

5

DIFFERENTIATION

SYLLABUS

- Graphical interpretation of
 - (i) $f'(x) > 0$, $f'(x) = 0$ and $f'(x) < 0$
 - (ii) $f''(x) > 0$ and $f''(x) < 0$
- Relating the graph of $y = f'(x)$ to the graph of $y = f(x)$
- Differentiation of simple functions defined implicitly or parametrically
- Determining the nature of the stationary points (local maximum and minimum points and points of inflexion) analytically, in simple cases, using the first derivative test or the second derivative test
- Locating maximum and minimum points using a graphing calculator or a graphing software
- Finding the approximate value of a derivative at a given point using a graphing calculator or a graphing software
- Problems involving tangents and normals to curves, including cases where the curve is defined implicitly or parametrically
- Local maxima and minima problems
- Connected rates of change problems

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LECTURE 1

Lesson Outline

- Recap on simple differentiation
- Definition of limits and derivative of simple functions from first principles
- Revision on some rules of differentiation

0 RECAP OF 'O' LEVEL DIFFERENTIATION

Type of function	Derivative
Constants	$\frac{d}{dx}(c) = 0$, where c is a constant
Polynomials	$\frac{d}{dx}(x^n) = nx^{n-1}$, where n is a real constant
Logarithmic	$\frac{d}{dx}(\ln x) = \frac{1}{x}$ Type equation here.
Exponential	$\frac{d}{dx}(e^x) = e^x$
Trigonometric	$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$

Self-Practice Exercise 1

Differentiate the following functions with respect to x :

(a) $\sin\left(2x + \frac{\pi}{2}\right)$

(b) $(2x+5)^3$

(c) $e^{x^2} + 2$

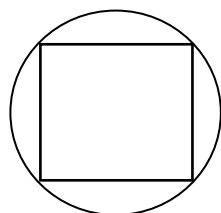
(d) $5x - \ln(3x+4)$

Ans: (a) $2\cos\left(2x + \frac{\pi}{2}\right)$ (b) $3(2x+5)^2(2)$ (c) $2xe^{x^2}$ (d) $5 - \frac{3}{3x+4}$

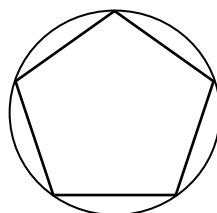
1 LIMITS

Examples

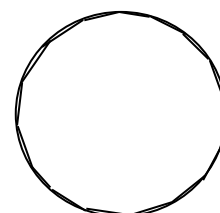
1. A regular polygon of n sides is inscribed in a circle. As n increases infinitely (i.e. as $n \rightarrow \infty$), what is the limit of the polygon?



$n = 4$



$n = 5$



$n \rightarrow \infty$

As $n \rightarrow \infty$, the polygon approaches a circle.

Let A_n be the area of the inscribed polygon with n sides. As n increases, it appears that A_n becomes closer and closer to the area of the circle. We say that the area of the circle, A is the **limit** of the areas of the inscribed polygons, and we write $A = \lim_{n \rightarrow \infty} A_n$.

2. To find the area of region in Figure 1, we approximate the desired area A by considering areas of n rectangles as shown in Figure 2. As n increases, the width of the rectangles decreases. We calculate A as the **limit** of these sums of areas of rectangles (this will be discussed in Chapter 10).

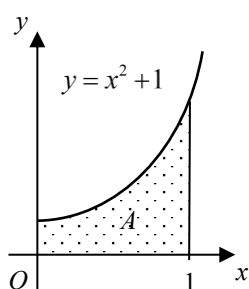


Figure 1

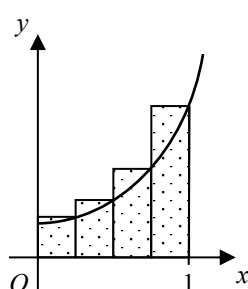


Figure 2

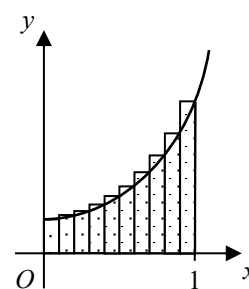
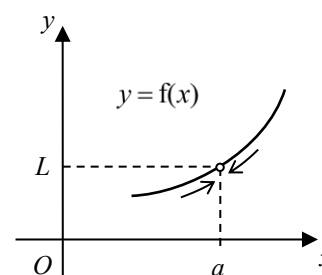


Figure 3

Definition

Given a function $f(x)$, if $f(x)$ approaches a fixed number L as x approaches a , then $\lim_{x \rightarrow a} f(x) = L$.



Example 1

Find the following limits without using the graphing calculator.

$$(a) \lim_{y \rightarrow 2} (y^2 + 5) \qquad (b) \lim_{x \rightarrow 0} \left(\frac{4x^2 + 3x}{x^2 - x} \right) \qquad (c) \lim_{x \rightarrow \infty} \left(\frac{4x^2 + 1}{x^2 - x} \right)$$

Solution

$$(a) \lim_{y \rightarrow 2} (y^2 + 5) \\ = 2^2 + 5 \\ = 9$$

$$(b) \lim_{x \rightarrow 0} \left(\frac{4x^2 + 3x}{x^2 - x} \right) \qquad (c) \lim_{x \rightarrow \infty} \left(\frac{4x^2 + 1}{x^2 - x} \right)$$

$$= \frac{0+3}{0-1} \\ = -3$$

$$= \frac{4+0}{1-0} \\ = 4$$

■

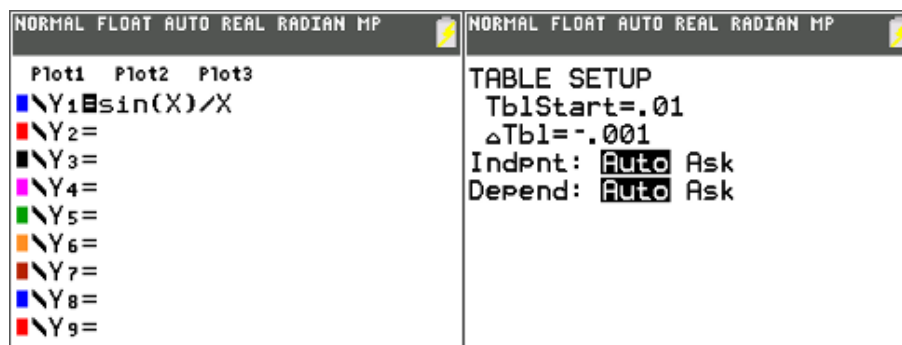
Example 2

Find the following limits using the graphing calculator.

(a) $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)$ (b) $\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right)$ (See Tutorial 10 Q1)

Solution

- 1) Press **[Y=]** and enter the function **sin (x) / x**.
- 2) For table setup, press **[2nd] [WINDOW]**. Change **TblStart** to **0.01** and **ΔTbl** to **-0.001**.



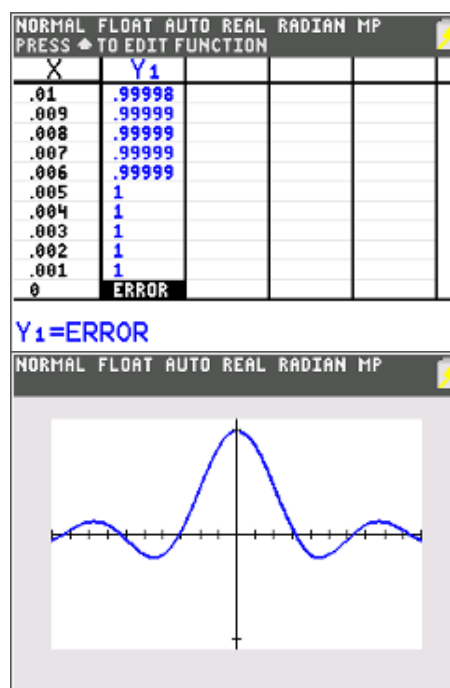
- 3) For table, press **[2nd] [GRAPH]**.

From the table obtained, we observe that when the value of x approaches 0, the function approaches 1.

Hence, $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1$.

Note

The graph of $y = \frac{\sin x}{x}$ has the following shape, but the graph is not defined at $x = 0$.



(a) Using GC, $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) =$

(b) $\lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) =$

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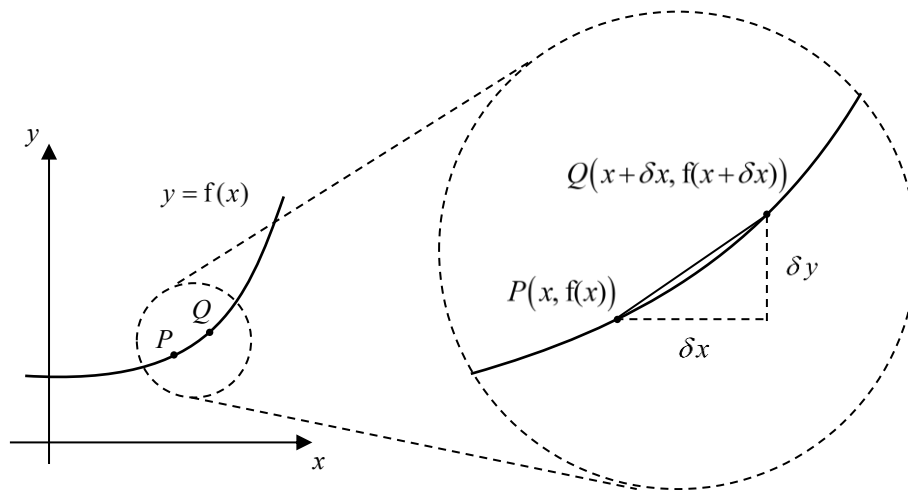
2 FIRST PRINCIPLES

The **gradient of a curve at any point** on the curve is defined as the **gradient of the tangent at that point**.

Let $P(x, f(x))$ be any point on the curve $y = f(x)$.

Let Q be another point on the curve near P ,

i.e., $Q(x + \delta x, f(x + \delta x))$ where δx is a small increment to the value of x .



$$\text{Gradient of the secant line } PQ = \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}.$$

As $Q \rightarrow P$, i.e. $\delta x \rightarrow 0$

The secant line $PQ \rightarrow$ The tangent at P

Gradient of the secant line $PQ \rightarrow$ Gradient of the tangent at P

Gradient of the curve at $P =$ gradient of the tangent at P ,

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Notation

1. We write $\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$ as $\frac{dy}{dx}$ or $\frac{d}{dx}(f(x))$ or $f'(x)$.
2. The process of obtaining $\frac{dy}{dx}$ is called differentiation.
3. $\frac{dy}{dx}$ is read as 'first derivative of y with respect to x '. It is the rate of change of y with respect to x .

This process is called differentiating from **First Principles**.

GeoGebra demonstration



<https://www.geogebra.org/m/swsuu5en>

4. $\frac{d}{dx}$ is an operator and is **not** a fraction or ratio and $\frac{dy}{dx} = \frac{d}{dx}(y)$.
5. To denote the value of the derivative at a specific point $x = a$, we may write $\left. \frac{dy}{dx} \right|_{x=a}$ or $f'(a)$.

Example 3

By considering the derivative as a limit, find the derivative of

- (i) x^2 (ii) $\frac{1}{x}$
- (iii) $\sin x$ (iv) e^x

Solution

(i) Let $f(x) = x^2$ (ii) $\frac{d}{dx} \left(\frac{1}{x} \right)$

$$f'(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{(x + \delta x)^2 - x^2}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{x^2 + 2x\delta x + (\delta x)^2 - x^2}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{\delta x(2x + \delta x)}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} (2x + \delta x)$$

$$= 2x$$

$$= \lim_{\delta x \rightarrow 0} \frac{\frac{1}{x + \delta x} - \frac{1}{x}}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{x - (x + \delta x)}{x(x + \delta x)\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{-\delta x}{x(x + \delta x)\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \frac{-1}{x(x + \delta x)}$$

$$= -\frac{1}{x^2}$$

Using the factor
formula: $\sin P - \sin Q =$
 $2 \cos \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$
(not in syllabus)

(iii) $\frac{d}{dx}(\sin x) = \lim_{\delta x \rightarrow 0} \frac{\sin(x + \delta x) - \sin x}{\delta x}$

$$= \lim_{\delta x \rightarrow 0} \frac{2 \cos \left(x + \frac{\delta x}{2} \right) \sin \frac{\delta x}{2}}{\delta x}$$

$$= \lim_{\delta x \rightarrow 0} \cos \left(x + \frac{\delta x}{2} \right) \lim_{\delta x \rightarrow 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$$= \lim_{\delta x \rightarrow 0} \cos \left(x + \frac{\delta x}{2} \right) \quad \text{since } \lim_{\delta x \rightarrow 0} \frac{\sin \frac{\delta x}{2}}{\frac{\delta x}{2}} = 1 \text{ (from Example 2(a))}$$

$$= \cos x$$

(iv) $\frac{d}{dx}(e^x) = \lim_{\delta x \rightarrow 0} \frac{e^{x + \delta x} - e^x}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{e^x(e^{\delta x} - 1)}{\delta x}$

$$= e^x \lim_{\delta x \rightarrow 0} \frac{e^{\delta x} - 1}{\delta x}$$

$$= e^x \quad \text{since } \lim_{\delta x \rightarrow 0} \left(\frac{e^{\delta x} - 1}{\delta x} \right) = 1 \text{ (from Example 2(b))}$$

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3 SOME RULES OF DIFFERENTIATION

General Results	Self-Practice Exercise 2
Sum of Functions $\frac{d}{dx}(f(x) \pm g(x))$ $= \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ $= f'(x) \pm g'(x)$	$y = x^{13} + x^5 - x^2 + 1$ $\Rightarrow \frac{dy}{dx} =$
Reciprocal of $\frac{dy}{dx}$ $\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$	$x = 2y + 6 \Leftrightarrow y = \frac{1}{2}x - 3$ $\Rightarrow \frac{dx}{dy} = \quad \text{and} \quad \frac{dy}{dx} =$
Product Rule $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$	$y = (3x^2 + 1)(2x + 7)$ $\Rightarrow \frac{dy}{dx} =$
Quotient Rule $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ (This result can be proven using the Product Rule.)	$y = \frac{x}{x^2 + 1}$ $\Rightarrow \frac{dy}{dx} =$
Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ i.e., $\frac{d}{dx}(fg(x)) = f'(g(x))g'(x)$	$y = \sqrt{x^2 + 5x + 3}$ $\Rightarrow \frac{dy}{dx} =$

Chain Rule for Composite Functions

Consider the composite function $y = fg(x)$.

Let $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{du} = f'(u) \quad \text{and} \quad \frac{du}{dx} = g'(x)$$

Hence,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = f'(u) \times g'(x) = f'(g(x))g'(x).$$

In using the Chain Rule, we work from the outside to the inside, i.e., we differentiate the outer function f and then we multiply by the derivative of the inner function.

$$\frac{d}{dx} \left(\underbrace{f}_{\text{outer function}} \left(\underbrace{g(x)}_{\text{evaluated at inner function}} \right) \right) = \underbrace{f'}_{\text{derivative of outer function}} \left(\underbrace{g(x)}_{\text{evaluated at inner function}} \right) \times \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Self-Practice Exercise 3

Find

$$(a) \frac{d}{dx} \left(x(x-1)^7 \right) \quad (b) \frac{d}{dx} \left(\frac{1}{(e^x + 1)^4} \right) \quad (c) \frac{d}{dx} \left(\frac{2 \sin x + 5}{\sqrt{x-3}} \right)$$

Solution

$$\begin{aligned} (a) \quad \frac{d}{dx} \left(x(x-1)^7 \right) &= \\ &= (x-1)^6 (7x + (x-1)) \\ &= (x-1)^6 (8x-1) \end{aligned}$$

$$(b) \quad \frac{d}{dx} \left(\frac{1}{(e^x + 1)^4} \right) = \frac{d}{dx} \left((e^x + 1)^{-4} \right)$$

$$= \frac{-4e^x}{(e^x + 1)^5}$$

$$\begin{aligned} (c) \quad \frac{d}{dx} \left(\frac{2 \sin x + 5}{\sqrt{x-3}} \right) &= \frac{4(x-3) \cos x - (2 \sin x + 5)}{2(x-3)^{\frac{3}{2}}} \quad \blacksquare \end{aligned}$$

Extra Practice

Annex A: Practice Question 1(a) – (e)

Lecture 2

Lesson Outline

- Discuss higher order derivatives
 - Discuss derivatives of trigonometric, inverse trigonometric, logarithmic and exponential functions
-

4 HIGHER ORDER DERIVATIVES, $\frac{d^n y}{dx^n}$

r	$\frac{d^r y}{dx^r}$	Description
1	$\frac{dy}{dx} = f'(x)$	1 st derivative
2	$\frac{d^2 y}{dx^2} = f''(x)$	2 nd derivative
3	$\frac{d^3 y}{dx^3} = f'''(x)$	3 rd derivative
4	$\frac{d^4 y}{dx^4} = f^{(4)}(x)$	4 th derivative
n	$\frac{d^n y}{dx^n} = f^{(n)}(x)$	n -th derivative

If $y = f(x)$ and $n \in \mathbb{Z}^+$, then

$$\frac{d^n y}{dx^n} = \frac{d}{dx} \left(\frac{d^{n-1} y}{dx^{n-1}} \right) = \frac{d}{dx} \left(f^{(n-1)}(x) \right).$$

For example, $y = 4x^3 + 3x^2 + 2x + 1$,

$$\frac{dy}{dx} = 12x^2 + 6x + 2,$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (12x^2 + 6x + 2) = 24x + 6.$$

Note that $\frac{d^2 y}{dx^2} \neq \left(\frac{dy}{dx} \right)^2$.

5 DERIVATIVES OF STANDARD FUNCTIONS

5.1 Derivatives of Trigonometric Functions

The second column is obtained by using chain rule.

$\frac{d}{dx} \sin x = \cos x$	$\frac{d}{dx} \sin(f(x)) = f'(x) \cos(f(x))$
$\frac{d}{dx} \cos x = -\sin x$	$\frac{d}{dx} \cos(f(x)) = -f'(x) \sin(f(x))$
$\frac{d}{dx} \tan x = \sec^2 x$	$\frac{d}{dx} \tan(f(x)) = f'(x) \sec^2(f(x))$
$\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$	$\frac{d}{dx} \cot(f(x)) = -f'(x) \operatorname{cosec}^2(f(x))$
$\frac{d}{dx} \sec x = \sec x \tan x$ (in MF27)	$\frac{d}{dx} \sec(f(x)) = f'(x) \sec(f(x)) \tan(f(x))$
$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$ (in MF27)	$\frac{d}{dx} \operatorname{cosec}(f(x))$ $= -f'(x) \operatorname{cosec}(f(x)) \cot(f(x))$

Notes

- all angles represented are in **radians**
- $\sin^n x = (\sin x)^n \neq \sin(x^n)$, $n \in \mathbb{Z}^+$. Thus

$$\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x, n \in \mathbb{Z}^+.$$

Similarly,

$$\frac{d}{dx} \cos^n x = -n \cos^{n-1} x \sin x, n \in \mathbb{Z}^+$$

$$\frac{d}{dx} \tan^n x = n \tan^{n-1} x \sec^2 x, n \in \mathbb{Z}^+$$

Example 4Differentiate the following functions with respect to x :

(a) $\sin x^\circ$ (b) $\tan x^2 \cot^3 x$ (c) $\cos(x^2 \sin x)$

Solution

(a) $\frac{d}{dx} \sin x^\circ =$

$$= \frac{\pi}{180} \cos\left(\frac{\pi}{180} x\right)$$

(b) $\frac{d}{dx} \tan x^2 \cot^3 x =$

$$= \cot^2 x (2x \sec^2 x^2 \cot x - 3 \tan x^2 \operatorname{cosec}^2 x)$$

(c) $\frac{d}{dx} \cos(x^2 \sin x) =$

$$= -\sin(x^2 \sin x)(2x \sin x + x^2 \cos x)$$
 ■

Extra Practice

Annex A: Practice Question 1(f) and (g)

5.2 Derivatives of Inverse Trigonometric Functions

The second column is obtained by using chain rule.

$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ (in MF27)	$\frac{d}{dx} \sin^{-1}(f(x)) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$
$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$ (in MF27)	$\frac{d}{dx} \cos^{-1}(f(x)) = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$
$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$ (in MF27)	$\frac{d}{dx} \tan^{-1}(f(x)) = \frac{f'(x)}{1+(f(x))^2}$

Note $\sin^{-1} x$ is an inverse function, whereas $\frac{1}{\sin x} = \operatorname{cosec} x$.

Example 5

Find $\frac{dy}{dx}$ for the following.

(a) $y = \cos^{-1} 2x$ (b) $y = \tan^{-1}(\sin x)$ (c) $y = \sin^{-1} \sqrt{1-x^2}$

Solution

$$(a) \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-(2x)^2}} \frac{d}{dx}(2x) = -\frac{2}{\sqrt{1-(2x)^2}}$$

$$(b) \quad \frac{dy}{dx} = \frac{\cos x}{1+\sin^2 x}$$

$$(c) \quad \frac{dy}{dx} = \frac{-x(1-x^2)^{-\frac{1}{2}}}{\sqrt{1-(1-x^2)}}$$

$$= -\frac{x}{|x|\sqrt{1-x^2}} \quad \left(\because \sqrt{x^2} = |x| \right)$$

■

Extra Practice

Annex A: Practice Question 1(h) and (i)

5.3 Derivatives of Logarithmic Functions

The second column is obtained by using chain rule.

$\frac{d}{dx} \ln x = \frac{1}{x}$	$\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$
$\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \quad (a > 1)$	$\frac{d}{dx} \log_a(f(x)) = \frac{f'(x)}{f(x) \ln a} \quad (a > 1)$

Note

The second formula is derived using the change of base formula for logarithms.

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx} \left(\frac{\ln x}{\ln a} \right) = \frac{1}{\ln a} \left(\frac{1}{x} \right) = \frac{1}{x \ln a}$$

Example 6

Differentiate with respect to x .

(a) $\ln(\tan x)$ (b) $\ln \sqrt{\frac{x^2-1}{x^2+1}}$ (c) $\log_5(x+x^2)$

Solution

(a) $\frac{d}{dx} \ln(\tan x) = \quad \quad \quad = \cot x \sec^2 x$

(b) $\frac{d}{dx} \left(\ln \sqrt{\frac{x^2-1}{x^2+1}} \right) =$

$$= \frac{1}{2} \left(\frac{2x}{x^2-1} - \frac{2x}{x^2+1} \right) = \frac{2x}{x^4-1}$$

(c) $\frac{d}{dx} \log_5(x+x^2) = \quad \quad \quad = \frac{1}{\ln 5} \left(\frac{1+2x}{x+x^2} \right) \quad \blacksquare$

Extra Practice

Annex A: Practice Question 1(j) and (k)

5.4 Derivatives of Exponential Functions

The second column is obtained by using chain rule.

$\frac{d}{dx} e^x = e^x$	$\frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}$
$\frac{d}{dx} a^x = a^x \ln a$ (Proof in Example 8b)	$\frac{d}{dx} a^{f(x)} = f'(x)a^{f(x)} \ln a$

Example 7

Differentiate the following with respect to x

(a) $e^{x+\sin x}$

(b) $8^{\sin x}$

Solution

(a) $\frac{d}{dx}(e^{x+\sin x}) = e^{x+\sin x} \frac{d}{dx}(x + \sin x) = e^{x+\sin x}(1 + \cos x)$

(b) $\frac{d}{dx}(8^{\sin x}) = (8^{\sin x} \ln 8) \frac{d}{dx} \sin x = (8^{\sin x} \ln 8) \cos x$ ■

LECTURE 3

Lesson Outline

- Differentiation of simple functions defined implicitly or parametrically

6 DERIVATIVES OF FUNCTIONS DEFINED IMPLICITLY

Notice that all differentiation carried out so far only involved equations of the form $y = f(x)$.

How then can an equation such as $y + xy + y^2 = 2$ be differentiated with respect to x ?

Exercise

Differentiate the following with respect to x .

- (a) y (b) y^2 (c) xy

Solution

$$(a) \quad \frac{d}{dx} y = \frac{dy}{dx}$$

$$(b) \quad \frac{d}{dx} y^2 = \frac{d}{dx} (y)^2 = 2y \frac{dy}{dx} \quad (\text{chain rule})$$

$$(c) \quad \frac{d}{dx} xy = x \frac{dy}{dx} + y \quad (\text{product rule}) \quad \blacksquare$$

Since it is not easy to make y the subject of the equation, i.e., y cannot be expressed explicitly in terms of x , we would do the differentiation implicitly with respect to x as follows:

$$\begin{aligned} y + xy + y^2 &= 2 \\ \Rightarrow \frac{d}{dx}(y) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(2) \\ \Rightarrow \frac{dy}{dx} + \left(x \frac{dy}{dx} + y \right) + 2y \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx}(1 + x + 2y) &= -y \\ \therefore \frac{dy}{dx} &= \frac{-y}{1 + x + 2y} \end{aligned}$$

Further ExercisesDifferentiate the following with respect to x .

(a) $x \frac{dy}{dx}$ (b) $\left(\frac{dy}{dx}\right)^2$ (c) $\sin y$

Solution

(a) $\frac{d}{dx}\left(x \frac{dy}{dx}\right) = x \frac{d^2y}{dx^2} + \frac{dy}{dx}$ (Product rule)

(b) $\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2 \frac{dy}{dx} \frac{d^2y}{dx^2}$ (Chain rule)

(c) $\frac{d}{dx} \sin y = \cos y \frac{dy}{dx}$ (Chain rule) ■

Example 8

Derive the following formulas using implicit differentiation.

(a) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ (b) $\frac{d}{dx}(a^x) = a^x \ln a$

Solution

Note that
 $\sin^{-1} x \neq \frac{1}{\sin x}$ and
 $\frac{1}{\sin x} = \operatorname{cosec} x$.

(a) Let $y = \sin^{-1} x \Rightarrow \sin y = x$

Differentiate implicitly w.r.t. x gives

$$\cos y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}.$$

Using $\sin^2 y + \cos^2 y = 1$, we have $\cos y = \pm\sqrt{1-\sin^2 y}$.Since $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$, i.e. $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$,

$$\therefore \cos y \geq 0 \Rightarrow \cos y = +\sqrt{1-\sin^2 y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

(b) Let $y = a^x$.

Taking natural logarithm gives

$$\ln y = x \ln a$$

$$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln a)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \ln a$$

$$\Rightarrow \frac{dy}{dx} = y \ln a$$

$$\therefore \frac{d}{dx}(a^x) = a^x \ln a$$

Whenever an exponent contains a variable, we can take natural logarithms on both sides before differentiating the equation.

■

Extra Practice

Annex A: Practice Question 1(l)

Example 9

Differentiate the following with respect to x .

(a) $y - x + 3e^{-y} = 3$

(b) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{4}$

(c) $y^x = x^y$

(d) $xy^2 = x^{\sin x}$

Solution

(a) $y - x + 3e^{-y} = 3$

Differentiate with respect to x ,

$$\frac{dy}{dx} = \frac{1}{(1 - 3e^{-y})}$$

(b) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{4}$

Differentiate with respect to x ,

$$\frac{dy}{dx} = -\frac{y^3}{x^3}$$

(c) $y^x = x^y$

Take natural logarithm gives $x \ln y = y \ln x$.

Differentiate with respect to x ,

$$\frac{dy}{dx} \left(\frac{x}{y} - \ln x \right) = \frac{y}{x} - \ln y$$

$$\frac{dy}{dx} = \frac{\frac{y}{x} - \ln y}{\frac{x}{y} - \ln x} = \frac{y^2 - xy \ln y}{x^2 - xy \ln x}$$

(d) $xy^2 = x^{\sin x}$

Take natural logarithm gives

Differentiate the equation with respect to x ,

$$\frac{1}{x} + \frac{2}{y} \left(\frac{dy}{dx} \right) = \cos x \ln x + \sin x \left(\frac{1}{x} \right)$$

Rearranging the terms,

$$\frac{dy}{dx} = \left(\cos x \ln x + \sin x \left(\frac{1}{x} \right) - \frac{1}{x} \right) \left(\frac{y}{2} \right)$$

■

Example 10

Given the curve $x^3 + y^3 = 2y$ such that $y \neq \pm\sqrt{\frac{2}{3}}$, evaluate $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ at the point $(1, 1)$.

Solution

$$x^3 + y^3 = 2y$$

Differentiate with respect to x ,

$$3x^2 + (3y^2 - 2)\frac{dy}{dx} = 0 \quad \text{---(1)}$$

Differentiate (1) with respect to x ,

$$(3y^2 - 2)\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 6x = 0 \quad \text{---(2)}$$

Differentiate (2) with respect to x ,

$$(3y^2 - 2)\frac{d^3y}{dx^3} + 18y\frac{dy}{dx}\frac{d^2y}{dx^2} + 6\left(\frac{dy}{dx}\right)^3 + 6 = 0 \quad \text{---(3)}$$

At $(1, 1)$,

$$(1): \quad 3(1^2) + (3(1^2) - 2)\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -3$$

$$(2): \quad (3 - 2)\frac{d^2y}{dx^2} + 6(1)(-3)^2 + 6(1) = 0 \Rightarrow \frac{d^2y}{dx^2} = -60$$

$$(3): \quad \Rightarrow \frac{d^3y}{dx^3} = -3084 \quad \blacksquare$$

Extra Practice

Annex A: Practice Question 2

7 DERIVATIVES OF FUNCTIONS DEFINED PARAMETRICALLY

In general, if a curve is defined parametrically by $x = f(t)$, $y = g(t)$, where t is a parameter, then the derivative $\frac{dy}{dx}$ can be found by **applying the chain rule**

$$\frac{dy}{dx} \times \frac{dx}{dt} = \frac{dy}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}.$$

Example 11

The parametric equations of a curve C are

$$x = e^t \sin t \text{ and } y = e^t \cos t \text{ where } -\pi \leq t \leq \pi.$$

- (a) Find the gradient(s) of the curve when $x = 0$.
 (b) Sketch the curve C , showing clearly the axial intercepts.

Solution

$$\begin{aligned} \text{(a)} \quad \frac{dx}{dt} &= & \frac{dy}{dt} &= \\ &= e^t (\sin t + \cos t) &= e^t (\cos t - \sin t) \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\cos t - \sin t}{\cos t + \sin t}$$

When $x = 0$,

$$\therefore \frac{dy}{dx} = 1$$

- (b) Intercepts:

When $x = 0$, $t = -\pi, 0, \pi$,

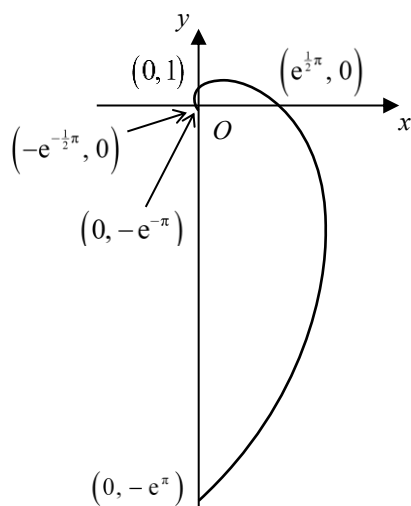
$$\therefore y = -e^{-\pi}, 1, -e^{\pi} \quad (y = -0.0432, 1, -23.1)$$

When $y = 0$,

$$\therefore x = -e^{-\frac{1}{2}\pi}, e^{\frac{1}{2}\pi} \quad (x = -0.208, 4.81)$$

The intercepts are,

$$(0, -e^{-\pi}), (0, 1), (0, -e^{\pi}), (-e^{-\frac{1}{2}\pi}, 0), (e^{\frac{1}{2}\pi}, 0)$$

**Extra Practice**

Annex A: Practice Question 3a, b and c

Example 12 [Prelim YJC P2 Q4 modified]

The parametric equations of a curve are $x = t + \ln t$ and $y = t + e^t$, $t > 0$. Show that the curve has no stationary point. Sketch the curve, indicating clearly all intercepts and asymptotes.

Solution

$$\frac{dx}{dt} = \quad \quad \quad \frac{dy}{dt} =$$

$$\therefore \frac{dy}{dx} = \frac{1+e^t}{1+\frac{1}{t}} = \frac{t(1+e^t)}{t+1}$$

Since $t > 0$, $\frac{dy}{dx} = \frac{t(1+e^t)}{t+1} > 0$. Thus the curve has no stationary point.

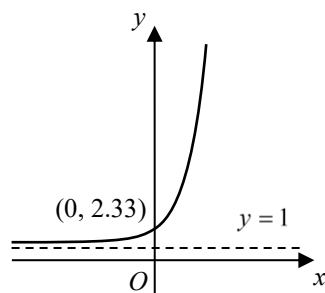
Intercept:

When $x = 0$, $t + \ln t = 0$, by GC, $t = 0.56714$, $\therefore y = 2.33$

Asymptotes:

Note that since $t > 0$, $y > 1$.

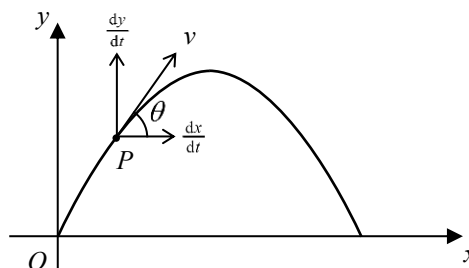
Consider as $x \rightarrow -\infty$, $t \rightarrow 0$, $\therefore y \rightarrow 1$. Therefore $y = 1$ is an asymptote.



7.1 Application to Motion of an Object

The direction at any time during the flight of a projectile can be analysed by considering the horizontal and vertical components of its velocity,

i.e. $\frac{dx}{dt}$ in the direction Ox , $\frac{dy}{dt}$ in the direction Oy .



If v is the velocity of the particle at time t , the direction of v makes an angle θ with Ox where $\tan \theta = \frac{dy}{dt} / \frac{dx}{dt} = \frac{dy}{dx}$.

If a particle is moving on a curve and is at the point P on the curve at time t , the velocity of the particle at time t is in the direction of the tangent to the curve at P .

Example 13

A ball is projected from O such that its horizontal and vertical distances from O at time t are $x = 10\sqrt{3}t$, $y = 10t - 4.9t^2$. What is the direction of the ball 2 seconds after projection?

Solution

$$x = 10\sqrt{3}t \qquad y = 10t - 4.9t^2$$

$$\frac{dx}{dt} = \qquad \frac{dy}{dt} =$$

$$\text{When } t = 2, \quad \frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{10 - (9.8)(2)}{10\sqrt{3}} = -0.554256$$

$$\therefore \tan \theta = -0.554256 \Rightarrow \theta = -29.0^\circ \text{ (1 d.p.)}$$

When $t = 2$, the ball makes an angle of 29.0° below the horizontal. ■

Example 14

A stone is thrown from the top of a cliff and hits the sea 20m from the bottom of the cliff. The position of the projectile at time t is given by $x = 5.79t$, $y = 3.34t - 4.9t^2$. Find the direction in which it is moving when it hits the sea.

Solution

$$x = 5.79t$$

$$y = 3.34t - 4.9t^2$$

$$\frac{dx}{dt} = 5.79$$

$$\frac{dy}{dt} = 3.34 - 9.8t$$

When $x = 20$, $t = 3.4542$,

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = -5.2693$$

$$\Rightarrow \theta = -79.3^\circ$$

The stone makes an angle of 79.3° the horizontal when it hits the sea. ■

LECTURE 4

Lesson Outline

- Discussion on increasing and decreasing functions and concavity.
 - Determination of stationary points and points of inflexion.
-

8 DIFFERENTIATION AND GRAPHS

8.1 Increasing and Decreasing Functions

A function f on (x_1, x_2) is **increasing** if for any $a, b \in (x_1, x_2)$, $f(a) < f(b)$ where $a < b$.

A function f on (x_1, x_2) is **decreasing** if for any $a, b \in (x_1, x_2)$, $f(a) > f(b)$ where $a < b$.

Consider $y = f(x)$ for $x \in (x_1, x_2)$.

If $\frac{dy}{dx} > 0$, then gradient is positive. Therefore y increases as x increases.

Hence $y = f(x)$ is increasing in the interval.

If $\frac{dy}{dx} < 0$, then gradient is negative. Therefore y decreases as x increases.

Hence $y = f(x)$ is decreasing in the interval.

For all $x \in (x_1, x_2)$, for a function f ,

- if $f'(x) > 0$, f is increasing over the interval (x_1, x_2) .
- If $f'(x) < 0$, f is decreasing over the interval (x_1, x_2) .

Example 15

Given that $f(x) = x + \frac{1}{x}$, determine the set of values of x for which f is

- (i) increasing, (ii) decreasing.

Solution

- (i) For f to be increasing, $f'(x) > 0$

$$1 - \frac{1}{x^2} > 0$$

$$\frac{x^2 - 1}{x^2} > 0$$

$$\frac{(x-1)(x+1)}{x^2} > 0$$

$$x < -1 \text{ or } x > 1$$

$$\{x : x \in \mathbb{R}, x < -1 \text{ or } x > 1\}$$

- (ii) For f to be decreasing, $f'(x) < 0$

$$1 - \frac{1}{x^2} < 0$$

$$\{x : x \in \mathbb{R}, x \neq 0, -1 < x < 1\}$$

■

Refer to the video for the graphical approach.

8.2 Concavity

Note that $f''(x) = \frac{d}{dx} f'(x)$ is the rate of change of the gradient with respect to x .

Concavity	2 nd Derivative	1 st Derivative	Shape	Interpretation
Concave Upwards	$f''(x) > 0$	$f'(x) > 0$		as x increases, gradient of $y = f(x)$ increases.
		$f'(x) < 0$		
Concave Downwards	$f''(x) < 0$	$f'(x) > 0$		as x increases, gradient of $y = f(x)$ decreases.
		$f'(x) < 0$		

8.3 Stationary Points

The equation $y = f(x)$ has stationary point at $(a, f(a))$, if $\frac{dy}{dx} = 0$ at $x = a$, i.e., $f'(a) = 0$.

$f(a)$ is also known as the **stationary value**.

Graphically, the tangent to the curve $y = f(x)$ at $(a, f(a))$ is parallel to the x -axis.

First Derivative Test

Given $\frac{dy}{dx} = 0$ at point where $x = a$. We test the value of $\frac{dy}{dx}$ at points around $x = a$.

Note

- Numerical values of a^- and a^+ , and the corresponding values of $\frac{dy}{dx}$ must be stated when using the first derivative test.

x	a^-	a	a^+	a^-	a	a^+	a^-	a	a^+	a^-	a	a^+
$\frac{dy}{dx}$	+	0	-	-	0	+	+	0	+	-	0	-
Shape		—			—			—			—	
Conclusion	$(a, f(a))$ is a maximum point			$(a, f(a))$ is a minimum point			$(a, f(a))$ is a stationary point of inflexion			$(a, f(a))$ is a stationary point of inflexion		

Second Derivative Test

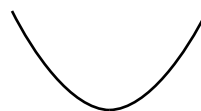
If $\frac{dy}{dx} = 0$ at the point where $x = a$ and

$\frac{d^2y}{dx^2}$	< 0	> 0	$= 0$
Conclusion	$(a, f(a))$ is a maximum point	$(a, f(a))$ is a minimum point	Inconclusive. This does NOT imply a point of inflexion. First derivative test must be done!

Note



Stationary point is maximum if $\frac{dy}{dx}$ decreases as x increases.



Stationary point is minimum if $\frac{dy}{dx}$ increases as x increases.

Example 16

A curve C has equation $y = 0.1x(x-5)(x+5)$.

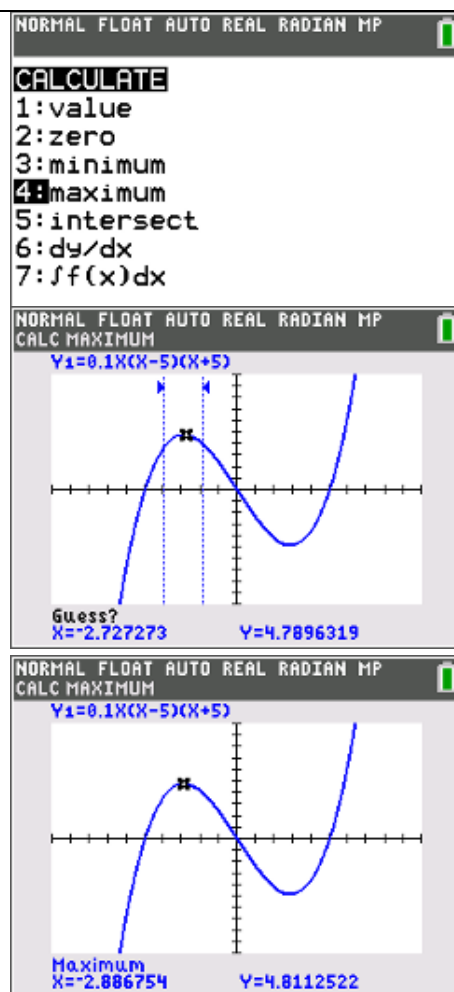
(a) Find the coordinates of the maximum point of C .

(b) Use GC to evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1$.

Solution

(a) To find maximum point using GC

- 1) Enter and plot the function **Y1 = 0.1X(X-5)(X+5)**.
- 2) For **[calc]**, press **[2nd][trace]**. Select **[4:maximum]** to calculate the maximum point of **Y1**.
- 3) You will be prompted to locate the lower bound. Press **[right]** and **[left]** to locate a point just to the left of the desired maximum and press **[ENTER]**.
- 4) You will be prompted to locate the upper bound. Press **[right]** and **[left]** to locate a point just to the right of the desired maximum and press **[ENTER]**.
- 5) You will be prompted to locate a guess at the maximum value. Shift cursor to a point as close as possible to the local maximum (or enter the number directly) and press **[ENTER]**.



Hence, the coordinates of the maximum point are $(-2.89, 4.81)$.

(b) To find $\frac{dy}{dx}$ at $x = 1$,

- 1) At Home Screen, press **[alpha][window]** and select **[3:nDeriv(]**.
- 2) Enter the variable **X** using **[X,T,θ,n]**, the function **0.1X(X-5)(X+5)** and the value **1** accordingly. Press **[ENTER]**.

NORMAL FLOAT AUTO REAL RADIAN MP 1:abs(2:sumation Σ(3:nDeriv(4:fnInt(5:logBASE(6:*J 7:nPr 8:nCr 9:! [FRAC] [Frac] [MTRX] [YVAR]	NORMAL FLOAT AUTO REAL RADIAN MP $\frac{d}{dx}(\text{[]}) _{x=}$	NORMAL FLOAT AUTO REAL RADIAN MP $\frac{d}{dx}(0.1X(X-5)(X+5)) _{X=1}$ ----- -2.1999999
--	--	--

Hence $\frac{dy}{dx}$ at $x = 1$ is -2.2 .

To find $\frac{d^2y}{dx^2}$ at $x = 1$,

- 1) At Home Screen, press **[alpha][window]** and select **[3:nDeriv()]**. Enter the variable **X** and the value **1** accordingly.
- 2) Within the function, press **[alpha][window]** and select **[3:nDeriv()]** again, at **X = X**. Enter the variable **X**, the function **0.1X(X-5)(X+5)** at **X = X** accordingly. Press **[ENTER]**.

NORMAL FLOAT AUTO REAL RADIAN MP $\frac{d}{dx}(\text{[]}) _{x=1}$	NORMAL FLOAT AUTO REAL RADIAN MP $\frac{d}{dx}(\frac{d}{dx}(\text{[]}) _{x=X}) _{x=1}$	NORMAL FLOAT AUTO REAL RADIAN MP $\frac{d}{dx}(\frac{d}{dx}(0.1X(X-5)(X+5)) _{X=X}) _{X=1}$ ----- .6
---	--	---

Hence, $\frac{d^2y}{dx^2}$ at $x = 1$ is 0.6 .

Alternatively, instead of keying the function explicitly into the command, we can key it into **[Y=]**.

- 1) Enter the function **Y1 = 0.1X(X-5)(X+5)**. Press **[2nd][mode]** to go to Home Screen.
- 2) Press **[alpha][window]** and select **[3:nDeriv()]**. Enter the function **X**, **Y1** and **1** respectively. Press **[ENTER]**.
- 3) Press **[alpha][window]** and select **[3:nDeriv()]**. Press **[ENTER]**. Instead of entering the function **Y1**, press **[alpha][window]** and select **[3:nDeriv()]**. Press **[ENTER]** again. (For **Y1**, press **[alpha][trace]** and select **[Y1]**.)

NORMAL FLOAT AUTO REAL RADIAN MP Plot1 Plot2 Plot3 Y1=0.1X(X-5)(X+5) Y2= Y3= Y4= Y5= Y6= Y7= Y8= Y9=	NORMAL FLOAT AUTO REAL RADIAN MP $\frac{d}{dx}(Y1) _{x=1}$ ----- -2.1999999	NORMAL FLOAT AUTO REAL RADIAN MP $\frac{d}{dx}(\frac{d}{dx}(Y1) _{X=X}) _{X=1}$ ----- .6
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8.4 Points of Inflexion

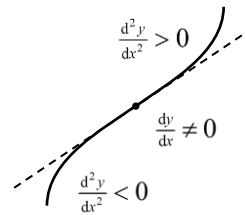
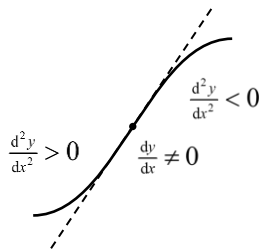
A **point of inflexion** is a point on the curve where the **concavity changes** (i.e., a point at which the curve is changing from concave upwards to concave downwards or vice versa).

At this point, $\frac{d^2y}{dx^2} = 0$ and the sign of the second derivative changes when moving through the point.

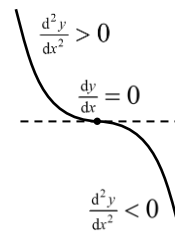
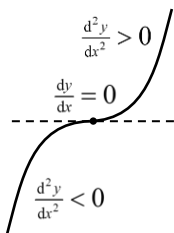
Notes

- A point of inflexion may or may not be a stationary point.
- There is usually a point of inflexion between any 2 consecutive turning points.

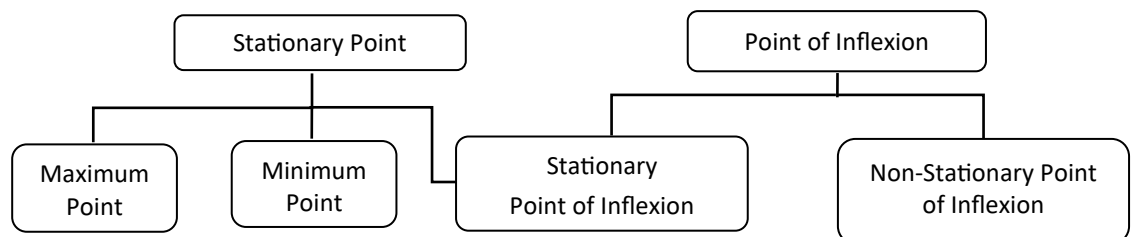
Non-stationary Point of Inflexion (not in syllabus)



Stationary Point of Inflexion



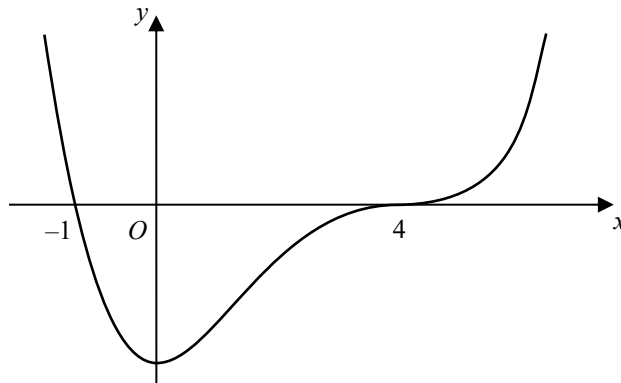
Note



Example 17

Given the graph of $y = f(x)$ in the diagram below, identify by shading the region where the graph is

- (i) concave upwards, (ii) decreasing.



■

Example 18

Given the graph of $y = x^3 + 2x^2 + 1$, find algebraically the range of values of x for which the graph is

- (i) increasing, (ii) concave upwards.

Solution

$$\frac{dy}{dx} =$$

$$\frac{d^2y}{dx^2} =$$

- (i) For the graph to be increasing

$$\frac{dy}{dx} \geq 0 \Rightarrow x > 0 \text{ or } x < -\frac{4}{3}$$

- (ii) For the graph to be concave upwards

$$\frac{d^2y}{dx^2} > 0 \Rightarrow x > -\frac{2}{3}$$

■

LECTURE 5

Lesson Outline

- Relationship between the graphs of $y = f'(x)$ and $y = f(x)$.

9 RELATING THE GRAPH OF $y = f'(x)$ TO THE GRAPH OF $y = f(x)$

Example 19

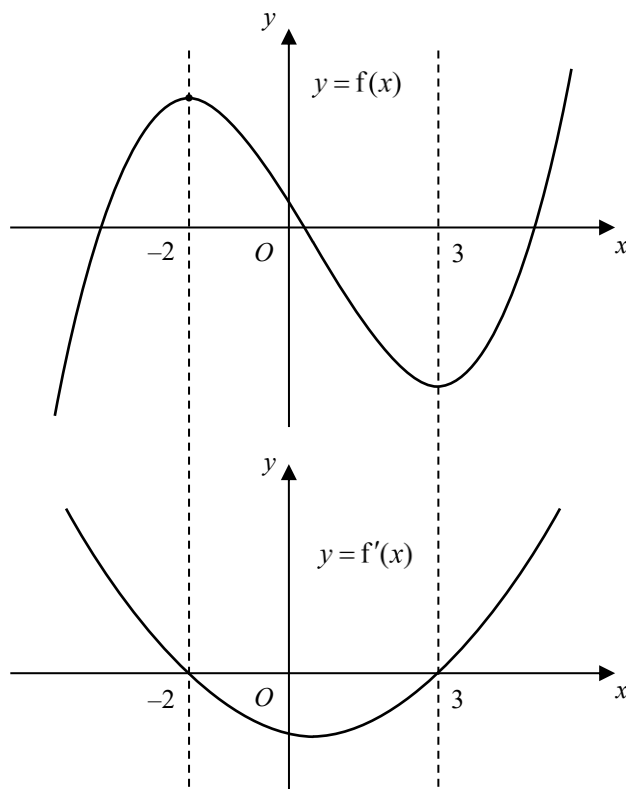
The function f is defined by $f(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 - 3x + 1$. Using a graphing calculator, sketch the graphs of $y = f(x)$ and its derivative function $y = f'(x)$.

- Find the x -coordinates of the turning points of the graph of $y = f(x)$ and write down the x -coordinates of the points of intersection of the graph of $y = f'(x)$ with the x -axis. Compare the results and explain your observations.
- Describe how you could deduce the nature of the turning points of $y = f(x)$ from the graph of $y = f'(x)$.
- For what values of x is the gradient of $y = f'(x)$
 - negative?
 - positive?

For each of the above set of values of x , is the graph of $y = f(x)$ concave upwards or concave downwards? Explain your observations.

Solution

$$f(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 - 3x + 1 \Rightarrow f'(x) = \frac{1}{2}x^2 - \frac{1}{2}x - 3$$



- (a) x -coordinates of the turning points of the graph of $y = f(x)$ are $x = -2, x = 3$.

x -coordinates of the points of intersection of the graph of $y = f'(x)$ with the x -axis are $x = -2, x = 3$.

Observation

The x -coordinates of the turning points on $y = f(x)$ correspond to the x -intercepts on $y = f'(x)$.

- (b) From $y = f'(x)$,

	$x \rightarrow -2^-$	$x = -2$	$x \rightarrow -2^+$
$y = f'(x)$	+	0	−

Therefore, maximum point of $y = f(x)$ occurs at $x = -2$.

	$x \rightarrow 3^-$	$x = 3$	$x \rightarrow 3^+$
$y = f'(x)$	−	0	+

Therefore, minimum point of $y = f(x)$ occurs at $x = 3$.

- (c) (i) gradient of $y = f'(x)$ is negative for $x < \frac{1}{2}$.

- (ii) gradient of $y = f'(x)$ is positive for $x > \frac{1}{2}$.

For $x < \frac{1}{2}$, graph of $y = f(x)$ is concave downwards.

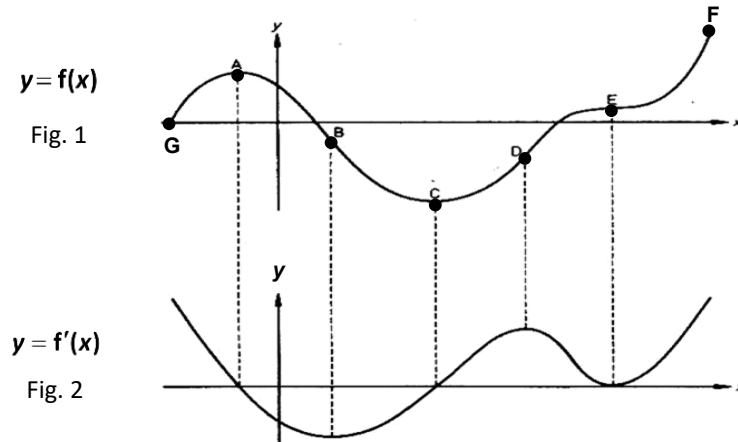
For $x > \frac{1}{2}$, graph of $y = f(x)$ is concave upwards.

Observation

$y = f(x)$ is concave downwards, $y = f'(x)$ is decreasing, since $f''(x) < 0$.

$y = f(x)$ is concave upwards, $y = f'(x)$ is increasing, since $f''(x) > 0$.

The diagram below shows the graph of a function $y = f(x)$ [Fig. 1] and the corresponding graph of the gradient function $y = f'(x)$ [Fig. 2] plotted against x .



Features of $y = f'(x)$ curve

- The x -intercepts of $y = f'(x)$:
 - Points A, C and E which are stationary points on $y = f(x)$ curve correspond to the x -intercepts on $y = f'(x)$ curve, since $f'(x) = 0$ at these points.
- Any asymptotes on $y = f'(x)$:
 - $y = f(x)$ has no asymptotes, hence $y = f'(x)$ has no asymptotes.
- The graph of $y = f'(x)$ is below or above the x -axis:
 - For the regions GA and CF, $y = f(x)$ curve is increasing, hence $y = f'(x)$ is above the x -axis.
 - For the region AC, $y = f(x)$ curve is decreasing, hence $y = f'(x)$ curve is below the x -axis.
- The turning points on $y = f'(x)$:
 - Points of inflexion B, D and E on $y = f(x)$ correspond to stationary points on $y = f'(x)$, i.e., $\frac{d}{dx}[f'(x)] = 0$ at these points.
- The concavity of $y = f'(x)$:
 - For the regions GB and DE, $y = f(x)$ curve is concave downwards, hence $y = f'(x)$ curve is decreasing, since $\frac{d}{dx}[f'(x)] < 0$ in these regions.
 - For the regions BD and EF, $y = f(x)$ curve is concave upwards, hence $y = f'(x)$ curve is increasing, since $\frac{d}{dx}[f'(x)] > 0$ in these regions.

Relationship between the graph of $y = f(x)$ and its derivative graph $y = f'(x)$

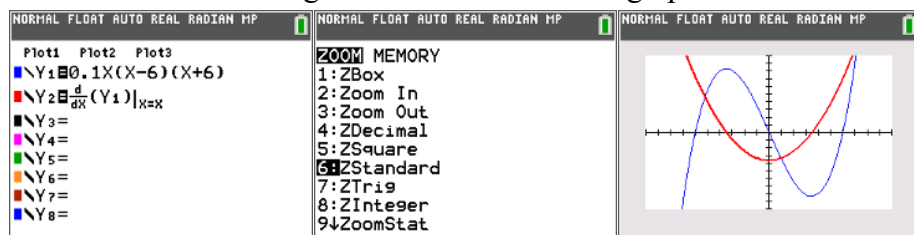
	Graph of $y = f(x)$		Graph of $y = f'(x)$
	Mathematical Feature	Geometrical Feature	
1	Stationary points $\frac{dy}{dx} = 0$ at $x = a$	Max, Min or stationary point of inflexion at $x = a$	Intercept at $x = a$
2(a)	Vertical asymptote at $x = p$		Vertical asymptote at $x = p$
2(b)	Horizontal asymptote at $y = q$		Horizontal asymptote at $y = 0$
2(c)	Oblique asymptote at $y = mx + c$		Horizontal asymptote at $y = m$
3(a)	$\frac{dy}{dx} > 0$ in interval (a, b)	Curve increasing in interval (a, b)	Curve above x -axis
3(b)	$\frac{dy}{dx} < 0$ in interval (a, b)	Curve decreasing in interval (a, b)	Curve below x -axis
4	$\frac{d^2y}{dx^2} = 0$ at $x = k$	Change in curvature of curve	There is a stationary point at $x = k$ i.e., max, min or inflexion point
5(a)	$\frac{d^2y}{dx^2} > 0$ in interval (a, b)	Curve is concave upwards in interval (a, b)	Curve is increasing in interval (a, b)
5(b)	$\frac{d^2y}{dx^2} < 0$ in interval (a, b)	Curve is concave downwards in interval (a, b)	Curve is decreasing in interval (a, b)

Example 20

Given $f(x) = 0.1x(x-6)(x+6)$, sketch the graphs of $y = f(x)$ and $y = f'(x)$.

Solution

- 1) Enter and plot the function **Y1 = 0.1X(X-6)(X+6)**. Press **[ENTER]**.
- 2) Enter and plot the function **Y2 = $\frac{d}{dx}(Y_1)_{x=x}$** . Press **[ENTER]**. For **nDeriv**, press **[alpha][window]** and select **[3:nDeriv()]**. For **Y1**, press **[alpha][trace]**.
- 3) Press **[zoom]** and select **[6:ZStandard]** to set the viewing window back to the default dimensions which are $[-10, 10]$ by $[-10, 10]$, with scale markings at each unit. This will graph both functions.

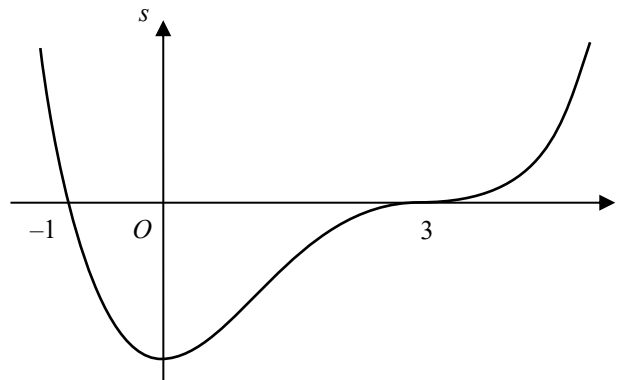
**Example 21**

If a particle is moving in a straight line such that, at time t , its displacement from a fixed point on that line is s , its velocity is v and its acceleration is a , then

$v = \frac{ds}{dt}$ = rate of change of displacement with respect to time, and

$a = \frac{dv}{dt}$ = rate of change of velocity with respect to time

The displacement-time graph of the particle is shown below.

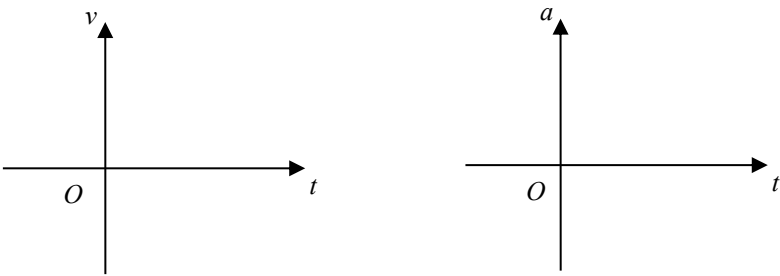


Sketch the

(i) velocity-time graph,

(ii) acceleration-time graph.

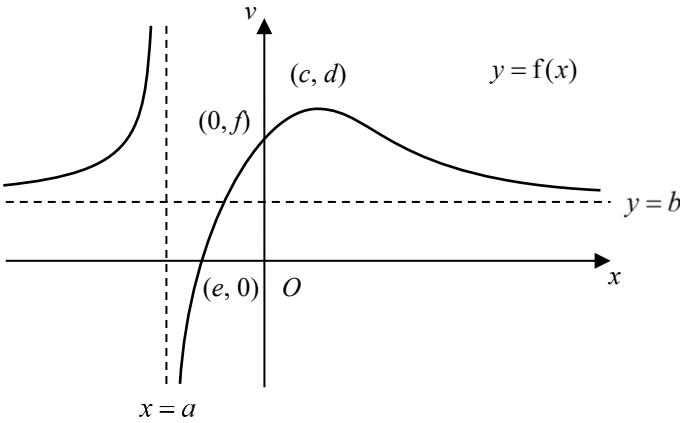
Solution



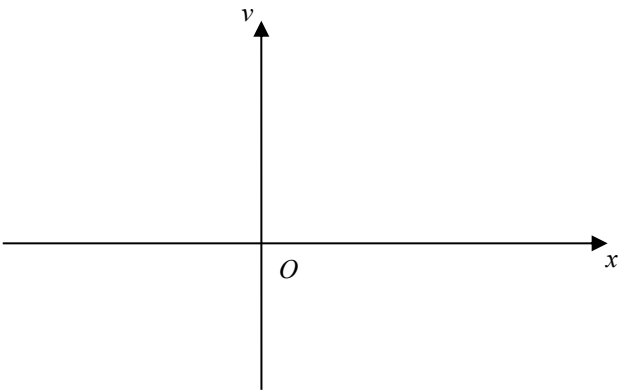
■

Example 22

Given the graph of $y = f(x)$ in the diagram below, sketch the graph of $y = f'(x)$.



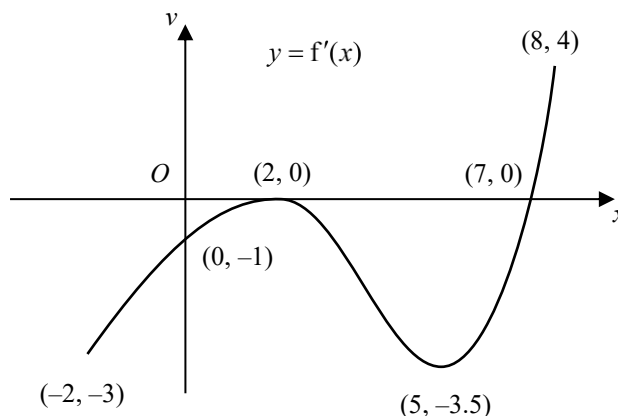
Solution



■

Example 23

The figure below shows the graph of $y = f'(x)$, the derivative of the function f , on the interval $-2 \leq x \leq 8$. The graph of $y = f'(x)$ has horizontal tangent lines at $x = 2$ and $x = 5$.



- (i) Write down the x -coordinates of all the stationary points of $y = f(x)$ for $-2 \leq x \leq 8$.
- (ii) What is the nature of each of the stationary points?
- (iii) Write down the intervals in which the graph of $y = f(x)$ is concave upwards and concave downwards.

Solution

- (i) x -coordinates of stationary point of $y = f(x)$: $x = 2, x = 7$
- (ii) At $x = 2$, it is a point of inflexion. At $x = 7$, it is a minimum point.
- (iii) Concave upwards for $-2 < x < 2$ or $5 < x < 8$, concave downwards for $2 < x < 5$ ■

LECTURE 6

Lesson Outline

- Applications of Differentiation
 - Gradients, tangents and normals
 - Maxima and minima
 - Rates of change
-

10 APPLICATIONS OF DIFFERENTIATION

10.1 Gradients, Tangents and Normals

The equation of the **tangent** to a curve $y = f(x)$ at $P(x_1, y_1)$ is

$$y - y_1 = m(x - x_1),$$

where m is the value of $\frac{dy}{dx}$ when $x = x_1$.

The equation of the **normal** to a curve $y = f(x)$ at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{1}{m}(x - x_1),$$

where m is the value of $\frac{dy}{dx}$ when $x = x_1$.

Heuristics for Tangents

- (A) In general, given a curve C , to find equation of tangent at a specific point (x_1, y_1) on curve C :

Step 1: Find $\frac{dy}{dx}$.

Step 2: **Compute the value of $\frac{dy}{dx}$ at the specific point (x_1, y_1) .**

Step 3: Use formula of line: $y - y_1 = m(x - x_1)$ or $y = mx + c$
(where m is the value of $\frac{dy}{dx}$ from step 2).

- (B) In general, given a curve C with **parametric equations**, to find point(s) on C given some conditions (e.g. tangent is parallel to x -axis), **find the value(s) of parameter t** when the conditions are met:

Step 1: Find $\frac{dy}{dx}$.

Step 2: **Find the value of parameter t using the conditions**
e.g. given that tangent is parallel to x -axis, put $\frac{dy}{dx} = 0$ to
find the value of t .

Step 3: Substitute the value of parameter t into x and y .

These heuristics can be applied for normals as well.

Notes

- (1) When the equations of a curve are given in parametric form, each point on the curve is associated with a parameter t .

E.g. Given a curve with parametric equations $x = \sin t$, $y = \cos t$,
 $0 \leq t \leq \pi$, the point $(0, -1)$ is such that $t = \pi$.

Thus to find any point on the curve, we need to first find the specific value of t .

If the question requires you to find, for example, the point where the tangent is parallel to the x -axis, we need to find the parameter t when this happens. This can be done by equating the gradient of tangent to 0.

- (2) Consider a curve C with the following parametric equations
 $x = k + \sin t$, $y = k \cos t$ where k is a fixed positive integer, $0 \leq t \leq \pi$.

The following are **equivalent statements**:

- (i) Find the equation of the tangent at the point with parameter p .
- (ii) Find the equation of tangent at $t = p$.
- (iii) Find the equation of tangent at the point $(k + \sin p, k \cos p)$.

Point with parameter p is **not** the point (p, p) .

Example 24

The curve C has equation $9x^2 - y^2 + 3xy = 5$.

- (i) Find the equations of the tangent and the normal to C at the point $P(1, -1)$.
- (ii) Find the equations of the tangents that are parallel to the y -axis.

[NJC/2019/Prelim/1/4(Modified)]

Solution

(i)

Steps:	Working
Step 1: Find $\frac{dy}{dx}$.	Differentiating $9x^2 - y^2 + 3xy = 5$ w.r.t. x , $18x - 2y \frac{dy}{dx} + 3x \frac{dy}{dx} + 3y = 0$ $2y \frac{dy}{dx} - 3x \frac{dy}{dx} = 18x + 3y$ $\frac{dy}{dx} = \frac{18x + 3y}{2y - 3x}$
Step 2: Compute the value of $\frac{dy}{dx}$ at $x = 1$.	When $x = 1$, gradient of tangent, $\frac{dy}{dx} = \frac{18 - 3}{-2 - 3} = -3$ Using $m_1 m_2 = -1$, Gradient of the normal $= \frac{1}{3}$
Step 3: Tangent $y - y_1 = m(x - x_1)$ or $y = mx + c$ Normal $y - y_1 = -\frac{1}{m}(x - x_1)$ or $y = -\frac{1}{m}x + c$, where $(x_1, y_1) = (1, -1)$	At P , $x = 1, y = -1, \frac{dy}{dx} = -3$ Equation of the tangent: $y + 1 = -3(x - 1) \Rightarrow y = -3x + 2$ Equation of the normal: $y + 1 = \frac{1}{3}(x - 1) \Rightarrow y = \frac{1}{3}x - \frac{4}{3}$

(ii)

Whenever tangents are parallel to y -axis, we take $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = 0$.

Steps: Heuristics	Working
Step 1: When tangents are parallel to y-axis, $\frac{dy}{dx}$ is undefined. So $\frac{dx}{dy} = 0$.	$\frac{dy}{dx} = \frac{18x + 3y}{2y - 3x}$ When tangents are parallel to y-axis, $2y - 3x = 0 \Rightarrow y = \frac{3}{2}x$
Step 2: Substitute condition obtained from $\frac{dx}{dy} = 0$ into equation of curve.	Substitute $y = \frac{3}{2}x$ into $9x^2 - y^2 + 3xy = 5$, $9x^2 - \left(\frac{3}{2}x\right)^2 + 3x\left(\frac{3}{2}x\right) = 5$ $\Rightarrow \frac{45}{4}x^2 = 5$ $\Rightarrow x = \pm \frac{2}{3}$ Thus equations of tangents parallel to y-axis are $x = \pm \frac{2}{3}$.

■

Example 25

A curve is defined by the parametric equations $x = 3t + 2$ and $y = t + \frac{3}{t}$, where t is a non-zero parameter.

- (i) Find the equation of the normal at the point P where $t = 1$.
- (ii) Find the equation of the tangent at the point P with parameter p and hence write down the equation of tangent at the point where $x = 11$.
- (iii) Determine whether the tangent in (ii) will cut the curve again.

[TPJC Promo 2009 Q10, Modified]

Solution

(i)

Steps: Heuristics	Working
Step 1: Find $\frac{dy}{dx}$	$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 1 - \frac{3}{t^2}$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \left(1 - \frac{3}{t^2}\right) \times \frac{1}{3} = \frac{1}{3} - \frac{1}{t^2}$

Step 2: Find $x, y, \frac{dy}{dx}$ when $t = 1$	When $t = 1, x = 5, y = 4,$ $\frac{dy}{dx} = \frac{1}{3} - \frac{1}{1^2} = -\frac{2}{3}$ $\therefore$ Gradient of normal $= \frac{3}{2}$
Step 3: Use the formula $y - y_1 = m(x - x_1)$ or $y = mx + c$ at the point where $t = 2$	Equation of normal at $(5, 4)$ is $y - 4 = \frac{3}{2}(x - 5) \Rightarrow y = \frac{3}{2}x - \frac{7}{2}$

(ii)

Steps: Heuristics	Working
Step 1: Find $\frac{dy}{dx}$.	$\frac{dx}{dt} = 3, \frac{dy}{dt} = 1 - \frac{3}{t^2}$ $\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{1}{3} \left(1 - \frac{3}{t^2} \right) = \frac{1}{3} - \frac{1}{t^2}$
Step 2: Point P has parameter p i.e. let $t = p$	When $t = p, \frac{dy}{dx} = \frac{1}{3} - \frac{1}{p^2}$
Step 3: Use the formula $y - y_1 = m(x - x_1)$ or $y = mx + c$ at the point P where $t = p$	When $t = p, x = 3p + 2, y = p + \frac{3}{p},$ $\frac{dy}{dx} = \frac{1}{3} - \frac{1}{p^2}$ Equation of tangent: $y - \left(p + \frac{3}{p} \right) = \left(\frac{1}{3} - \frac{1}{p^2} \right) (x - (3p + 2))$
Step 4: When $x = 11, p = 3.$ Substitute $p = 3$ into equation of tangent obtained	When $x = 11, 3p + 2 = 11 \Rightarrow p = 3.$ Thus $y = 3 + \frac{3}{3} = 4, \text{ and } \frac{dy}{dx} = \frac{1}{3} - \frac{1}{3^2} = \frac{2}{9}$ Equation of tangent: $y - 4 = \frac{2}{9}(x - 11) \Rightarrow y = \frac{2}{9}x + \frac{14}{9}$

Important to find the value of p as this allows you to find values of y and $\frac{dy}{dx}$.

(iii)

Steps: Heuristics	Working
Step 1: Use equation of tangent obtained in (ii) to find equation of tangent.	Equation of tangent when $x = 11$ is $y = \frac{2}{9}x + \frac{14}{9}.$
Step 2: Find the value of parameter t using the conditions i.e. find value of t when the tangent at point P cuts the curve again	Assuming conditions are met i.e. tangent cuts the curve again: Find the value of t such that the tangent meets the curve i.e. substitute equations of curve $x = 3t + 2$ and $y = t + \frac{3}{t}$ into equation of tangent: $t + \frac{3}{t} = \frac{2(3t + 2)}{9} + \frac{14}{9}$ $\Rightarrow 3t^2 - 18t + 27 = 0$ $\Rightarrow 3(t - 3)^2 = 0$ $\Rightarrow t = 3$
Step 3: Substitute the value of parameter t into x and y .	When $t = 3$, $x = 11$, $y = 4$ gives the coordinates of the point where the tangent is. Thus the tangent will not cut the curve again.

We cannot equate the parametric equations of curve with equation of tangent.
 Instead, substitute the parametric equations into equation of tangent.

■

Example 26

The coordinates of a point on a curve are given in terms of a parameter t by the equations $x = a \sec t$ and $y = a \tan t$, $a > 0$, $0 < t < \frac{\pi}{2}$.

- (i) Find the equation of the tangent at the point $(a \sec \theta, a \tan \theta)$.
- (ii) The tangent at the point $P(2a, \sqrt{3}a)$ meets the axes at S and T . Find the area of triangle OST .

Solution

$$\begin{aligned} \text{(i)} \quad \frac{dx}{dt} &= \frac{dy}{dt} = \\ \frac{dy}{dx} &= \frac{\sec t}{\tan t} = \frac{\frac{1}{\cos t}}{\frac{\sin t}{\cos t}} = \operatorname{cosec} t \end{aligned}$$

At the point $(a \sec \theta, a \tan \theta)$, $t = \theta$, $\therefore \frac{dy}{dx} =$

Using $y - y_1 = m(x - x_1)$,

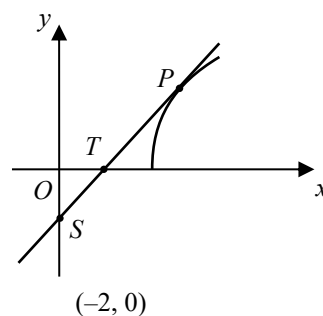
Equation of the tangent:

$$y = \operatorname{cosec} \theta x + a(\tan \theta - \sec \theta \operatorname{cosec} \theta)$$

- (ii) At $P(2a, \sqrt{3}a)$,

$$a \tan \theta = \sqrt{3}a \Rightarrow \theta = \frac{\pi}{3}$$

Equation of tangent:



At S : $x = 0$, $y = -\frac{a}{\sqrt{3}} \Rightarrow OS = \frac{a}{\sqrt{3}}$

At T : $y = 0$, $x = \frac{a}{2} \Rightarrow OT = \frac{a}{2}$

Area of triangle $OST = \frac{1}{2} \left(\frac{a}{\sqrt{3}} \right) \left(\frac{a}{2} \right) = \frac{1}{4\sqrt{3}} a^2 = \frac{\sqrt{3}}{12} a^2$ unit². ■

10.2 Maxima and Minima

Heuristics for Maxima & Minima

In general, to find maximum or minimum V

Step 1: Express V in terms of 1 or more variables using a formula.

(e.g. Volume of a cone = $\frac{1}{3}\pi r^2 h$; Surface area of a cone = $\pi r l$)

Step 2: Express V in terms of **one** variable only before differentiating i.e. eliminate the 2nd variable using another equation.

(e.g. Volume of a cone = $\frac{1}{3}\pi r^2 h$; eliminate, for example, h by expressing h in terms of r using similar triangles or other identities)

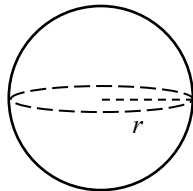
Step 3: Put $\frac{dV}{dr} = 0$ to find specific value of r .

Step 4: Using the specific value of r , show that $\frac{d^2V}{dr^2} > 0$ for minimum V

or $\frac{d^2V}{dr^2} < 0$ for maximum V (or use 1st derivative test)

Step 5: Substitute the specific value of r into V to find the maximum or minimum V .

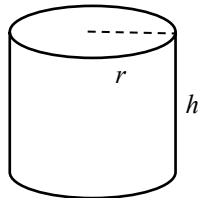
Mensuration Formulae



Sphere

$$\text{Volume} = \frac{4}{3}\pi r^3$$

$$\text{Surface area} = 4\pi r^2$$

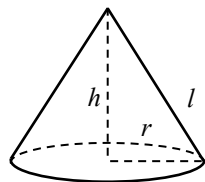


Cylinder

$$\text{Volume} = \pi r^2 h$$

$$\text{Curved Surface Area} = 2\pi r h$$

$$\text{Total Surface Area of closed cylinder} = 2\pi r h + 2\pi r^2$$



Cone

$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

$$\text{Curved Surface Area} = \pi r l$$

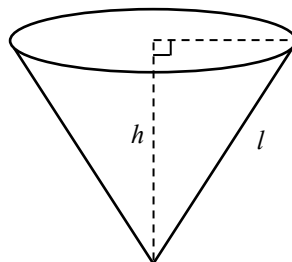
$$\text{Total Surface Area} = \pi r l + \pi r^2$$

Example 27

As shown in the diagram, a cone-shaped paper cup with radius, r , height, h , and slanted length, l , is to hold $30\pi \text{ cm}^3$ of water.

Find the least amount of paper required to produce such a cup.

[Volume of a cone = $\frac{1}{3}\pi r^2 h$; Surface area of a cone = $\pi r l$] [JJC Promo 2009 Q7]



Solution

Steps: Heuristics	Working
Step 1: Let the area of paper used be A. To find minimum A, express A in terms of 1 or more variables using a formula.	Surface area, $A = \pi r l$
Step 2: Express A in terms of one variable only before differentiating i.e. eliminate the 2nd variable, l using another equation (constraint). In this question, $A = \pi r l$. We try to eliminate l by expressing l in terms of r using Pythagoras Theorem.	Since $l^2 = h^2 + r^2$ (Using Pythagoras Theorem) $l = \sqrt{h^2 + r^2}$ $\therefore A = \pi r \sqrt{h^2 + r^2}$ Given that volume, $V = \frac{1}{3}\pi r^2 h = 30\pi \Rightarrow h = \frac{90}{r^2}$ $A = \pi r \sqrt{\left(\frac{90}{r^2}\right)^2 + r^2}$ $= \frac{\pi \sqrt{8100 + r^6}}{r}$
Step 3: Put $\frac{dA}{dr} = 0$ to find specific value of r.	$\frac{dA}{dr} = \pi \frac{r \left(\frac{1}{2}\right) (8100 + r^6)^{-\frac{1}{2}} (6r^5) - \sqrt{8100 + r^6}}{r^2}$ $= \pi \frac{2r^6 - 8100}{r^2 \sqrt{8100 + r^6}}$ When $\frac{dA}{dr} = 0$, $\pi \frac{2r^6 - 8100}{r^2 \sqrt{8100 + r^6}} = 0$ $r^6 = 4050 \Rightarrow r = \sqrt[6]{4050} = 3.9925$

Note that r and l are variables. We need to eliminate one of them before differentiating A .

Identify the constraints here (**Pythagoras Theorem and fixed volume of 30π**). Eliminate h by expressing h in terms of r using the constraint that $V = 30\pi$.

When using first derivative test, we must specify values of r and $\frac{dA}{dr}$.

Step 4:

Using the specific value of r , show $\frac{d^2A}{dr^2} > 0$ for minimum A .

Alternatively, use the first derivative test.

Using GC, $\frac{d^2A}{dr^2} = 21.8 > 0$ where $r = \sqrt[6]{4050}$.

Alternatively, using the first derivative test:

r	3.95	$\sqrt[6]{4050}$	4
$\frac{dA}{dr}$	-0.929	0	0.164

Thus minimum point occurs at $r = \sqrt[6]{4050}$.

Step 5:

Substitute the specific value of r into A to find the maximum or minimum A .

Least amount of paper required

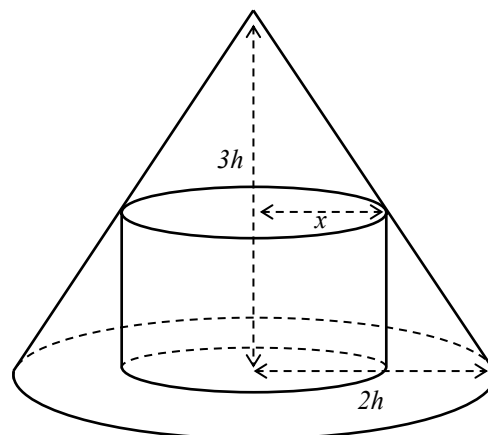
$$\begin{aligned}
 A &= \frac{\pi\sqrt{8100+r^6}}{r} \\
 &= \frac{\pi\sqrt{8100+4050}}{\sqrt[6]{4050}} \\
 &= 86.7 \text{ cm}^2 \text{ (3 s.f.)}
 \end{aligned}$$

■

Example 28

A closed circular cylinder of radius x is inscribed in a right circular cone of radius $2h$ and vertical height $3h$, where h is a constant. One circular end of the cylinder lies on the base of the cone and the circumference of the other circular end is in contact with the inner surface of the cone.

[IJC 2012 Promo Q4, Modified]



- (i) Show that $V = 3\pi x^2 \left(h - \frac{1}{2}x\right)$, where V is the volume of the cylinder. [2]
- (ii) If x is made to vary, find the maximum volume of the cylinder in terms of h . [4]

Solution

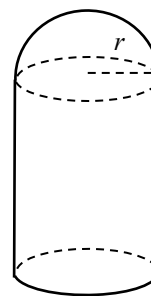
Steps: Heuristics	Working
Step 1: We start off by expressing V in terms of 2 variables, x and y.	(i) Let y be the height of the cylinder. $V = \pi x^2 y$
Step 2: Observe from similar triangles that x and y are related, hence V can be expressed in terms of x only.	By similar triangles, $\frac{3h - y}{3h} = \frac{x}{2h}$ $\Rightarrow 3h - y = \frac{3}{2}x$ $\Rightarrow y = 3h - \frac{3}{2}x$ $\therefore V = \pi x^2 y = 3\pi x^2 \left(h - \frac{1}{2}x \right) \text{ (shown)}$
Step 3: Solve $\frac{dV}{dx} = 0$ to find x.	(ii) For maximum V, $\frac{dV}{dx} = 3\pi h(2x) - \frac{3}{2}\pi(3x^2) = 0$ $\Rightarrow x \left(6h - \frac{9}{2}x \right) = 0$ $\Rightarrow x = 0 \text{ (rejected) or } x = \frac{4}{3}h$
Step 4: Show that when $x = \frac{4}{3}h$, $\frac{d^2V}{dx^2} < 0 \text{ i.e. } V \text{ is maximum}$	When $x = \frac{4}{3}h$, $\frac{d^2V}{dx^2} = 6\pi h - 12\pi h = -6\pi h < 0.$
Step 5: Find maximum V using the value of x found.	$V = \pi \left(\frac{4}{3}h \right)^2 \left(3h - \left(\frac{3}{2} \right) \left(\frac{4}{3}h \right) \right)$ $= \pi \left(\frac{16}{9}h^2 \right) (h) = \frac{16}{9}\pi h^3$ Hence maximum V is $\frac{16}{9}\pi h^3$.

Example 29

A cylindrical silo with a hemispherical rooftop is constructed to store rubbish.

The cost of each unit area of the rooftop is fixed at \$21 and the cost of each unit area of the curved surface of the cylinder is fixed at \$7. The total cost of the rooftop and the curved surface is fixed at \$2100.

Given that the radius of the hemisphere is r and the volume of the structure is V , find, by differentiation, the exact cost of the hemispherical top when V is at its maximum.



[Curved surface area of sphere = $4\pi r^2$; volume of sphere = $\frac{4}{3}\pi r^3$]

[SRJC 2010 Prelim P1 Q6, Modified]

Solution

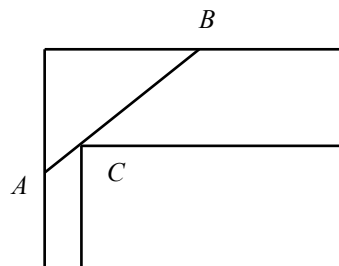
Steps: Heuristics	Working
Step 1: Since V is to be maximum, we first express V in terms of 2 variables, r and h.	Let h be the height of the cylinder. $V = \dots\dots\dots(1)$
Step 2: Using the information on cost, form an equation between r and h. Then, by expressing h in terms of r, V can be expressed in terms of r only.	Total cost of rooftop and surface area, $= 42\pi r^2 + 14\pi rh$ $\Rightarrow h = \frac{2100 - 42\pi r^2}{14\pi r}$ $\Rightarrow h = \frac{150}{\pi r} - 3r \quad \dots\dots(2)$ Substitute (2) into (1) $V = \pi r^2 h + \frac{2}{3}\pi r^3$ $= \frac{2}{3}\pi r^3 + 150r - 3\pi r^3$ $= 150r - \frac{7}{3}\pi r^3$

Step 3: Solve $\frac{dV}{dr} = 0$ to find r.	For maximum V, $\Rightarrow r = \sqrt{\frac{150}{7\pi}} \text{ (since } r > 0 \text{)}$
Step 4: Show that when $r = \sqrt{\frac{150}{7\pi}}, \frac{d^2V}{dr^2} < 0$ i.e. V is maximum	When $r = \sqrt{\frac{150}{7\pi}}$ Therefore the volume is maximum when $r = \sqrt{\frac{150}{7\pi}}.$
Step 5: Find the cost of hemispherical top using the value of r found.	Cost of hemispherical top when volume is maximum $= 2\pi r^2(21) = 2\pi \left(\frac{150}{7\pi}\right)(21) = \900

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Example 30

One hallway which is 1m wide meets another hallway which is 8m wide in a right angle. Assuming that the ladder is carried horizontally and that the thickness of the ladder is negligible, find the shortest possible length of AB , touching the corner C . Hence find the length of the longest ladder which can be carried horizontally around the corner?



Solution

Step 1:
We start off by expressing length of L in terms of 2 variables, a and b .

Step 2:
We re-express L in terms of one variable only using properties of right-angled triangle. Now we are ready to differentiate L with respect to θ .

Let the length of the ladder be $L = a + b$

Observe from the smaller right-angle triangle that

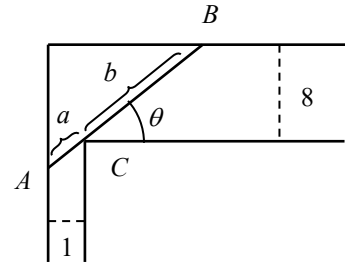
$$\cos \theta = \frac{1}{a} \Rightarrow a = \frac{1}{\cos \theta}$$

Similarly from the larger right-angle triangle,

$$\sin \theta = \frac{8}{b} \Rightarrow b = \frac{8}{\sin \theta}$$

$$\text{Hence } L = a + b = \frac{1}{\cos \theta} + \frac{8}{\sin \theta} = \sec \theta + 8 \operatorname{cosec} \theta$$

We now want to find the minimum value of L as θ varies.



Step 3:
Solve $\frac{dL}{d\theta} = 0$ to find θ .

Differentiating L with respect to θ and equating to zero:

$$\frac{dL}{d\theta} = \sec \theta \tan \theta - 8 \operatorname{cosec} \theta \cot \theta = 0$$

$$\Rightarrow \sec \theta \tan \theta = 8 \operatorname{cosec} \theta \cot \theta$$

$$\Rightarrow \frac{\tan \theta}{\cos \theta} = \frac{8}{\sin \theta \tan \theta}$$

$$\Rightarrow \tan^3 \theta = 8$$

$$\Rightarrow \tan \theta = 2$$

Step 4:
Show that when $\tan \theta = 2$, $\frac{d^2L}{d\theta^2} > 0$ i.e. L is minimum.

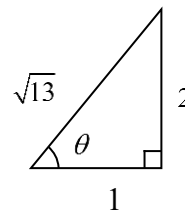
Check that when $\tan \theta = 2$, $\frac{d^2L}{d\theta^2} = 33.5 > 0$

Hence the minimum length of L occurs when $\tan \theta = 2$.

Hence minimum length L

$$= \frac{1}{\cos \theta} + \frac{8}{\sin \theta} = \frac{1}{\frac{1}{\sqrt{5}}} + \frac{8}{\frac{2}{\sqrt{5}}} = 5\sqrt{5}$$

Hence the length of the longest ladder = $5\sqrt{5}$.



10.3 Rates of Change

$\frac{dx}{dt}$ is the rate of change of x with respect to (w.r.t.) t .

$\frac{dx}{dt} > 0$ implies that x increases as t increases.

$\frac{dx}{dt} < 0$ implies that x decreases as t increases.

Heuristics for Rates of Change

In general, to find rates of change:

Step 1: Identify the rate of change that the question requires you to solve for. (e.g. rate of change of volume of a sphere with respect to time, $\frac{dV}{dt}$).

Step 2: Identify any other rates of change given in the question. (e.g. rate of change of radius with respect to time, $\frac{dr}{dt} = 0.5$.)

Step 3: Use chain rule to identify what need to be found. (e.g. $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$, need to find $\frac{dV}{dr}$)

Step 4: Express V in terms of one or more variables using a formula. (e.g. volume of a sphere $\frac{4}{3}\pi r^3$)

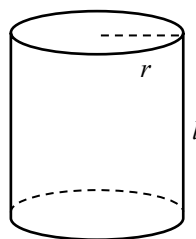
Step 5: Differentiate V with respect to radius r .

Step 6: Use chain rule to the required rate of change.

Example 31

A circular cylinder is expanding in such a way that, at time t seconds, the length of the cylinder is $20x$ cm and the area of the cross-section is x cm². Given that, when $x = 5$, the area of the cross-section is increasing at a rate of 0.025 cm²s⁻¹, find the rate of increase at this instant of

- (i) the length of the cylinder,
- (ii) the volume of the cylinder,
- (iii) the radius of the cylinder.



[N01/I/17a]

Solution**Method**

- (i) At time t , length of cylinder, $l = 20x$ where x is area of cross-section

Given when $x = 5$, $\frac{dx}{dt} = 0.025$.

To find $\frac{dl}{dt}$, use chain rule,

$$\frac{dl}{dt} = \frac{dl}{dx} \times \frac{dx}{dt}$$

Now, since $l = 20x$, $\frac{dl}{dx} = 20$

Thus

$$\frac{dl}{dt} = 20(0.025) = 0.5 \text{ cm s}^{-1}$$

- (ii) To find $\frac{dV}{dt}$, use

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

Now, volume of cylinder

$$V = x(20x) = 20x^2$$

$$\therefore \frac{dV}{dx} = 40x$$

When $x = 5$,

$$\frac{dV}{dt} = 40(5)(0.025) = 5 \text{ cm}^3 \text{ s}^{-1}$$

- (iii) To find $\frac{dr}{dt}$, use $\frac{dr}{dt} = \frac{dr}{dx} \times \frac{dx}{dt}$

Area of circular cross section

$$x = \pi r^2 \Rightarrow \frac{dx}{dr} = 2\pi r.$$

When $x = 5$, $\pi r^2 = 5 \Rightarrow r = \sqrt{\frac{5}{\pi}}$

since $r > 0$

$$\begin{aligned} \text{Thus } \frac{dr}{dt} &= \frac{dr}{dx} \times \frac{dx}{dt} = \frac{1}{2\pi r} \frac{dx}{dt} \\ &= \frac{1}{2\pi\sqrt{\frac{5}{\pi}}} (0.025) \\ &= 0.00315 \text{ cm s}^{-1} \end{aligned}$$

Alternative Method

- (i) At time t , length of cylinder, $l = 20x$ where x is area of cross-section

Given when $x = 5$, $\frac{dx}{dt} = 0.025$.

Since $l = 20x$, differentiate with respect to t

$$\begin{aligned} \frac{dl}{dt} &= 20 \frac{dx}{dt} \\ &= 20(0.025) = 0.5 \text{ cm s}^{-1} \end{aligned}$$

- (ii) $V = xl = x(20x) = 20x^2$

Differentiate with respect to t :

$$\frac{dV}{dt} = 40x \frac{dx}{dt}$$

When $x = 5$,

$$\frac{dV}{dt} = 40(5)(0.025) = 5 \text{ cm}^3 \text{ s}^{-1}$$

Alternatively, $V = xl$

Differentiate with respect to t

$$\begin{aligned} \frac{dV}{dt} &= l \frac{dx}{dt} + x \frac{dl}{dt} \\ &= 20x \frac{dx}{dt} + x \frac{dl}{dt} \end{aligned}$$

When $x = 5$,

$$\begin{aligned} \frac{dV}{dt} &= 100(0.025) + 5(0.5) \\ &= 5 \text{ cm}^3 \text{ s}^{-1} \end{aligned}$$

- (iii) $x = \pi r^2 \Rightarrow \frac{dx}{dt} = 2\pi r \frac{dr}{dt}$

When $x = 5$, $r^2 = \frac{5}{\pi}$

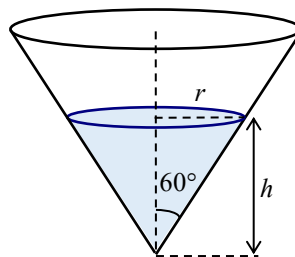
Hence $0.025 = 2\pi\sqrt{\frac{5}{\pi}} \frac{dr}{dt}$

$$\begin{aligned} \frac{dr}{dt} &= \frac{1}{2\pi\sqrt{\frac{5}{\pi}}} (0.025) \\ &= 0.00315 \text{ cm s}^{-1} \end{aligned}$$

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Example 32

An inverted right circular conical tank has a 60° semi-vertex angle. Water is leaking out of the tank from the vertex at a rate of $3 \text{ cm}^3 \text{ s}^{-1}$.



- (i) Find the rate of decrease of the depth of water when the depth is 4 cm.
 (ii) Find also the rate at which the area of the water surface is decreasing at that instant.

Solution

- (i) To find $\frac{dh}{dt}$, use $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

Given that $\frac{dV}{dt} = -3$, let us find

$$\frac{dh}{dV}.$$

Now, volume of water in the

$$\text{cone, } V = \frac{1}{3} \pi r^2 h \text{ ----- (1)}$$

To find $\frac{dh}{dV}$, we express V in

terms of one variable (i.e. eliminate r).

$$\text{Using } \tan 60^\circ = \frac{r}{h} \Rightarrow \sqrt{3} = \frac{r}{h}$$

$$\Rightarrow r = \sqrt{3}h \text{ ----- (2)}$$

Substitute (2) into (1),

$$V = \quad = \pi h^3$$

$$\Rightarrow \frac{dV}{dh} = 3\pi h^2$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} \\ = -\frac{1}{\pi h^2}$$

When $h = 4$,

$$\frac{dh}{dt} = -\frac{1}{\pi(4)^2} = -\frac{1}{16\pi}.$$

Thus when the depth is 4 cm, the rate of decrease of the depth

of water is $\frac{1}{16\pi} \text{ cm s}^{-1}$.

- (ii) To find $\frac{dA}{dt}$, use

From (i), $\frac{dh}{dt} = -\frac{1}{16\pi}$, let us

$$\text{find } \frac{dA}{dh}.$$

Now from (2),

$$A = \quad = 3\pi h^2$$

$$\Rightarrow \frac{dA}{dh} = 6\pi h$$

So

$$= 6\pi h \times \left(-\frac{1}{16\pi} \right) \\ = -\frac{6}{h}$$

$$\text{When } h = 4, \frac{dA}{dt} = -\frac{6}{4} = -\frac{3}{2}$$

Thus the rate of decrease of the area of water surface is

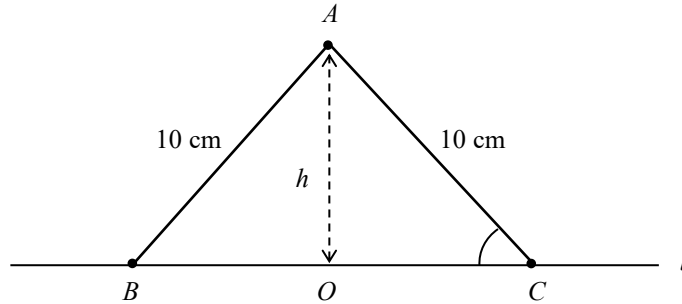
$$\frac{3}{2} \text{ cm}^2 \text{ s}^{-1}.$$

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Example 33

The diagram below shows an isosceles triangle ABC with fixed lengths AB and AC of 10 cm each and $\angle ACB = \angle ABC = \theta$ radians.

A is a variable point which is at a height h cm directly above the point O while B and C are variable points which move horizontally along the line l .



Given that A descends vertically towards the point O such that the area of triangle ABC is decreasing at a constant rate of $0.7 \text{ cm}^2/\text{s}$, determine, at the instant when A is 6 cm above O ,

- (a) the rate of change of θ ,
 (b) the rate at which C is moving away from O .

Solution

Method

(a) To find $\frac{d\theta}{dt}$,

$$\text{Use } \frac{dP}{dt} = \frac{dP}{d\theta} \times \frac{d\theta}{dt}.$$

Given $\frac{dP}{dt} = -0.7$, now,

$$P = \frac{1}{2}(10)(10)\sin(\pi - 2\theta)$$

$$= 50\sin(2\theta)$$

$$\Rightarrow \frac{dP}{d\theta} = 100\cos 2\theta$$

When $h = 6$,

$$\sin \theta = \frac{6}{10} = \frac{3}{5}, \text{ thus}$$

$$\cos 2\theta = 1 - 2\sin^2 \theta$$

$$= 1 - 2\left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\therefore \frac{dP}{d\theta} = 100\left(\frac{7}{25}\right) = 28$$

Alternative Method

(a) Let $P = \text{Area of } \triangle ABC$
 $\angle BAC = \pi - 2\theta$
 Now,
 $P = \frac{1}{2}(10)(10)\sin(\pi - 2\theta)$
 $= 50\sin(2\theta)$
 $\Rightarrow \frac{dP}{dt} = 100\cos 2\theta \frac{d\theta}{dt}$
 When $h = 6$,
 $\sin \theta = \frac{6}{10} = \frac{3}{5}$, thus
 $\cos 2\theta = 1 - 2\sin^2 \theta$
 $= 1 - 2\left(\frac{3}{5}\right)^2 = \frac{7}{25}$
 Given $\frac{dP}{dt} = -0.7$,
 $-0.7 = 100\left(\frac{7}{25}\right)\frac{d\theta}{dt}$
 $\Rightarrow \frac{d\theta}{dt} = -0.025 \text{ rad s}^{-1}$

$$\text{So } \frac{dP}{dt} = \frac{dP}{d\theta} \times \frac{d\theta}{dt}$$

$$\Rightarrow -0.7 = 28 \times \frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = -0.025 \text{ rad s}^{-1}$$

(b) Let $OC = x = 10 \cos \theta$

To find $\frac{dx}{dt}$, use

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$

$$\Rightarrow \frac{dx}{dt} = -10 \sin \theta \frac{d\theta}{dt}$$

$$= -10 \left(\frac{3}{5} \right) (-0.025)$$

$$= 0.15 \text{ cm s}^{-1}$$

(b) Let $OC = x = 10 \cos \theta$,

$$\frac{dx}{dt} = -10 \sin \theta \frac{d\theta}{dt}$$

$$= -10 \left(\frac{3}{5} \right) (-0.025)$$

$$= 0.15 \text{ cm s}^{-1}$$

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ANNEX A**Practice Questions on Differentiation**

1 Differentiate each of the following functions with respect to x .

(a) $\sqrt{3-4x^2}$

(b) $\frac{1}{(3x^2+5)^2}$

(c) $\frac{x}{x^2+1}$

(d) $\frac{e^{2x}}{1+e^x}$

(e) $e^{2x} \cos x$

(f) $\sin(x^3+4)$

(g) $\cos \sqrt{x}$

(h) $\sin^{-1} \frac{x}{10}$

(i) $\tan^{-1} \sqrt{x}$

(j) $\ln \frac{4}{x^2}$

(k) $\ln(\sin^2 x)$

(l) $3^x x^3$

2 Express the gradient of the following curves in terms of x and y .

(a) $x^3 + y^3 = 9$

(b) $\tan x + \tan y = 3$

(c) $4x^2 - y^2 = 7$

3 Find $\frac{dy}{dx}$ for the following parametric equations.

(a) $x = 120t - 4t^2, y = 60t - 6t^2$

(b) $x = 2t + \sin 2t, y = \cos 2t, 0 < t < \frac{\pi}{2}$

(c) $x = \frac{1}{1+t^2}, y = \tan^{-1} t$

Answers

1 (a) $-\frac{4x}{\sqrt{3-4x^2}}$ (b) $-\frac{12x}{(3x^2+5)^3}$ (c) $\frac{1-x^2}{(x^2+1)^2}$

(d) $\frac{e^{2x}(2+e^x)}{(1+e^x)^2}$ (e) $e^{2x}(2\cos x - \sin x)$ (f) $3x^2 \cos(x^3+4)$

(g) $-\frac{\sin \sqrt{x}}{2\sqrt{x}}$ (h) $\frac{1}{\sqrt{100-x^2}}$ (i) $\frac{1}{2(1+x)\sqrt{x}}$

(j) $-\frac{2}{x}$ (k) $2 \cot x$ (l) $3^x x^2 (3 + x \ln 3)$

2 (a) $-\frac{x^2}{y^2}$ (b) $-\frac{\sec^2 x}{\sec^2 y}$ (c) $\frac{4x}{y}$

3 (a) $\frac{15-3t}{30-2t}$ (b) $-\tan t$ (c) $-\frac{1+t^2}{2t}$