate of h prawns per month.

(e) Modify the differential equation (*) to take into account the monthly sale of prawns. [1]

(f) What happens in the long run if h is larger than the rate of growth found in part **(d)**? [1]