1 **(a)** Differentiate the following with respect to x and simplify your answer as a single fraction, where necessary.

(i) $y = \ln\left(\dfrac{e^{4x^2}}{\sqrt{x+1}}\right)$ [3]

$$y = \ln\left(\frac{e^{4x^2}}{\sqrt{x+1}}\right)$$

$$y = \ln e^{4x^2} - \ln\sqrt{x+1}$$

$$= 4x^2 - \frac{1}{2}\ln(x+1)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x - \frac{1}{2}\left(\frac{1}{x+1}\right)$$

$$= \frac{16x^2+16x-1}{2(x+1)}$$

1 **(a)** **(ii)** $y = \dfrac{2x}{\sqrt{1+2x}}$ [3]

$$y = \frac{2x}{\sqrt{1+2x}}$$
$$= \frac{2x}{(1+2x)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{2(1+2x)^{\frac{1}{2}} - 2x\left(\frac{1}{2}\right)(1+2x)^{-\frac{1}{2}}(2)}{1+2x}$$
$$= \frac{2(1+2x)^{\frac{1}{2}} - 2x(1+2x)^{-\frac{1}{2}}}{1+2x}$$
$$= \frac{2(1+2x)^{-\frac{1}{2}}[1+2x-x]}{1+2x}$$
$$= \frac{2(1+x)}{\sqrt{(1+2x)^3}} \text{ or } \frac{2(1+x)}{(1+2x)^{\frac{3}{2}}}$$

OR

$$\frac{dy}{dx} = 2(1+2x)^{-\frac{1}{2}} - 2x\left(\frac{1}{2}\right)(1+2x)^{-\frac{3}{2}}(2)$$
$$= 2(1+2x)^{-\frac{3}{2}}[1+2x-x]$$
$$= \frac{2(1+x)}{\sqrt{(1+2x)^3}} \text{ or } \frac{2(1+x)}{(1+2x)^{\frac{3}{2}}}$$

(b) Given that $y = x^3 e^{(x+1)^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 e^{(x+1)^2}\left[\mathrm{f}(x)\right]$, find $\mathrm{f}(x)$. [3]

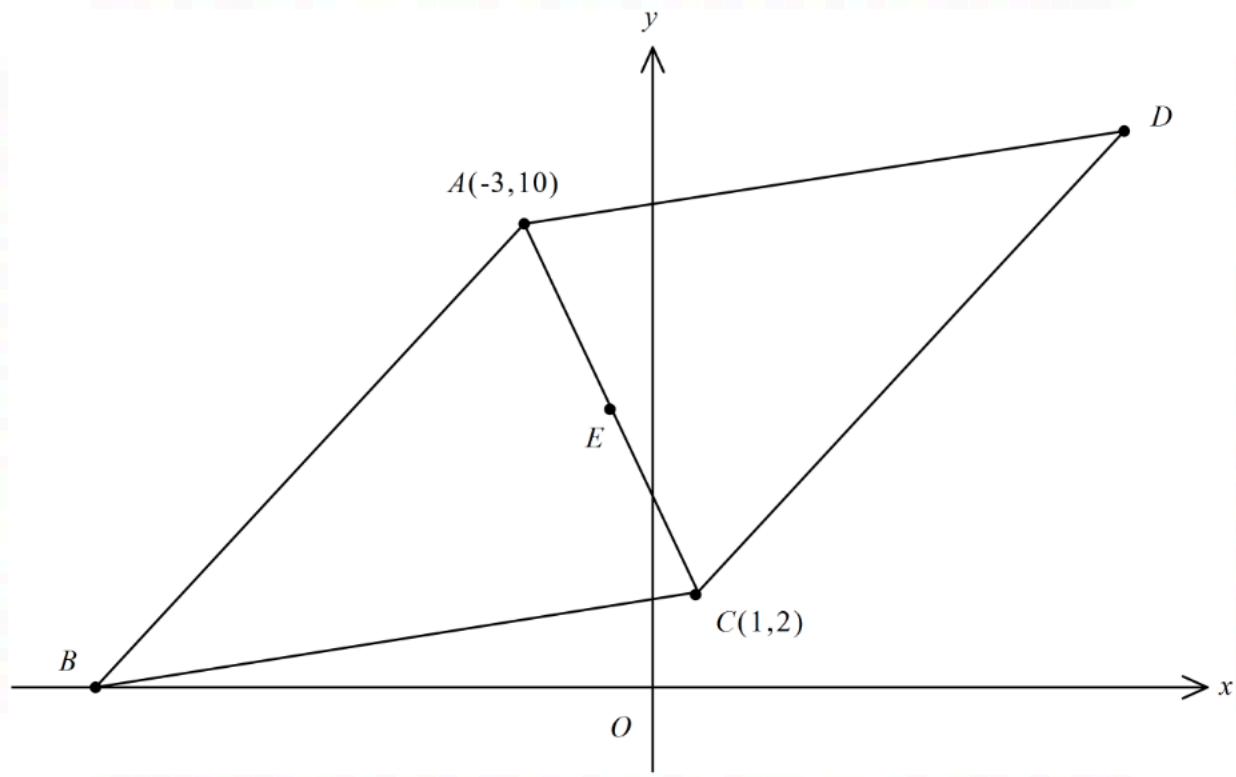
$$y = x^3 e^{(x+1)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 e^{(x+1)^2} + x^3(2)(x+1)e^{(x+1)^2}$$
$$= x^2 e^{(x+1)^2}\left[3 + 2x(x+1)\right]$$
$$= x^2 e^{(x+1)^2}\left[2x^2 + 2x + 3\right]$$

$$\mathrm{f}(x) = 2x^2 + 2x + 3$$

2 The points $A\ (-3,\ 10)$ and $C\ (1,\ 2)$ are vertices of a rhombus $ABCD$.

The point B lies on the x-axis and E is the mid-point of AC.



(a) Calculate the coordinates of E. [1]

$$\text{coordinates of } E = \left(\frac{-3+1}{2}, \frac{10+2}{2}\right)$$
$$= (-1,\ 6)$$

2 **(b)** Calculate the coordinates of B. [3]

Let B be $(a,0)$:

$$\text{Grad of } AC = \frac{10-2}{-3-1} = -2$$

$$\text{Grad of } BE = \frac{1}{2}$$

$$\frac{6-0}{-1-a} = \frac{1}{2}$$

$$-12 = a+1$$

$$a = -13$$

$$B \text{ is } (-13,\ 0)$$

OR

Let B be $(a,0)$:

$$\text{length of } AB = \sqrt{(-3-a)^2 + (10-0)^2}$$

$$\text{length of } BC = \sqrt{(1-a)^2 + (2-0)^2}$$

$$\text{length of } AB = \text{length of } BC$$

$$\sqrt{(-3-a)^2 + 100} = \sqrt{(1-a)^2 + 4}$$

$$9 + 6a + a^2 + 100 = 1 - 2a + a^2 + 4$$

$$8a = -104$$

$$a = -13$$

$$B \text{ is } (-13,\ 0)$$

(c) Calculate the coordinates of D. [2]

Let D be (x, y)

Midpt of BD = Midpt of AC

$$\left(\frac{x-13}{2}, \frac{y+0}{2}\right) = (-1, 6)$$

$$\frac{x-13}{2} = -1; \frac{y+0}{2} = 6$$

$$x = 11, y = 12$$

D is $(11,12)$

3 **(a)** By completing the square, express $-2x^2-8x+c$ in the form $p(x+q)^2+r$, where p and q are constants, and r is an expression in terms of c. [2]

$$
\begin{aligned}
&-2x^2-8x+c \\
&=-2\left(x^2+4x\right)+c \\
&=-2\left(x^2+4x+4\right)+8+c \\
&=-2(x+2)^2+8+c
\end{aligned}
$$

(b) The maximum value of $-2x^2-8x+c$ is 10.

(i) Find the value of c. [1]

$$
\begin{aligned}
8+c&=10 \\
c&=2
\end{aligned}
$$

(ii) State the value of x when the maximum value occurs. [1]

Max value occurs when $x=-2$

4 A polynomial can be expressed by $6x^3 - 13x^2 + Ax + 34 \equiv (2x^2 + Bx + 4)(3x + 7) + 5x + C$.

(a) Find the values of A, B and C. [4]

$6x^3 - 13x^2 + Ax + 34 \equiv (2x^2 + Bx + 4)(3x + 7) + 5x + C$

Comparing coefficients of

x^2: $3B + 14 = -13$

$B = -9$

x: $A = 7(-9) + 12 + 5$

$A = -46$

Comparing constants: $C + 28 = 34$

$C = 6$

(b) Find the remainder when $6x^3 - 13x^2 + Ax + 34$ is divided by $x - 4$. [2]

Let $f(x) = 6x^3 - 13x^2 - 46x + 34$

remainder $= f(4)$

$= 6(4)^3 - 13(4)^2 - 46(4) + 34$

$= 26$

5 A curve has equation $y = \mathrm{f}(x)$ where $\mathrm{f}''(x) = 5 - \dfrac{5}{(x-2)^2}$.

Given that $\mathrm{f}(3) = \mathrm{f}'(3) = 5$, find $\mathrm{f}(x)$. [5]

$\mathrm{f}'(x) = 5x + \dfrac{5}{x-2} + c$, where c is an arbitrary constant

$$\mathrm{f}'(3) = 5(3) + \frac{5}{3-2} + c$$

$$5 = 15 + 5 + c$$

$$c = -15$$

$$\mathrm{f}'(x) = 5x + \frac{5}{x-2} - 15$$

$\mathrm{f}(x) = \dfrac{5}{2}x^2 + 5\ln|x-2| - 15x + d$, where d is an arbitrary constant

$$\mathrm{f}(3) = \frac{5}{2}(3)^2 + 5\ln|3-2| - 15(3) + d$$

$$5 = -\frac{45}{2} + d$$

$$d = \frac{55}{2}$$

$$\mathrm{f}(x) = \frac{5}{2}x^2 + 5\ln|x-2| - 15x + \frac{55}{2}$$

6 The equation of the curve is $y = \dfrac{a\left(ax^2 - 1\right)}{3x + b}$, where a and b are non-zero constants.

The gradient of the tangent to the curve at $\left(0,\ 2\right)$ is 6.

(a) Find the value of a and of b. [5]

$$y = \frac{a\left(ax^2 - 1\right)}{3x + b}$$

at $x = 0, y = 2,$

$$2 = \frac{a(-1)}{b}$$

$$a = -2b$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\left[\frac{(2ax)(3x+b) - \left(ax^2 - 1\right)(3)}{(3x+b)^2}\right]$$

$$= a\left[\frac{6ax^2 + 2abx - 3ax^2 + 3}{(3x+b)^2}\right]$$

at $x = 0, \dfrac{\mathrm{d}y}{\mathrm{d}x} = 6$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6$$

$$a\left(\frac{3}{b^2}\right) = 6$$

$$-2b\left(\frac{3}{b^2}\right) = 6$$

$$b = -1$$

$$a = 2$$

6 **(b)** Show that there is no tangent on the curve which is parallel to the line $y = 2$.

[2]

$$\frac{dy}{dx} = 2\left[\frac{(4x)(3x-1)-(2x^2-1)(3)}{(3x-1)^2}\right]$$

$$= 2\left[\frac{6x^2-4x+3}{(3x-1)^2}\right]$$

For $\frac{dy}{dx} = 0$,

$6x^2 - 4x + 3 = 0$

Discriminant $= (-4)^2 - 4(6)(3)$

$= -56 < 0$ (no real roots)

OR

$$x = \frac{4 \pm \sqrt{4^2 - 4(6)(3)}}{2(6)}$$

$$= \frac{4 \pm \sqrt{-56}}{12} \quad \text{(no real roots)}$$

$\therefore \frac{dy}{dx} \neq 0$, no such tangent exists.

OR

$$6x^2-4x+3=6\left(x^2-\frac{4}{6}x\right)+3$$

$$=6\left(x-\frac{1}{3}\right)^2+3-6\left(\frac{1}{3}\right)^2$$

$$=6\left(x-\frac{1}{3}\right)^2+2\frac{1}{3}$$

since $6\left(x-\frac{1}{3}\right)^2>0, x\neq\frac{1}{3}.$

$$6\left(x-\frac{1}{3}\right)^2+2\frac{1}{3}>2\frac{1}{3}>0$$

and $(3x-1)^2>0,\ \frac{\mathrm{d}y}{\mathrm{d}x}>0$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}\neq 0,$ no such tangent exists.

7 **(a)** Find $\displaystyle\int \frac{7e^{x}-5e^{-x}+2}{3e^{3x}}\,dx.$ [2]

$$\int \frac{7e^{x}-5e^{-x}+2}{3e^{3x}}\,dx$$
$$=\int \frac{7}{3}e^{-2x}-\frac{5}{3}e^{-4x}+\frac{2}{3}e^{-3x}\,dx$$
$$=\frac{7}{3}\left(-\frac{1}{2}\right)e^{-2x}-\frac{5}{3}\left(-\frac{1}{4}\right)e^{-4x}+\frac{2}{3}\left(-\frac{1}{3}\right)e^{-3x}+c, \text{ where } c \text{ is an arbitrary constant}$$
$$=-\frac{7}{6e^{2x}}+\frac{5}{12e^{4x}}-\frac{2}{9e^{3x}}+c$$

(b) Find $\displaystyle\int \frac{4}{\sqrt[3]{(2-5x)^{2}}}\,dx.$ [2]

$$\int \frac{4}{\sqrt[3]{(2-5x)^{2}}}\,dx=\int 4(2-5x)^{-\frac{2}{3}}\,dx$$
$$=\frac{4(2-5x)^{\frac{1}{3}}}{\left(\frac{1}{3}\right)(-5)}+c$$
$$=-\frac{12}{5}\sqrt[3]{2-5x}+c, \quad c\text{: constant}$$

8 **(a)** Solve the equation $5^x = \sqrt{5e^{4x}}$. [3]

$$
\begin{aligned}
5^x &= \sqrt{5e^{4x}} \\
(5^x)^2 &= 5e^{4x} \\
\frac{5^{2x}}{5} &= e^{4x} \\
5^{2x-1} &= e^{4x} \\
(2x-1)\ln 5 &= 4x \\
(2\ln 5 - 4)x &= \ln 5 \\
x &= \frac{\ln 5}{2(\ln 5 - 2)} \\
&= -2.06
\end{aligned}
$$

(b) Solve the equation to find x in terms of a, where $a > 0$.

$2\ln(2x) = 2 + \ln a^2$ [3]

$$
\begin{aligned}
2\ln(2x) &= 2 + \ln a^2 \\
\ln(2x)^2 - \ln a^2 &= 2 \\
\ln\left(\frac{2x}{a}\right)^2 &= 2 \\
2\ln\left(\frac{2x}{a}\right) &= 2 \\
\frac{2x}{a} &= e \\
x &= \frac{ae}{2}
\end{aligned}
$$

OR

$$2\ln(2x) = 2 + \ln a^2$$

$$2x = e^{\frac{2+\ln a^2}{2}}$$

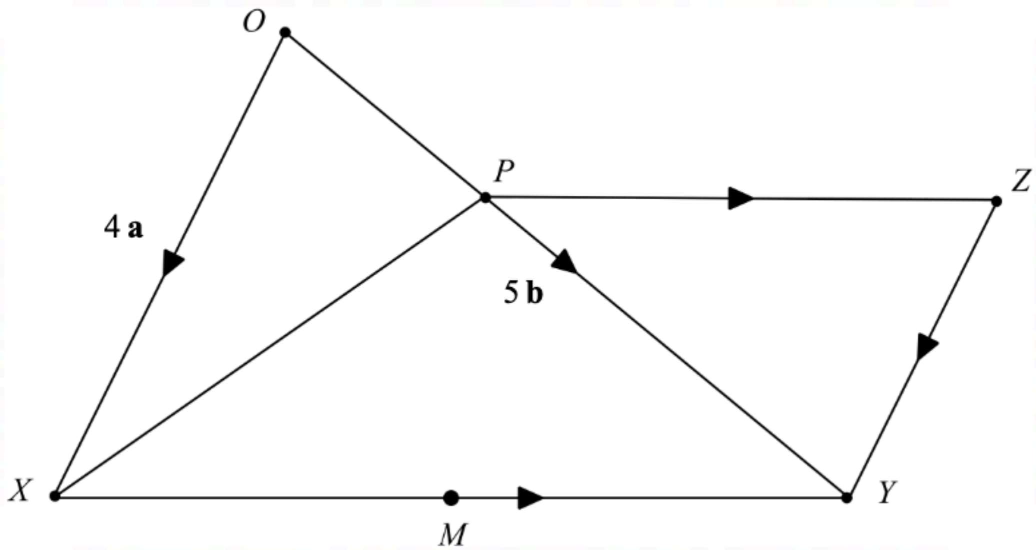
$$x = \frac{1}{2}e^{\frac{2+\ln a^2}{2}}$$

$$x = \frac{1}{2}\left(e^{1+\ln a}\right)$$

$$x = \frac{a\mathrm{e}}{2}$$

9 In the diagram below, OX is parallel to ZY and PZ is parallel to XY.

O, P and Y lie on a straight line.
M is the midpoint of X and Y.
The position vectors of X and Y relative to O are $4\mathbf{a}$ and $5\mathbf{b}$ respectively.



(a) Show that $\overrightarrow{OP} = m\mathbf{b}$ where m is a real constant. [1]

Since O, P and Y lie on a straight line, OP is parallel to OY,

$\overrightarrow{OP} = k(5\mathbf{b})$ for some $k \in \mathbb{R}$

$\therefore \overrightarrow{OP} = m\mathbf{b}, m = 5k$ is a real constant

(b) Given that $\overrightarrow{OP} = (2-n)\mathbf{a} + n\mathbf{b}$ where n is a real constant, find the value of n. [2]

$(2-n)\mathbf{a} + n\mathbf{b} = m\mathbf{b}$
$2-n=0; m=n$
$\therefore n=2$

9 **(c)** Find the numerical value of $\dfrac{\text{area of triangle } PYZ}{\text{area of triangle } YOX}$. [1]

ΔPYZ is similar to ΔYOX and $\overrightarrow{OP} = 2\mathbf{b}$

$\overrightarrow{PY} = 3\mathbf{b}$

$$\frac{\text{Area of } \Delta PYZ}{\text{Area of } \Delta YOX} = \left(\frac{3}{5}\right)^2$$
$$= \frac{9}{25}$$

(d) Find $\overrightarrow{PM}$. [3]

Method 1

$$\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OX} + \overrightarrow{OY}\right)$$
$$= \frac{1}{2}(4\mathbf{a} + 5\mathbf{b}) = 2\mathbf{a} + \frac{5}{2}\mathbf{b}$$
$$\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP}$$
$$= 2\mathbf{a} + \frac{5}{2}\mathbf{b} - 2\mathbf{b}$$
$$= 2\mathbf{a} + \frac{1}{2}\mathbf{b}$$

Method 2

$$\overrightarrow{YM} = \frac{1}{2}\overrightarrow{YX}$$
$$= \frac{1}{2}(4\mathbf{a} - 5\mathbf{b}) = 2\mathbf{a} - \frac{5}{2}\mathbf{b}$$
$$\overrightarrow{PM} = \overrightarrow{PY} + \overrightarrow{YM}$$
$$= 3\mathbf{b} + 2\mathbf{a} - \frac{5}{2}\mathbf{b}$$
$$= 2\mathbf{a} + \frac{1}{2}\mathbf{b}$$

10 Let $\mathrm{f}(x) = 2x^2 + 3x - 3$ and $\mathrm{g}(x) = 4x^3 + ax^2 - 7x + b$, where a and b are constants.

When $\mathrm{g}(x)$ is divided by $\mathrm{f}(x)$, the quotient and remainder are equal.

(a) Find the quotient when $\mathrm{g}(x)$ is divided by $\mathrm{f}(x)$. [3]

Let the quotient be $2x + k$, where k is a constant.

$4x^3 + ax^2 - 7x + b = (2x+k)(2x^2 + 3x - 3) + 2x + k$

Comparing coefficient of x:

$$-6 + 3k + 2 = -7$$

$$k = -1$$

Thus, the quotient is $2x - 1$.

Alternative (1)

$$\begin{array}{r|l}
 & \phantom{4x^3 + ax^2 -{}} 2x - 1 \\
\hline
2x^2 + 3x - 3 & 4x^3 + ax^2 - 7x + b \\
 & \underline{4x^3 + 6x^2 - 6x} \\
 & (a-6)x^2 - x + b \\
 & \underline{ -2x^2 - 3x + 3} \\
 & \underline{ 2x + (b-3)}
\end{array}$$

Since quotient = remainder, thus, quotient $= 2x - 1$

Alternative (2)

$$
\begin{array}{r|l}
 & 2x+\tfrac{1}{2}(a-6) \\
\hline
2x^2+3x-3 & 4x^3+ax^2-7x \qquad +b \\
 & \underline{4x^3+6x^2-6x} \\
 & (a-6)x^2-x \qquad +b \\
 & \underline{(a-6)x^2+\tfrac{3}{2}(a-6)x-\tfrac{3}{2}(a-6)} \\
 & \underline{\left[-1-\tfrac{3}{2}(a-6)\right]x+b-\tfrac{3}{2}(a-6)}
\end{array}
$$

Since quotient = remainder,

$$
\begin{aligned}
-1-\tfrac{3}{2}(a-6) &= 2 \\
a &= 4
\end{aligned}
$$

Thus, quotient $= 2x-1$

(b) Show that $g(x)=(2x-1)^2(x+2)$. [2]

$$
\begin{aligned}
g(x) &= (2x-1)(2x^2+3x-3)+2x-1 \\
&= (2x-1)(2x^2+3x-2) \\
&= (2x-1)(2x-1)(x+2) \\
&= (2x-1)^2(x+2)
\end{aligned}
$$

10 **(c)** Express $\dfrac{2x^3-5x+1}{g(x)}$ as partial fractions. [5]

Let $\dfrac{2x^3-5x+1}{(x+2)(2x-1)^2} \equiv \dfrac{1}{2}+\dfrac{A}{x+2}+\dfrac{B}{2x-1}+\dfrac{C}{(2x-1)^2}$

By Cover Up Rule:

$x=-2: A=\dfrac{-16+10+1}{25}=-\dfrac{1}{5}$

$x=\dfrac{1}{2}:\qquad C=\dfrac{\frac{1}{4}-\frac{5}{2}+1}{\frac{5}{2}}=-\dfrac{1}{2}$

Subst $x=0$: $\quad \dfrac{1}{2}=\dfrac{1}{2}-\dfrac{1}{10}-B-\dfrac{1}{2}$

$B=-\dfrac{3}{5}$

$\therefore \dfrac{2x^3-5x+1}{(x+2)(2x-1)^2} \equiv \dfrac{1}{2}-\dfrac{1}{5(x+2)}-\dfrac{3}{5(2x-1)}-\dfrac{1}{2(2x-1)^2}$

11 A company has 3 outlets, A, B and C, selling chicken, mushroom and vegetable puffs.

The table shows the number of puffs sold on Friday and the selling price of each puff.

	Outlet A	Outlet B	Outlet C	Selling Price
Chicken	53	$2x-17$	72	\$3.50
Mushroom	33	x	42	\$3.20
Vegetable	x	50	75	\$2.80

The number of puffs of each type sold on Friday at each outlet is represented by a 3×3 matrix $\mathbf{N}=\begin{pmatrix}53 & 2x-17 & 72\\33 & x & 42\\x & 50 & 75\end{pmatrix}$.

The selling price of each type of puff is represented by a row matrix **P**.

(a) A matrix **T** represents the product of matrices **N** and **P**.

The elements of **T** represent the total amount, in dollars, collected from selling all types of puffs on Friday by outlet A, B and C respectively.

Find the matrix **T**, leaving your answer in terms of x. [3]

$$\mathbf{P}=\begin{pmatrix}3.5 & 3.2 & 2.8\end{pmatrix}$$

$$\mathbf{T}=\mathbf{PN}$$

$$=\begin{pmatrix}3.5 & 3.2 & 2.8\end{pmatrix}\begin{pmatrix}53 & 2x-17 & 72\\33 & x & 42\\x & 50 & 75\end{pmatrix}$$

$$=\begin{pmatrix}291.1+2.8x & 10.2x+80.5 & 596.4\end{pmatrix}$$

11 **(b)** The total amount of money collected by outlets A and B is \$891.60.

Given that $\mathbf{TU} = (891.6)$, write down the matrix $\mathbf{U}$ and find the value of x. [3]

$$U = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$596.44 + 291.1 + 2.8x + 10.2x + 80.5 = 891.6$$
$$x = 40$$

(c) On Saturday, there was a surge in sales of 20%, 30% and 15% in the outlets A, B and C respectively, compared to Friday.

Write down a matrix $\mathbf{S}$ such that the elements in $\mathbf{TS}$ give the extra amount collected by each outlet on Saturday compared to Friday. [1]

$$\mathbf{S} = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.15 \end{pmatrix}$$

12 **(a)** Three numbers are chosen from the set $\{1,2,3,4,5,6,7,8,9\}$ at random and without replacement.

Find the probability that the sum of the three numbers is odd. [3]

$$\begin{aligned}\mathrm{P}(\text{sum is odd}) &= \mathrm{P}(\text{odd,odd,odd}) + \mathrm{P}(\text{odd,even,even}) \\ &= \frac{5}{9}\times\frac{4}{8}\times\frac{3}{7} + 3\left(\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\right) \\ &= \frac{10}{21}\end{aligned}$$

(b) Alvin chooses two numbers from the set $\{1,2,3,4,5,6,7,8,9\}$ at random and without replacement.

Independently, Ben also chooses two numbers from the set $\{1,2,3,4,5,6,7,8,9\}$ at random and without replacement.

(i) Find the probability that the numbers chosen by Alvin and Ben have at least one number in common. [2]

$$\begin{aligned}\mathrm{P}(\text{at least one number in common}) &= 1 - \mathrm{P}(\text{no numbers in common}) \\ &= 1 - \frac{9}{9}\times\frac{8}{8}\times\frac{7}{9}\times\frac{6}{8} \\ &= \frac{5}{12}\end{aligned}$$

12 **(b)** **(ii)** Find the probability that exactly two prime numbers are chosen by Alvin and none chosen by Ben. [2]

$$\begin{aligned}\text{P}(\text{exactly 2 prime numbers, Alvin}) &= \text{P}(\text{Alvin choose 2 prime and Ben choose 2 non-prime}) \\ &= \frac{4}{9}\times\frac{3}{8}\times\frac{5}{9}\times\frac{4}{8} \\ &= \frac{5}{108}\end{aligned}$$

(iii) Find the probability that exactly two prime numbers are chosen by Alvin and Ben. [2]

$$\begin{aligned}\text{P}(\text{exactly 2 prime numbers}) &= \text{P}(\text{Alvin or Ben choose 2 prime numbers}) \\ &\quad + \text{P}(\text{Each chooses 1 prime number}) \\ &= 2\times\frac{5}{108} \\ &\quad + \frac{4}{9}\times\frac{5}{8}\times 2\times\frac{4}{9}\times\frac{5}{8}\times 2 \\ &= \frac{65}{162}\end{aligned}$$