RAFFLES INSTITUTION
2022 YEAR 5 COMMON TEST (MODIFIED)

MATHEMATICS **9758**

2 hours

Answer **all** the questions.

1 It is given that $\mathrm{f}(x) = ax^3 - bx^2 + bx + c$ and $\mathrm{g}(x) = ax + d$, where a, b, c and d are constants. Given that the curve with equation $y = \mathrm{f}(x)$ has a stationary point at $(1,1)$ and it intersects the curve with equation $y = \mathrm{g}(x)$ at $(3,17)$, determine the values of a, b, c and d. [5]

2 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on BA produced such that $AB : AC = 3 : 2$.

(i) Find $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and show that the area of triangle OAC can be written as $k|\mathbf{a} \times \mathbf{b}|$, where k is a constant to be found. [3]

(ii) Given that $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and the magnitude of $\mathbf{a} + \mathbf{b}$ is $\sqrt{3}$, find the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$. [3]

3 The line L_1 has equation $x = 3,\ \dfrac{6-y}{3} = z$ and the line L_2 has equation $x - 2 = y - 6,\ z = 7$.

(i) Find the acute angle between L_1 and L_2. [2]

(ii) Show that L_1 and L_2 are skew lines. [2]

(iii) Let P_1 and Q_1 be two points on L_1 such that distance between them is $\sqrt{10}$. Let P_2 and Q_2 be the two respective points on L_2 such that P_1P_2 and Q_1Q_2 are minimum. Find the exact distance between P_2 and Q_2. [2]

4 **(a)** Differentiate $\log_3\left(2x^2 + 1\right)$ with respect to x. [2]

(b) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, given that $y = \dfrac{\tan^{-1} x}{1 + x^2}$ [2]

(c) Given that $x = 1 - \cos\theta$, $y = 2\sin\theta$, where $0 < \theta < \pi$, find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of θ. [2]

5 **(a)** Without using a calculator, solve the inequality

$$\frac{2x^3+x^2-5x+4}{x+1} > x. \qquad [4]$$

Hence solve the inequalities

(i) $$\frac{2|x|^3+x^2-5|x|+4}{|x|+1} > |x|, \qquad [1]$$

(ii) $$\frac{2\cos^3 x-\cos^2 x-5\cos x-4}{1-\cos x} < \cos x. \qquad [2]$$

(b) Show that there are no solutions to the equation

$$b|x-1| = x-b, \text{ where } b>1 \text{ is a fixed constant.} \qquad [4]$$

6 The function f is defined by

$$\mathrm{f}: x \mapsto (x-3)^2+2, \quad x\in\mathbb{R}, \quad x \geq k.$$

(i) Given that the function f^{-1} exists, state the least value of k. [1]

Use the value of k found in part **(i)** for the rest of this question.

(ii) Without finding f^{-1}, sketch on the same diagram the graphs of $y=\mathrm{f}(x)$, $y=\mathrm{f}^{-1}(x)$ and $y=\mathrm{f}^{-1}\mathrm{f}(x)$. You should label your graphs clearly. [3]

(iii) Hence find the exact solution(s) of the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [3]

The function g is defined by

$$\mathrm{g}: x \mapsto \sqrt{x}+h, \quad x\in\mathbb{R}, \quad x \geq 0.$$

(iv) Find the set of values of h such that composite function fg exists and write down an expression for $\mathrm{fg}(x)$ in terms of h. [3]

7 **Do not use a calculator in answering this question.**

The complex number z is such that $\arg(z)=\theta$ where $-\frac{\pi}{2}<\theta<-\frac{\pi}{4}$.

(i) In an Argand diagram, mark and label clearly the points P, Q and R representing the complex numbers z, $\mathrm{i}z$ and $z+\mathrm{i}z$ respectively. [2]

(ii) State the geometrical representation of $OPRQ$ and express $\arg(z+\mathrm{i}z)$ in terms of θ and π. [2]

(iii) It is now given that $z=2-2\sqrt{3}\mathrm{i}$. Show that

$$\sin\left(\frac{\pi}{12}\right)=\frac{\sqrt{3}-1}{2\sqrt{2}}. \qquad [5]$$

8 **(a)** A manufacturer can produce q thousand units of a particular machine per week at the unit price $\$p$ according to the equation $q^2 - 3qp + p^2 = 5$, where $p > 3$ and $q > 3$.

Find the rate of change of the supply of the machines when the quantity produced is 4000 units, and the unit price is \$11 which is increasing at a rate of \$0.10 per week. [3]

(b) Isocost lines and isoquant curves are studied in economics. A general class of isoquant curves are the Cobb-Douglas functions of the form $x^{\alpha} y^{\beta} = k$, where α, β and k are positive constants.

(i) Show that $\dfrac{x}{y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) = -\dfrac{\alpha}{\beta}$. [2]

The point at which the isocost line is a tangent to the isoquant curve is called an equilibrium point.

(ii) The isocost line $27x + 2y = 6$ is tangential to the isoquant curve $x^{\frac{1}{3}} y^{\frac{2}{3}} = k$, where $x > 0$, $y > 0$ and k is a positive constant.

Find the coordinates of the equilibrium point and the value of k. [7]