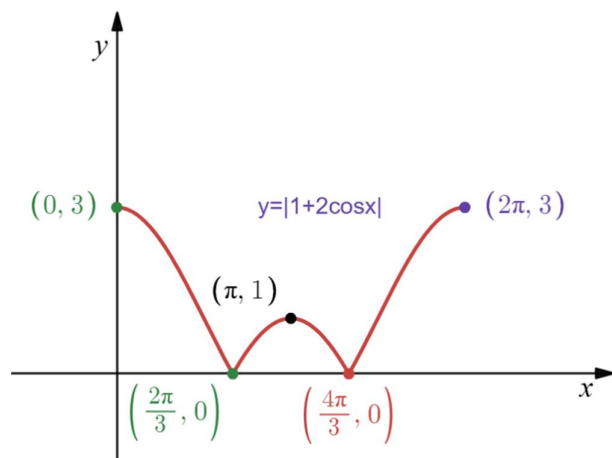


Q1



From graph, $0 < k \leq 1$

Q2

By similar triangles,

$$\frac{r}{h} = \frac{6}{12} = \frac{1}{2} \Rightarrow r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

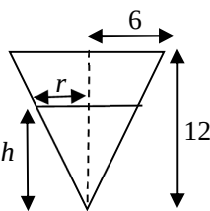
$$\frac{dV}{dh} = \frac{1}{4}\pi h^2$$

Now, $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$$-2 = \frac{1}{4}\pi(4)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{1}{2\pi} \approx -0.159 \text{ cm / min (3 s.f.)}$$

Rate at which the h is decreasing $= \frac{1}{2\pi} \text{ cm / min}$



Q3

(i) Area of $R = \int_0^4 (x-5) - (-\sqrt{25-x^2}) \, dx = 5.59$

(ii) Volume of solid $= \pi \int_{-5}^{-3} 25 - y^2 \, dy + \text{Volume of cylinder} - \text{Volume of cone}$

$$\begin{aligned} &= \pi \int_{-5}^{-3} 25 - y^2 \, dy + \pi(4)^2(2) - \frac{1}{3}\pi(4)^2(4) \\ &= 88.0 \end{aligned}$$

Q4

$$\text{Surface area, } 1500 = 2\pi r^2 + 2\pi rh$$

$$h = \frac{1500 - 2\pi r^2}{2\pi r} = \frac{750 - \pi r^2}{\pi r}$$

$$\text{Volume, } V = \pi r^2 \left(\frac{750 - \pi r^2}{\pi r} \right) = 750r - \pi r^3$$

$$\text{At stationary point, } \frac{dV}{dr} = 750 - 3\pi r^2 = 0$$

$$3\pi r^2 = 750 \quad r = \sqrt{\frac{250}{\pi}} \quad \text{or} \quad r = -\sqrt{\frac{250}{\pi}} \quad (\text{NA since } r > 0)$$

$$\frac{d^2V}{dr^2} = -6\pi r < 0 \quad (\text{since } r > 0). \text{ Hence, } V \text{ is maximum.}$$

$$\begin{aligned} \text{When } r &= \sqrt{\frac{250}{\pi}}, & V &= 750r - \pi r^3 \\ & & &= r(750 - \pi r^2) \\ & & &= \sqrt{\frac{250}{\pi}} \left(750 - \pi \left(\frac{250}{\pi} \right) \right) \\ & & &= 500\sqrt{\frac{250}{\pi}} \\ & & &= 2500\sqrt{\frac{10}{\pi}} \\ & & & \quad (a = 2500) \end{aligned}$$

Q5 (i)(a) $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is perpendicular to the plane.

(i)(b) d is the perpendicular distance from the origin to the plane.

(ii)(a)

$$(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v}) = \mathbf{0}$$

$$\Rightarrow (\mathbf{u} \times \mathbf{u}) - (\mathbf{v} \times \mathbf{v}) - (\mathbf{v} \times \mathbf{u}) + (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$$

Since $\mathbf{u} \times \mathbf{u} = \mathbf{v} \times \mathbf{v} = \mathbf{0}$ and $-(\mathbf{v} \times \mathbf{u}) = \mathbf{u} \times \mathbf{v}$,
we have $2(\mathbf{u} \times \mathbf{v}) = \mathbf{0}$.

Since $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$ are parallel.

$$(ii)(b) |(\mathbf{u} - \mathbf{v}) \times (\mathbf{u} + \mathbf{v})| = 2|\mathbf{u} \times \mathbf{v}| = 2|\mathbf{u}||\mathbf{v}|\sin 90^\circ = 2$$

Q6(a) Since $-2+3i$ is a root,

$$(-2+3i)^3 + a(-2+3i)^2 - 7(-2+3i) + b = 0.$$

$$\Rightarrow 60 - 12i + a(-5 - 12i) + b = 0$$

Comparing real parts, $-5a + b = -60$ -----(1)

Comparing imaginary parts, $-12a = 12$

$$\Rightarrow a = -1$$

Sub into (1): $b = -65$

Method 1

Using GC, the other two roots are $-2-3i$ and 5.

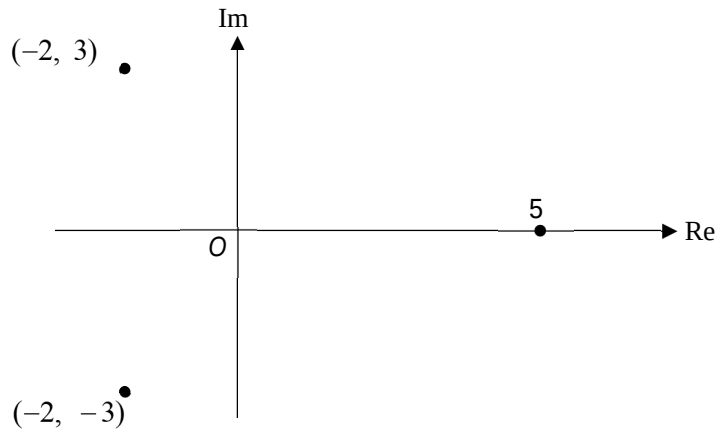
Method 2 (When use of calculator not allowed)

Since $-2+3i$ is a root, and the coefficients in the cubic equation are all real, $-2-3i$ is another root. The last root is a real number c

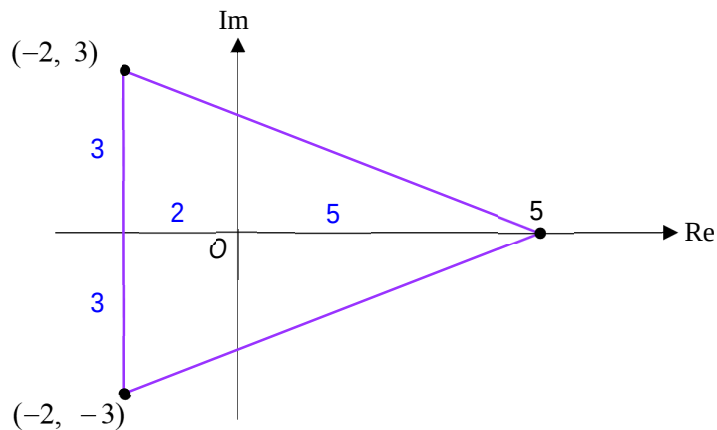
$$\begin{aligned} z^3 - z^2 - 7z - 65 &= [z - (-2+3i)][z - (-2-3i)][z - c] \\ &= (z^2 + 4z + 13)(z - c) \end{aligned}$$

Comparing constant term, last root $= c = 5$

(b)



(c)



$$\text{Area of triangle } PQR = \frac{1}{2}(6)(2+5) = 21 \text{ units}^2$$

Q7 (a)(i) $20v_{n+1} = 17v_n + 14 \Rightarrow v_{n+1} = \frac{17v_n + 14}{20}$

From GC, $v_{15} = 4.495$ (3 decimal places)

NORMAL FLOAT AUTO REAL RADIAN MP		NORMAL FLOAT AUTO REAL RADIAN MP	
INITIAL CONDITION		PRESS * FOR $\Delta T b$	
Plot1	Plot2	Plot3	
TYPE: SEQ(n)	SEQ(n+1)	SEQ(n+2)	
nMin=1			
$u(n+1) = \frac{17u(n) + 14}{20}$			
u(1)=3			
u(2)=			
$v(n+1) =$			
v(1)=			
v(2)=			

n	u			
8	4.1324			
9	4.2125			
10	4.2806			
11	4.3385			
12	4.3878			
13	4.4296			
14	4.4652			
15	4.4954			
16	4.5211			
17	4.5429			
18	4.5615			

n=15

(ii) As $n \rightarrow \infty$, $v_n \rightarrow L$ and $v_{n+1} \rightarrow L$

$$20L = 17L + 14$$

$$3L = 14$$

$$L = \frac{14}{3} \text{ (exact)}$$

(b)(i) $1^2 + 2^2 + 3^2 + \dots + 4n^2$
 $= 1^2 + 2^2 + 3^2 + \dots + (2n)^2$
 $= \sum_{r=1}^{2n} r^2$
 $= \frac{1}{6}(2n)(2n+1)(4n+1)$
 $= \frac{1}{3}n(2n+1)(4n+1)$

(b)(ii) $(n+1)^2 + (n+2)^2 + \dots + 4n^2$
 $= \sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^n r^2$
 $= \frac{1}{3}n(2n+1)(4n+1) - \frac{1}{6}n(n+1)(2n+1)$
 $= \frac{1}{6}n(2n+1)[2(4n+1) - (n+1)]$
 $= \frac{1}{6}n(2n+1)(7n+1)$

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2$$

$$= \sum_{r=1}^{2n} r^2 - \left[2^2 + 4^2 + 6^2 + \dots + (2n)^2 \right] \quad \text{using (i)}$$

$$= \sum_{r=1}^{2n} r^2 - 2^2 \left[1^2 + 2^2 + 3^2 + \dots + n^2 \right]$$

$$= \frac{1}{3}n(2n+1)(4n+1) - 4 \sum_{r=1}^n r^2$$

$$\begin{aligned}
&= \frac{1}{3}n(2n+1)(4n+1) - 4\left[\frac{1}{6}n(n+1)(2n+1)\right] \\
&= \frac{1}{3}n(2n+1)(4n+1) - \frac{2}{3}n(n+1)(2n+1) \\
&= \frac{1}{3}n(2n+1)((4n+1) - 2(n+1)) \\
&= \frac{1}{3}n(2n+1)(2n-1)
\end{aligned}$$

Q8 (a)(i) $\int \cos^2 2x \, dx = \int \frac{1 + \cos 4x}{2} \, dx$

$$= \frac{x}{2} + \frac{1}{8} \sin 4x + C$$

(a)(ii)

$$\begin{aligned}
\int \frac{xe^{x^2}}{\sqrt{e^{x^2} + 2}} \, dx &= \frac{1}{2} \int \frac{2xe^{x^2}}{\sqrt{e^{x^2} + 2}} \, dx \\
&= \frac{1}{2} \frac{(e^{x^2} + 2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\
&= \sqrt{e^{x^2} + 2} + C
\end{aligned}$$

(a)(iii)

$$u = \tan^{-1}(x^2) \quad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{2x}{1+x^4} \quad v = \frac{x^2}{2}$$

$$\begin{aligned}
&\int x \tan^{-1}(x^2) \, dx \\
&= \frac{x^2}{2} \tan^{-1}(x^2) - \int \frac{x^2}{2} \left(\frac{2x}{1+x^4} \right) \, dx \\
&= \frac{x^2}{2} \tan^{-1}(x^2) - \frac{1}{4} \int \frac{4x^3}{1+x^4} \, dx \\
&= \frac{x^2}{2} \tan^{-1}(x^2) - \frac{1}{4} \ln(1+x^4) + C
\end{aligned}$$

(b) $x = 4 \cos^2 \theta$

$$\frac{dx}{d\theta} = -8 \sin \theta \cos \theta$$

When $x = 2$, $4\cos^2 \theta = 2$

$$\cos \theta = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$

When $x = 3$, $4\cos^2 \theta = 3$

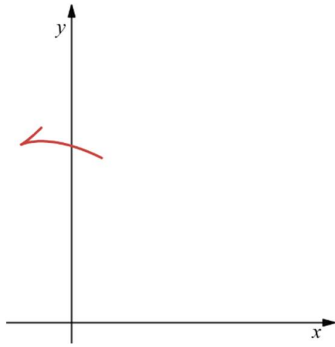
$$\cos \theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6}$$

$$\begin{aligned} & \int_2^3 \frac{1}{x\sqrt{4-x}} \, dx \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{1}{4\cos^2 \theta \sqrt{4-4\cos^2 \theta}} (-8\sin \theta \cos \theta) \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{4\cos^2 \theta \sqrt{4-4\cos^2 \theta}} (8\sin \theta \cos \theta) \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin \theta}{\cos \theta \sqrt{1-\cos^2 \theta}} \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos \theta} \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec \theta \, d\theta \\ &= \left[\ln |\sec \theta + \tan \theta| \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\ &= \ln \left| \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right| - \ln \left| \sec \frac{\pi}{6} + \tan \frac{\pi}{6} \right| \\ &= \ln |\sqrt{2} + 1| - \ln \left| \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right| \\ &= \ln (\sqrt{2} + 1) - \ln \sqrt{3} \\ &= \ln \frac{\sqrt{2} + 1}{\sqrt{3}} \text{ (exact)} \end{aligned}$$

Q9

(i)



$$x = 3t^2 - 4t + \frac{1}{2} \quad y = 3t^3 - \frac{9}{2}t^2 + 2t + \frac{49}{18}$$

$$\frac{dx}{dt} = 6t - 4 = 2(3t - 2)$$

$$\frac{dy}{dt} = 9t^2 - 9t + 2 = (3t - 1)(3t - 2)$$

$$\frac{dy}{dx} = \frac{(3t - 1)(3t - 2)}{2(3t - 2)} = \frac{1}{2}(3t - 1), t \neq \frac{2}{3}$$

(ii) At the stationary point, $\frac{dy}{dx} = 0$

$$\frac{1}{2}(3t - 1) = 0$$

$$t = \frac{1}{3}$$

When $t = \frac{1}{3}$,

$$x = 3\left(\frac{1}{3}\right)^2 - 4\left(\frac{1}{3}\right) + \frac{1}{2} = -\frac{1}{2}$$

$$y = 3\left(\frac{1}{3}\right)^3 - \frac{9}{2}\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right) + \frac{49}{18} = 3$$

Coordinates of Q $\left(-\frac{1}{2}, 3\right)$

Equation of tangent is $y = 3$

(iii) When $y = 3$, $3t^3 - \frac{9}{2}t^2 + 2t + \frac{49}{18} = 3$

Using GC, $t = \frac{5}{6}$ or $t = \frac{1}{3}$ (NA) (Reject since it is pt Q)

$$\text{When } t = \frac{5}{6}, \quad x = 3\left(\frac{5}{6}\right)^2 - 4\left(\frac{5}{6}\right) + \frac{1}{2} = -\frac{3}{4}$$

$\therefore$ coordinates of R is $\left(-\frac{3}{4}, 3\right)$.

$$\frac{dy}{dx} = \frac{1}{2} \left(3 \left(\frac{5}{6} \right) - 1 \right) = \frac{3}{4}$$

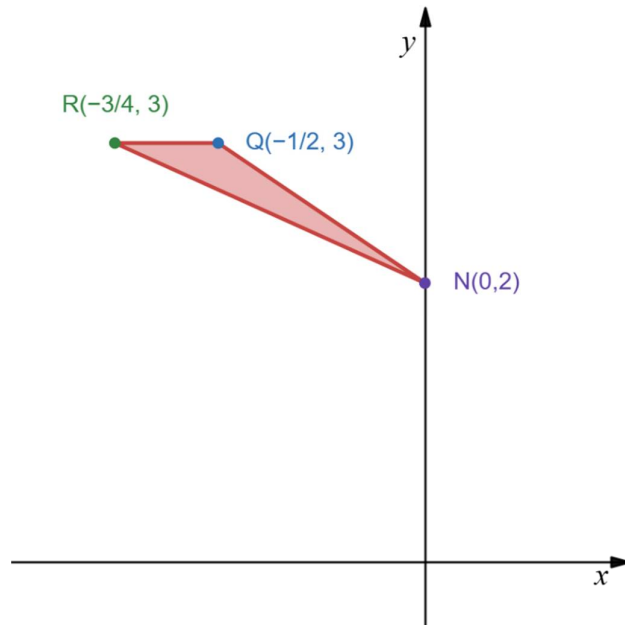
Equation of normal at R :

$$y - 3 = -\frac{4}{3} \left(x + \frac{3}{4} \right)$$

$$y = -\frac{4}{3}x + 2$$

(iv)

Coordinates of $N(0, 2)$.



$$\begin{aligned} \text{Area of } \triangle QNR &= \frac{1}{2} \left(\left| -\frac{3}{4} - \left(-\frac{1}{2} \right) \right| \right) (|3 - 2|) \\ &= \frac{1}{2} \left(\frac{1}{4} \right) (1) \\ &= \frac{1}{8} \text{ unit}^2 \end{aligned}$$

Q10

(a)

Acute angle between L_1 and L_2

$$= \cos^{-1} \frac{\left| \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right|}{\left\| \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\| \left\| \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\|}$$

$$= \cos^{-1} \frac{5}{\sqrt{2}\sqrt{14}}$$

$$= 19.1^\circ$$

(b)

Suppose L_1 and L_2 intersect. Then $\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ c \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}.$

$$\lambda + 2\mu = 0 \quad \text{-----}(1)$$

$$\mu = -2 \quad \text{-----}(2)$$

$$\lambda + 3\mu = -c \quad \text{-----}(3)$$

Sub (2) into (1): $\lambda = 4$

Sub into (3): $c = 2$

Since L_1 and L_2 do not intersect, $c \neq 2$.

(c)

Let B be the point $(4, 1, 0)$ on L_1 .

Perpendicular distance from A to L_1

$$= \frac{\left| \overrightarrow{BA} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right|} = \frac{\left| \begin{pmatrix} 0 \\ 2 \\ c \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right|}{\sqrt{2}}$$

$$= \frac{\left| \begin{pmatrix} -2 \\ c \\ -2 \end{pmatrix} \right|}{\sqrt{2}} = \frac{\sqrt{4 + c^2 + 4}}{\sqrt{2}} = \sqrt{\frac{1}{2}(c^2 + 8)}$$

(d)

Perpendicular distance from A to l

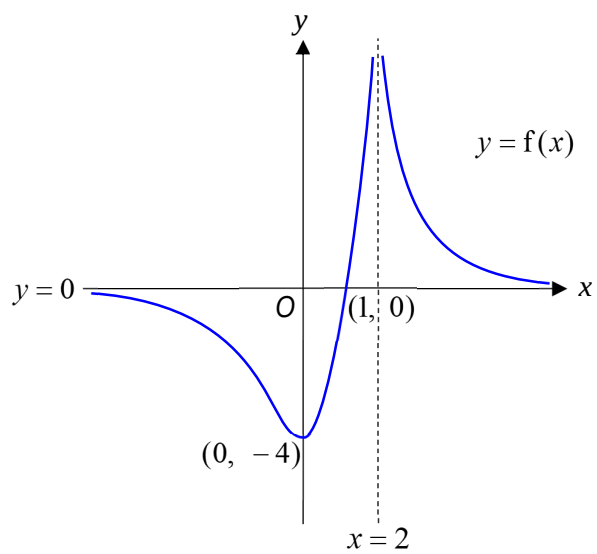
$$= \frac{\left| \overrightarrow{OA} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right|} = \frac{\left| \begin{pmatrix} 4 \\ 3 \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right|}{\sqrt{2}} = \frac{|4 + c|}{\sqrt{2}}$$

$$\text{Since } \sqrt{\frac{1}{2}(c^2 + 8)} = \frac{|4 + c|}{\sqrt{2}},$$

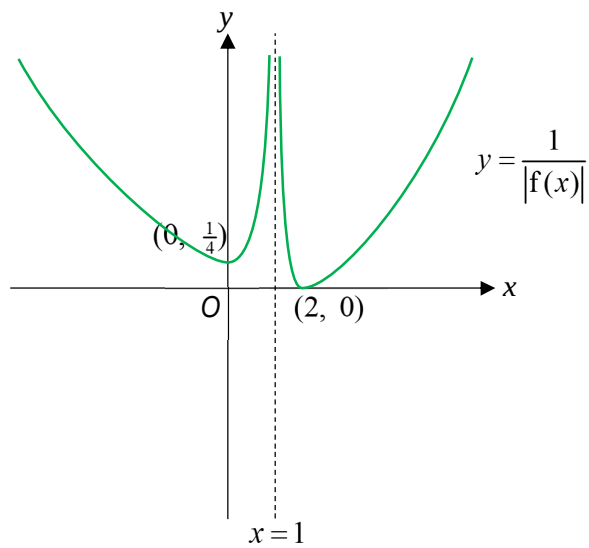
$$c^2 + 8 = c^2 + 8c + 16.$$

$$\Rightarrow c = -1$$

Q11(i) (a)



(b)



(ii) $a=1, b=\frac{1}{3}$

$$y = g(x) \rightarrow y = g(3x) \rightarrow y = g\left[3\left(x + \frac{1}{3}\right)\right] = g(3x+1)$$

$$c = \frac{1}{3}, d = \frac{1}{3}$$

Q12 (i) $\frac{dx}{dt} = 2x - kx^2$

When $x = 8$, $\frac{dx}{dt} = 0$

$$0 = 2(8) - k(8)^2$$

$$64k = 16$$

$$k = \frac{1}{4}$$

$$\frac{dx}{dt} = 2x - \frac{x^2}{4} \text{ (Shown)}$$

(ii) $\frac{dx}{dt} = 2x - \frac{x^2}{4} - 1.75$

$$= -\frac{1}{4}(x^2 - 8x + 7)$$

$$= -\frac{1}{4}[(x-4)^2 - 4^2 + 7]$$

$$= -\frac{1}{4}[(x-4)^2 - 9]$$

$$= -\frac{1}{4}[(x-4)^2 - 3^2] \text{ where } a = 3 \text{ (shown)}$$

(iii) $\frac{dx}{dt} = -\frac{1}{4}[(x-4)^2 - 3^2]$

$$\int \frac{1}{(x-4)^2 - 3^2} dx = -\frac{1}{4} \int 1 dt$$

$$\frac{1}{2(3)} \ln \left| \frac{(x-4)-3}{(x-4)+3} \right| = -\frac{1}{4}t + c$$

$$\ln \left| \frac{x-7}{x-1} \right| = -\frac{3}{2}t + 6c$$

$$\frac{x-7}{x-1} = \pm e^{-\frac{3}{2}t+6c} = Ae^{-\frac{3}{2}t}, \quad A = \pm e^{6c}$$

When $t = 0$, $x = 15$:

$$\frac{15-7}{15-1} = A \Rightarrow A = \frac{4}{7}$$

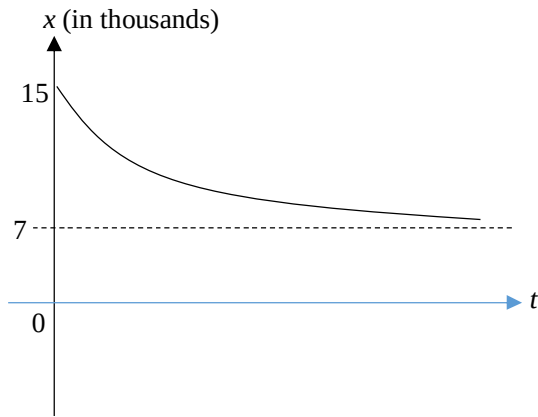
$$\frac{x-7}{x-1} = \frac{4}{7}e^{-\frac{3}{2}t}$$

$$x-7 = \frac{4}{7}e^{-\frac{3}{2}t}x - \frac{4}{7}e^{-\frac{3}{2}t}$$

$$x - \frac{4}{7}e^{-\frac{3}{2}t}x = 7 - \frac{4}{7}e^{-\frac{3}{2}t}$$

$$x = \frac{7 - \frac{4}{7}e^{-\frac{3}{2}t}}{1 - \frac{4}{7}e^{-\frac{3}{2}t}} = \frac{49 - 4e^{-\frac{3}{2}t}}{7 - 4e^{-\frac{3}{2}t}}$$

(iv)



As $t \rightarrow \infty$, $e^{-\frac{3}{2}t} \rightarrow 0$ $x \rightarrow 7$

The number of fish will decrease and approach 7000 in the long run.