RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Tutorial 7C: Maclaurin Series

Section A (Basic Questions)

1 EJC Promo 9758/2022/Q2

It is given that $y = \mathrm{e}^{2x}\cos x$.

(a) Show that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 4\dfrac{\mathrm{d}y}{\mathrm{d}x} - 5y$. [3]

(b) Find the Maclaurin series for y up to the term in x^2. [2]

(c) Hence, show that the Maclaurin series for $\mathrm{e}^x\sqrt{\cos x}$ is $1 + x + \dfrac{1}{4}x^2$, up to the term in x^2. [2]

(a) $y = \mathrm{e}^{2x}\cos x$

Differentiate wrt *x*:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{2x}(-\sin x) + (\cos x)(2\mathrm{e}^{2x}) = -\mathrm{e}^{2x}\sin x + 2y \quad --(1)$$

Differentiate wrt *x*:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\mathrm{e}^{2x}(\cos x) - \sin x(2\mathrm{e}^{2x}) + 2\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -y - 2\left(2y - \frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \textbf{(From (1): } \mathrm{e}^{2x}(\sin x) = 2y - \frac{\mathrm{d}y}{\mathrm{d}x}\textbf{)}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 4\frac{\mathrm{d}y}{\mathrm{d}x} - 5y \text{ (shown)}$$

(b) $x = 0, y = 1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 4(2) - 5(1) = 3$

Thus, $y = 1 + 2x + \dfrac{3}{2!}x^2 + \ldots = 1 + 2x + \dfrac{3}{2}x^2 + \ldots$

(c)

$$\mathrm{e}^x\sqrt{\cos x} = \left(\mathrm{e}^{2x}\cos x\right)^{\frac{1}{2}}$$

$$= \left(1 + 2x + \frac{3}{2}x^2 \ldots\right)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}\left(2x + \frac{3}{2}x^2\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(2x + \frac{3}{2}x^2\right)^2 + \ldots$$

$$= 1 + x + \frac{3}{4}x^2 - \frac{1}{8}(4x^2 + \ldots) = 1 + x + \frac{1}{4}x^2 + \ldots$$

2 TMJC Promo 9758/2022/Q4

It is given that $\mathrm{f}(x)=\dfrac{\sqrt{4-x}}{1+2x}$.

(i) Using standard series from the List of Formulae (MF 27), find the series expansion for $\mathrm{f}(x)$, up to and including the term in x^2. Give the coefficients as exact fractions in their simplest form. [4]

(ii) Hence, by using $x=-\dfrac{1}{6}$, find an estimation for $\sqrt{6}$. Leave your answer as an exact fraction in its simplest form. [2]

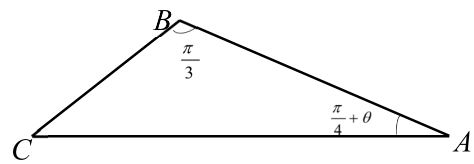
(i)	$\mathrm{f}(x)=\dfrac{\sqrt{4-x}}{1+2x}$ $=\sqrt{4}\left(1-\dfrac{x}{4}\right)^{\frac{1}{2}}(1+2x)^{-1}$ $=2\left(1+\left(\dfrac{1}{2}\right)\left(-\dfrac{x}{4}\right)+\dfrac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\dfrac{x}{4}\right)^2+\ldots\right)\left(1+(-1)(2x)+\dfrac{(-1)(-2)}{2!}(2x)^2+\ldots\right)$ $=2\left(1-2x+4x^2-\dfrac{x}{8}+\dfrac{x^2}{4}-\dfrac{x^2}{128}+\ldots\right)$ $=2-\dfrac{17}{4}x+\dfrac{543}{64}x^2+\ldots \quad \left(\text{up to } x^2\right)$
(ii)	Using $x=-\dfrac{1}{6}$, $\dfrac{\sqrt{4-\left(-\frac{1}{6}\right)}}{1+2\left(-\frac{1}{6}\right)}\approx 2-\dfrac{17}{4}\left(-\dfrac{1}{6}\right)+\dfrac{543}{64}\left(-\dfrac{1}{6}\right)^2$ $\dfrac{\sqrt{\frac{25}{6}}}{\frac{2}{3}}\approx\dfrac{2261}{768}$ $\dfrac{5}{\sqrt{6}}\approx\dfrac{2261}{1152}$ $\dfrac{\sqrt{6}}{5}\approx\dfrac{1152}{2261}$ OR $\dfrac{5\sqrt{6}}{6}\approx\dfrac{2261}{1152}$ $\sqrt{6}\approx\dfrac{5760}{2261}$ $\qquad$ $\sqrt{6}\approx\dfrac{2261}{960}$ **Note:** Need not give both answers, just one of the above answers will do,

3 MI PU2 P2 Promo 9758/2022/Q2

In triangle ABC, angle A is $\left(\frac{\pi}{4}+\theta\right)$ radians and angle B is $\frac{1}{3}\pi$ radians. Show that when θ is sufficiently small for terms in θ^3 and higher powers of θ to be neglected,

$$\frac{AC}{BC} \approx \frac{\sqrt{6}}{2}\left(1-\theta+k\theta^2\right)$$

where k is a constant to be found. [6]

3	Using sine rule, $\frac{AC}{\sin B}=\frac{BC}{\sin A} \Rightarrow \frac{AC}{BC}=\frac{\sin B}{\sin A}$ $\frac{AC}{BC}=\frac{\sin\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{4}+\theta\right)}$ $=\frac{\left(\frac{\sqrt{3}}{2}\right)}{\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta}$ $=\frac{\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{\sqrt{2}}\left(\cos\theta+\sin\theta\right)}$ $=\frac{\sqrt{6}}{2}\left(\cos\theta+\sin\theta\right)^{-1}$  Therefore, when θ is sufficiently small, $\frac{AC}{BC} \approx \frac{\sqrt{6}}{2}\left(1-\frac{\theta^2}{2}+\theta\right)^{-1}$ $=\frac{\sqrt{6}}{2}\left(1-\left(\frac{\theta^2}{2}-\theta\right)\right)^{-1}$ $=\frac{\sqrt{6}}{2}\left(1+\left(\frac{\theta^2}{2}-\theta\right)+\left(\frac{\theta^2}{2}-\theta\right)^2+\ldots\right)$ $\approx \frac{\sqrt{6}}{2}\left(1-\theta+\frac{\theta^2}{2}+\theta^2\right)$ $=\frac{\sqrt{6}}{2}\left(1-\theta+\frac{3}{2}\theta^2\right)$ where $k=\frac{3}{2}$