MOE H3 Math Mathematical Proofs and Reasoning

Problem Set 4

1. Show that there exists integers x and y that satisfy

$$(2n+1)x + (9n+4)y = 1$$

for every integer n.

2. Prove that for every pair of irrational numbers p and q such that $p < q$, there is an irrational x such that $p < x < q$.

3. Show that there is one and only one integer t such that $t, t+2, t+4$ are all prime numbers.

4. Given n real numbers $a_1, a_2, \ldots, a_n$. Show that there exists an a_i $(1 \leq i \leq n)$ such that a_i is greater than or equal to the mean (average) value of the n numbers.

5. Prove that there are infinitely many prime numbers that are congruent to 3 modulo 4.

6. Prove that, for any positive integer n, there is a perfect square m^2 (m is an integer) such that $n \leq m^2 \leq 2n$.

7. (Y2022 Q3)
For any real number s, the greatest integer less than or equal to s is denoted by $\lfloor s \rfloor$. For example, $\lfloor 3.7 \rfloor = 3$ and $\lfloor 5 \rfloor = 5$.

   (a) For any positive integer a and real number t, it is given that t can be written as $an + p$, where n is an integer and $a > p \geq 0$. Prove that

$$\int_0^a \left\lfloor \frac{x+t}{a} \right\rfloor dx = t.$$

   (b) For any positive integer a and b and real number x

      (i) prove that

$$\left\lfloor \frac{\left\lfloor \frac{x}{a} \right\rfloor}{b} \right\rfloor = \left\lfloor \frac{x}{ab} \right\rfloor,$$

      (ii) find

$$\int_0^{ab} (fg(x) - gf(x))dx$$

      where $f(x) = \left\lfloor \dfrac{x+a}{b} \right\rfloor$ and $g(x) = \left\lfloor \dfrac{x+b}{a} \right\rfloor$.

8. (Y2024 Q7)
Let $S = \{1, 2, ..., 50\}$ and let D be a subset of S of size 27.

(a) Show that there are 25 subsets of S of the form $\{a, a+5\}$ whose union is S.
Apply the pigeonhole principle to prove that D must contain two numbers that differ by exactly 5.

(b) Prove that D must contain two numbers that differ by exactly 6.
Show that D does not necessarily contain two numbers that differ by exactly 7.

(c) Determine the maximum possible size of a subset of S that contains no four consecutive numbers.

(d) Determine the maximum possible size of a subset of S that contains no two numbers whose sum is a multiple of 10.

9. A function $f : X \to Y$ is said to be *injective* if

$$\forall x_1, x_2 \in X, \quad f(x_1) = f(x_2) \implies x_1 = x_2.$$

A function $f : X \to Y$ is said to be *surjective* if

$$\forall y \in Y, \quad \exists x \in X \text{ such that } f(x) = y.$$

Use the pigeonhole principle and proof by contradiction to prove: given nonempty finite sets X and Y with $|X| = |Y|$, a function $f : X \to Y$ is injective if and only if it is surjective.

Hints

1. *Hint for Question 1.* Since this is must be true for all n, compare coefficients of n.
2. *Hint for Question 3.* Existence part is by construction; uniqueness part: show that one of $t, t+2, t+4$ must be divisible by 3.
3. *Hint for Question 4.* Prove by contradiction by assuming all the n numbers are less than the average value.
4. *Hint for Question 5.* The idea is similar to the proof for infinitely many primes. Use the fact that integers of the form $4k+1$ are closed under multiplication.
5. *Hint for Question 6.* Prove by contradiction.
6. *Hint for Question 7.* For part (b), write $x = kab + r$ for some integer k and $0 \leq r < ab$.