# H2 Mathematics (9758)
# Chapter 7 Differentiation
# Extra Practice Solutions

| Q1 | Solutions |
|---|---|
| (a) | $(x+1)y+x^4y^2=1$<br>Differentiate wrt $x$,<br>$(x+1)\dfrac{\mathrm{d}y}{\mathrm{d}x}+y(1)+x^4 2y\dfrac{\mathrm{d}y}{\mathrm{d}x}+y^2(4x^3)=0$<br>$\dfrac{\mathrm{d}y}{\mathrm{d}x}\left[(x+1)+2yx^4\right]=-y(1+4x^3y)$<br>$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-y(1+4x^3y)}{(x+1)+2x^4y}$ |
| (b) | $\ln xe^x=y\ln x^2$, $x>0$<br>$\Rightarrow \ln x+\ln e^x=2y\ln x$<br>$\Rightarrow \ln x+x=2y\ln x$<br>Differentiate wrt $x$,<br>$\dfrac{1}{x}+1=2y\left(\dfrac{1}{x}\right)+2(\ln x)\dfrac{\mathrm{d}y}{\mathrm{d}x}$<br>$2\ln x\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1+x-2y}{x}$<br>$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1+x-2y}{2x\ln x}$ |
| (c) | $e^{2y}-e^{x^2+y^2}=\dfrac{1}{y}$<br>Differentiate wrt $x$,<br>$e^{2y}\left(2\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)-e^{x^2+y^2}\left(2x+2y\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=-y^{-2}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$<br>$\dfrac{\mathrm{d}y}{\mathrm{d}x}(2e^{2y}-2ye^{x^2+y^2}+y^{-2})=2xe^{x^2+y^2}$<br>$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2xe^{x^2+y^2}}{2e^{2y}-2ye^{x^2+y^2}+y^{-2}}$ |
| (d) | $\dfrac{y^2}{x}+(x-y)^2-\cos(xy)=3$<br>Differentiate wrt $x$, |

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| | $\dfrac{x\left(2y\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)-y^2}{x^2}+2(x-y)\left(1-\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)-(-\sin(xy))\left(x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\right)=0$ <br> $\dfrac{2y}{x}\dfrac{\mathrm{d}y}{\mathrm{d}x}-\left(\dfrac{y}{x}\right)^2+2(x-y)-2(x-y)\dfrac{\mathrm{d}y}{\mathrm{d}x}+x\sin(xy)\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\sin(xy)=0$ <br> $\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(\dfrac{2y}{x}-2(x-y)+x\sin(xy)\right)=\left(\dfrac{y}{x}\right)^2-2(x-y)-y\sin(xy)$ <br> $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\left(\dfrac{y}{x}\right)^2-2(x-y)-y\sin(xy)}{\dfrac{2y}{x}-2(x-y)+x\sin(xy)}$ |

| **Q2** | **Solutions** |
|---|---|
| | $x^2y+xy^2+54=0$ <br> Differentiating w.r.t $x$: <br> $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}+2xy+y^2+2xy\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ <br> $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{2xy+y^2}{x^2+2xy}=-\dfrac{y(2x+y)}{x(x+2y)}$ <br> Gradient $=-1$: $-\dfrac{2xy+y^2}{x^2+2xy}=-1$ <br> $2xy+y^2=x^2+2xy$ <br> $y^2=x^2$ <br> $y=x$ or $y=-x$ <br> When $y=-x$: <br> $-x^3+x^3+54=0$ (contradiction $\therefore$ reject $y=-x$) <br> When $y=x$: <br> $x^3+x^3+54=0$ <br> $x^3=-27$ <br> $x=-3$ <br> Hence there is only one such point $(-3,-3)$. |

| Q3 | Solutions |
|---|---|
| (a) | $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(\cos^{-1}(3x))\right]=\cos(\cos^{-1}(3x))\cdot\frac{-3}{\sqrt{1-9x^2}}=\frac{-9x}{\sqrt{1-9x^2}}$ |
| (b) | $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}(\cos x)\right)$<br>$=\frac{1}{\sqrt{1-(\cos x)^2}}(-\sin x)$<br>$=-\frac{\sin x}{\sqrt{\sin^2 x}}$<br>$=-\frac{\sin x}{\lvert\sin x\rvert}$<br>$=\begin{cases}\frac{-\sin x}{-\sin x} & \text{if } \sin x<0\\ \frac{-\sin x}{\sin x} & \text{if } \sin x>0\end{cases}$<br>$=\begin{cases}1 & \text{if } \sin x<0\\ -1 & \text{if } \sin x>0\end{cases}$ |

| Q4 | Solutions |
|---|---|
| | $\frac{\mathrm{d}x}{\mathrm{d}\theta}=1-\cos\theta,\quad \frac{\mathrm{d}y}{\mathrm{d}\theta}=\sin\theta$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}\theta}\times\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{\sin\theta}{1-\cos\theta}$<br>At point where $\theta=\alpha$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}$<br>$\Rightarrow\frac{\sin\alpha}{1-\cos\alpha}=\frac{1}{2}$<br>$\Rightarrow 2\sin\alpha=1-\cos\alpha$<br>$\Rightarrow 2\sin\alpha+\cos\alpha=1$ (shown) |

| Q5 | Solutions |
|---|---|
| (i) | $x^2 - xy + y^2 - 9 = 0$<br>Differentiate wrt x : $2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$<br>$(2y - x)\frac{dy}{dx} = y - 2x$<br>$(2y - x)\frac{dy}{dx} = y - 2x$ |
| (ii) | At stationary points, $\frac{dy}{dx} = 0$. We get $y - 2x = 0 \Rightarrow y = 2x$<br>Substitute $y = 2x$ to eqn of $C$:<br>$x^2 - 2x^2 + 4x^2 - 9 = 0$<br>$3x^2 - 9 = 0$<br>$\Rightarrow x = \pm\sqrt{3}$ and hence $y = \pm 2\sqrt{3}$<br>The **exact** coordinates of the stationary points are $\left(\sqrt{3}, 2\sqrt{3}\right)$ and $\left(-\sqrt{3}, -2\sqrt{3}\right)$ |
| (iii) | Differentiate $(2y - x)\frac{dy}{dx} = y - 2x$ wrt $x$ :<br>$\left(2\frac{dy}{dx} - 1\right)\frac{dy}{dx} + (2y - x)\frac{d^2y}{dx^2} = \frac{dy}{dx} - 2$<br>At stat points, , $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = \frac{-2}{(2y - x)}$<br>At $\left(\sqrt{3}, 2\sqrt{3}\right)$, we have 2y-x > 0. $\frac{d^2y}{dx^2} = \frac{-2}{(2y - x)} < 0$. Max point<br>At $\left(-\sqrt{3}, -2\sqrt{3}\right)$, we have 2y-x < 0, $\frac{d^2y}{dx^2} = \frac{-2}{(2y - x)} > 0$ Min point |

| Q6 | Solutions |
|---|---|

$\mathrm{f}(x) = x^2 e^{x^2}$, for $x \in \mathbb{R}$,

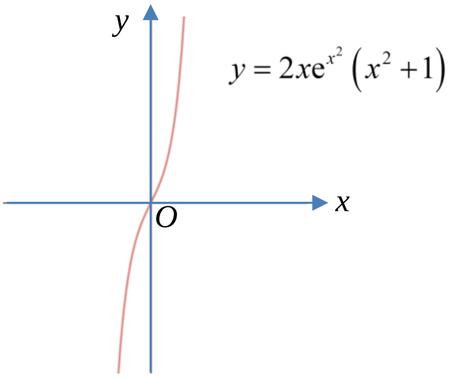
$$\mathrm{f}'(x) = x^2\left(2xe^{x^2}\right) + 2xe^{x^2}$$

$$= 2xe^{x^2}\left(x^2+1\right)$$

For the function to be increasing,

$$\mathrm{f}'(x) = 2xe^{x^2}\left(x^2+1\right) > 0$$

**<u>Method 1:</u>** By GC,




From the sketch, for $2xe^{x^2}\left(x^2+1\right) > 0$, $\therefore x > 0$

**<u>Method 2:</u>**

Since $x^2+1>0$ and $e^{x^2} > 0$, for all $x \in \mathbb{R}$,

$\therefore x > 0$

| Q7 | Solutions |
|---|---|
| **(i)** | When $y = 0$, $3x^2 = 48 \Rightarrow x = \pm 4$<br>$A = \frac{1}{2} \times 8 \times y = 4y$ |
| **(ii)** | $10y\frac{\mathrm{d}y}{\mathrm{d}x} - 3x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y + 6x = 0$ ---- (*)<br>$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y-6x}{10y-3x}$ |
| **(iii)** | Since $A$ has a stationary value,<br>$\frac{\mathrm{d}A}{\mathrm{d}x} = 0$<br>$\Rightarrow 4\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 0$<br>$\frac{3y-6x}{10y-3x} = 0 \Rightarrow 3y - 6x = 0 \Rightarrow y = 2x$<br>$5(2x)^2 - 3x(2x) + 3x^2 - 48 = 0$<br>$17x^2 = 48 \Rightarrow x = 4\sqrt{\frac{3}{17}}$ (reject $x = -\sqrt{\frac{48}{17}}$ $\because y = 2x$ and $y > 0 \Rightarrow x > 0$) |

| **Q8** | **Solutions** |
|---|---|
| | $x = \mathrm{e}^t \sin t$<br>$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^t \sin t + \mathrm{e}^t \cos t$<br>$= \mathrm{e}^t \left(\sin t + \cos t\right)$<br>$y = \mathrm{e}^{-t} \cos t$<br>$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{e}^{-t}\left(-\sin t\right) - \mathrm{e}^{-t}\cos t$<br>$= -\mathrm{e}^{-t}\left(\sin t + \cos t\right)$<br>$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\mathrm{e}^{-t}\left(\sin t + \cos t\right)}{\mathrm{e}^{t}\left(\sin t + \cos t\right)}$<br>$= -\frac{1}{\mathrm{e}^{2t}}$<br>For $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$,<br>$\frac{\mathrm{d}y}{\mathrm{d}x} \ne 0$, therefore C has no stationary points. |

| **Q9** | **Solutions** |
|---|---|
| **(a)** | $\mathrm{f}(x) = \mathrm{e}^{\mathrm{g}(x)}$<br>For $\mathrm{f}'(x) = \mathrm{g}'(x)\mathrm{e}^{\mathrm{g}(x)} = 0$,<br>Since $\mathrm{e}^{\mathrm{g}(x)} > 0$, $\therefore \mathrm{g}'(x) = 0$<br>Since $\mathrm{g}'\left(\frac{\pi}{2}\right)$=0, $x = \frac{\pi}{2}$.(shown)<br>OR<br>$\mathrm{f}'(x) = \mathrm{g}'(x)\mathrm{e}^{\mathrm{g}(x)}$<br>Since $\mathrm{g}'\left(\frac{\pi}{2}\right)$=0 $\therefore \mathrm{f}'\left(\frac{\pi}{2}\right) = \mathrm{g}'\left(\frac{\pi}{2}\right)\mathrm{e}^{\mathrm{g}\left(\frac{\pi}{2}\right)} = 0$ .<br>$\mathrm{f}''(x) = \mathrm{g}'(x)\mathrm{g}'(x)\mathrm{e}^{\mathrm{g}(x)} + \mathrm{e}^{\mathrm{g}(x)}\mathrm{g}''(x)$<br>At $x = \frac{\pi}{2}$,<br>$\mathrm{f}''\left(\frac{\pi}{2}\right) = \left[\mathrm{g}'\left(\frac{\pi}{2}\right)\right]^2 \mathrm{e}^{\mathrm{g}\left(\frac{\pi}{2}\right)} + \mathrm{e}^{\mathrm{g}\left(\frac{\pi}{2}\right)}\mathrm{g}''\left(\frac{\pi}{2}\right) = 0 + \mathrm{e}(-1) = -\mathrm{e} < 0$<br>It is a maximum point at $x = \frac{\pi}{2}$ |
| **(b)** | $\mathrm{f}(x) = \mathrm{e}^{\sin x}$<br>$\mathrm{f}'(x) = \cos x \mathrm{e}^{\sin x}$<br>Since $\cos x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\mathrm{e}^{\sin x} > 0$ for all real values of *x*,<br>$\mathrm{f}'(x) = \cos x \mathrm{e}^{\sin x} > 0$. Therefore f(x) is increasing on the interval for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. |