

Module I Functions & Graphs

Graphing Techniques

1 [2023/ASRJC/Promo/2]

The curve C has equation $y = \frac{x^2 + 2}{x - 1}$, where $x \neq 1$.

- Draw a sketch of the curve C , clearly indicate the equation(s) of its asymptote(s) and the coordinates of any intersection with the axes. [3]
- By drawing an additional graph on the diagram drawn in (i), state the number of real root(s) of the equation $x^2 + 2 = (x - 1)(x^2 + 1)$. [2]

2 [2023/JPJC/Promo/6]

The curve C has equation $y = \frac{2x - 9}{x^2 - 9}$.

- Sketch C , stating clearly the equations of the asymptotes, the coordinates of turning points and axial intercepts. [4]
- By adding a suitable graph in the same diagram in (i), find the number of real roots of the equation $x^4 - 9x^2 + 2x - 9 = 0$. [2]

3 [2023/TMJJC/Promo/5(part)]

A curve has equation $y = \frac{2x - 3}{(2x - 1)(2x + 5)}$.

- Write down the x -coordinate of the x -intercept. [1]
- Find, using an algebraic method, the values that y can take. [4]
- Sketch graph of $y = \frac{2x - 3}{(2x - 1)(2x + 5)}$, indicating clearly the main relevant features of the curve.

4 [2023/RVHS/Promo/5]

Inequalities

The curve C has equation $y = \frac{4x - 32}{(3x + 1)(x - 5)}$, $x \in \mathbb{R}$, $x \neq -\frac{1}{3}, 5$.

- Show, algebraically, that $y \leq \frac{1}{16}$ or $y \geq 1$. [3]
- Sketch the graph of curve C , stating the equations of the asymptotes, coordinates of its turning points and points where it crosses the axes. [3]
- On the same diagram in part (ii), sketch the line with equation $y = 1 - \frac{x}{8}$. Hence, or otherwise, solve the inequality $\frac{4x - 32}{(3x + 1)(x - 5)} \geq 1 - \frac{x}{8}$. [2]

[2023/HCI/Promo/10]

Differentiation

It is given that curve C has the equation $y = ax + \frac{2a}{x+1}$, where $a > 0, x \neq -1$.

- (a) Using an algebraic method, find the exact range of values of y for which there are points on C , leaving your answer in terms of a . [4]
- (b) Find, by differentiation, the exact x -coordinates of the turning point(s) on C . [2]
- (c) Sketch C , clearly indicating the coordinates of the axial intercept(s), stationary point(s) and equations of any asymptote(s). [3]

6 [2023/VJC/Promo/10]

Conics

The curve C has equation $y = \frac{3x^2 + ax + 3}{x+1}$, where a is a constant, $a \neq 6$.

- (a) Find the set of values of a such that C does not cut the x -axis. [2]
- (b) It is given that the equation of C is $y = \frac{3x^2 - 6x + 3}{x+1}$. Sketch C and give the equations of any asymptotes. Also, state the coordinates of any points where C crosses the axes and of any turning points. [4]
- (c) Let k be a positive constant. By sketching a suitable graph in the same diagram in part (b), find an inequality satisfied by k such that the equation

$$k^2(x-1)^2 + \left(\frac{3x^2 - 6x + 3}{x+1} + 12 \right)^2 = k^2$$

has 2 real and distinct roots.

[2]

7 [2023/TJC/Promo/7]

Differentiation

The curve C has equation $y = \frac{x^2 + 2x + a^2}{(x+a)^2}$, where a is a real constant and $a \neq 0, 1$.

- (a) Show that the x -coordinate of the stationary point on C is a . [3]
 - (b) By completing the square for $x^2 + 2x + a^2$, or otherwise, find the exact range of values of a for which C always lies above the x -axis. [3]
- It is given that $a > 1$.
- (c) Find the equations of the asymptotes of C . [2]
 - (d) Sketch C , giving the coordinates of the axial intercept and the stationary point. [2]



Answers

1	(ii) 1	2	(ii) 2
3	(i) $\frac{3}{2}$ (ii) $y \leq \frac{1}{18}$ or $y \geq \frac{1}{2}$	4	(iii) $-\frac{1}{3} < x < 5$ or $x \geq 8$
5	(a) $y \leq -2\sqrt{2}a - a$ or $y \geq 2\sqrt{2}a - a$ (b) $x = -1 \pm \sqrt{2}$	6	(a) $\{a \in \mathbb{R} : -6 < a < 6\}$ (c) $k > 12$
7	(b) $a < -1$ or $a > 1$ (c) $x = -a$ and $y = 1$		

Functions

- 1 [2023/HCI/Promo/6(a)]
A piecewise function is given by

$$f(x) = \begin{cases} \cos\left(\frac{x}{2}\right) & \text{for } 0 \leq x < 2\pi, \\ \frac{2}{\pi}x - 5 & \text{for } 2\pi \leq x < 3\pi, \end{cases}$$

and that $f(x) = f(x + 3\pi)$ for all real values of x .

Sketch the graph of $y = f(x)$ for $-\frac{\pi}{2} \leq x < 4\pi$, labelling the coordinates of the end-points clearly. [3]

- 2 [2023/HCI/Promo/3]
The function h is defined by

$$h: x \mapsto \ln(3x^2 - 9) \quad \text{for } x \in \mathbb{R}, \quad x < -\sqrt{3} \text{ or } x > \sqrt{3}.$$

- (a) Explain why h does not have an inverse. [1]

If the domain of h is further restricted to $-3 \leq x < \lambda$, where $\lambda \in \mathbb{R}$ such that h^{-1} will exist.

- (b) State the largest exact value of λ . [1]

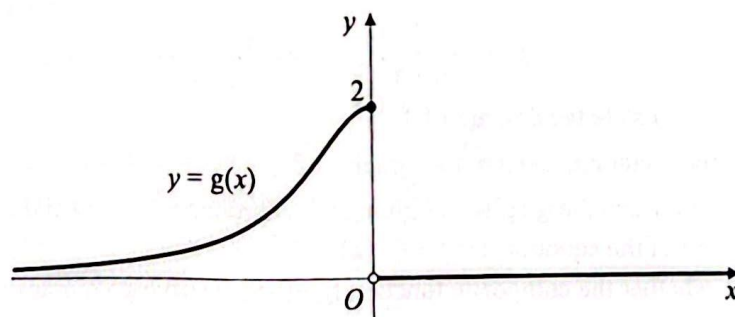
Use the value of λ found in part (b) for the rest of the parts of this question.

- (c) Find $h^{-1}(x)$ and state the domain of h^{-1} . [3]

- (d) Find the set of values of x for which $h^{-1}h(x) = h h^{-1}(x)$. [1]

- 3 [2023/EJC/Promo/7]
A function g is defined by

$$g(x) = \begin{cases} \frac{2}{1+x^2} & \text{if } x \leq 0, \\ 0 & \text{if } x > 0. \end{cases}$$



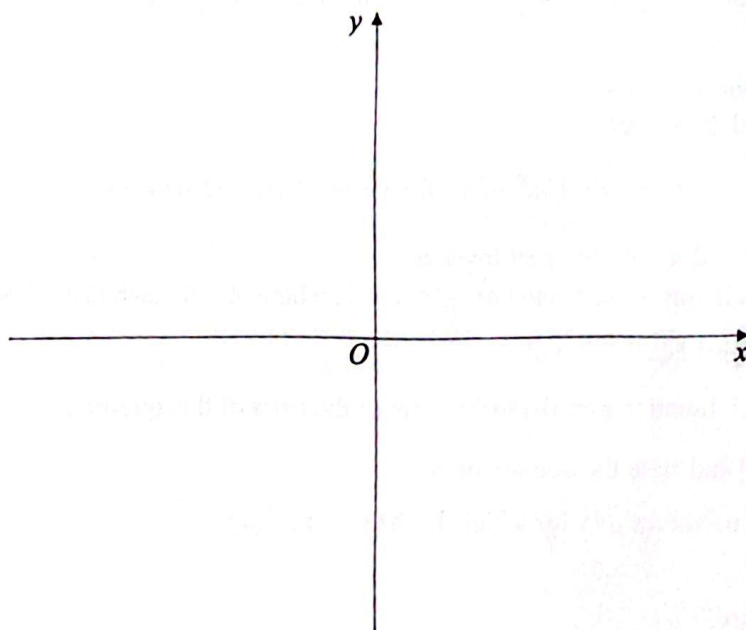
- (a) Give a reason why g does not have an inverse. [1]



- (b) The function g^{-1} exists if the domain of g is restricted to $x \leq k$. State the greatest possible value of k . [1]

In the rest of the question, the domain of g is $x \leq k$, where k takes the value determined in part (b).

- (c) Find $g^{-1}(x)$ and state the domain of g^{-1} . [3]
- (d) Sketch, on the axes given below, the graphs of $y = g(x)$ and $y = g^{-1}(x)$. Label the two graphs clearly.
Write down the equation of the line in which the graph of $y = g(x)$ must be reflected in order to obtain the graph of $y = g^{-1}(x)$. [3]



4 [2023/TJC/Promo/8]

Functions f and g are defined by

$$f: x \mapsto (x-2)^2 - 1, \quad x \in \mathbb{R}, x \leq 1,$$

$$g: x \mapsto \frac{1}{1-x}, \quad x \in \mathbb{R}, x > 1.$$

- (a) Find $f^{-1}(x)$ and state the domain of f^{-1} . [3]
- (b) On the same diagram, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, showing clearly the relationship between the graphs and giving the coordinates of the axial intercepts. Hence find the exact solution of the equation $f(x) = f^{-1}(x)$. [5]
- (c) Determine whether the composite function fg exists, justifying your answer. [2]

5 [2023/ACJC/Promo/5]

(a) Functions f and g are defined by

$$f(x) = e^{x+1} - a \text{ for } x \in \mathbb{R}, x \leq 4,$$

$$g(x) = 2x + 5 \text{ for } x \in \mathbb{R}.$$

where a is a constant.

(i) Find $f^{-1}(x)$ and state its domain, leaving your answer in terms of a . [3]

(ii) It is given that the composite function $g^{-1}f$ exists and $g^{-1}f(-3) = 15$. Without finding $g^{-1}f$, find the value of a . [2]

(b) A curve has equation $y = h(x)$, where

$$h(x) = \begin{cases} 2^x & \text{for } 0 < x \leq 2, \\ 6 - \frac{1}{2}x^2 & \text{for } 2 < x \leq 3, \end{cases}$$

and that $h(x) = h(x-3)$ for all real values of x .

(i) Find $h(9)$. [1]

(ii) Sketch the graph of $y = h(x)$ for $-1 \leq x \leq 4$. [4]

6 [2023/TMJJC/Promo/2]

Functions f and g are defined by

$$f: x \mapsto \frac{2x+7}{x-2}, \quad x \in \mathbb{R}, x \neq 2,$$

$$g: x \mapsto (x-1)^2 - 2, \quad x \in \mathbb{R}, 0 < x < 3.$$

(i) Show that the composite function fg exists. [2]

(ii) Define fg in a similar form and find its range. [3]

7 [2023/RVHS/Promo/3]

The function f is given by $f: x \mapsto \frac{1}{x-2} + 5, x > 2$.

(i) Explain why f^{-1} exists. [1]

(ii) Find f^{-1} in similar form. [3]

(iii) Find ff in a similar form and the exact range of ff . [3]



- 8 [2023/SAJC/Promo/8]
The function f is defined by

$$f: x \mapsto \frac{-x-3}{x+1}, \text{ for } x \in \mathbb{R}, x \neq k.$$

- (i) State the value of k and explain why this value has to be excluded from the domain of f . [2]
(ii) Find $f^{-1}(x)$. Hence find $f^2(x)$. [3]

The function g is defined by

$$g: x \mapsto \sqrt{x+1}, x \in \mathbb{R}, x > -1.$$

- (iii) Find the range of fg . [2]

- 9 [2023/NYJC/Promo/5]
The function g is defined by

$$g: x \rightarrow e^{(x-\frac{1}{2})^2}, x \in \mathbb{R}, -1 < x < a.$$

- (a) State the greatest value of a for which the function g^{-1} exists. [1]
For the rest of the question, let $a = 0$.
(b) Sketch the graphs of $y = g(x)$, $y = g^{-1}(x)$ and $y = g^{-1}g(x)$ on the same diagram, showing clearly the relationship between them. You should also state the coordinates of the end points of the graphs. [3]

The function h is defined by

$$h: x \rightarrow 1 + \ln x, x \in \mathbb{R}, x > 0.$$

- (c) (i) Find $hg(x)$ and state its domain. [2]
(ii) Hence, or otherwise, find the exact value of $(hg)^{-1}(\frac{3}{2})$. [2]

- 10 [2023/ASRJC/Promo/7]
The functions f and g are defined by

$$f: x \mapsto \begin{cases} -x-2, & -5 \leq x < 0, \\ (x-1)^2, & 0 \leq x \leq 5, \end{cases}$$
$$g: x \mapsto |x|-1, \quad -5 \leq x \leq 5.$$

- (i) Find the value of $f(0)$. [1]
(ii) Sketch the graph of f , indicating all axial intercepts and end-points of the graph. Hence explain why f^{-1} does not exist. [4]
(iii) Explain why fg exists. Hence find the exact range of fg . [2]
(iv) If the domain of f is restricted to $[-5, 0)$, find the set of values of x that satisfy the equation $f^{-1}(x) = f(x)$. [1]

11 [2023/JPJC/Promo/9]

The function f is given by

$$f: x \mapsto \sqrt{9-x^2}, \quad -3 < x \leq 3.$$

(i) Show that f does not have an inverse. [1]

(ii) The function f^{-1} exist if the domain of f is restricted to $-3 < x \leq h$. State the largest value of h . [1]

In the rest of the question, the domain of f is restricted to $-3 < x \leq h$, where h is the value stated in (ii).

(iii) Find f^{-1} in a similar form. [3]

(iv) Sketch, on the same diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$. [2]

The function g is defined by $g: x \mapsto \frac{1}{x}, x \neq 0$.

(v) Find $gf(x)$ and the range of gf . [2]

12 [2023/DHS/Promo/9part]

(a) The functions f and g are defined by

$$f: x \mapsto \frac{1}{x+a}, \quad x \in \mathbb{R}, \quad x \neq -a,$$

$$g: x \mapsto x^2 + 2, \quad x \in \mathbb{R},$$

where a is a positive constant.

(i) Show that f^{-1} exists and define f^{-1} in a similar form. [3]

(ii) Show that the composite function gf exists and find its exact range. [3]

(b) The function h is defined by

$$h: x \mapsto \begin{cases} \tan^{-1} x & \text{for } -1 < x \leq 1, \\ -\frac{\pi}{4}x + \frac{\pi}{2} & \text{for } 1 < x \leq 3. \end{cases}$$

It is further given that $h(x+4) = h(x)$ for all real values of x .

Sketch the graph of $y = h(x)$ for $-2 \leq x \leq 6$. [3]

13 [2023/YIJC/Promo/9]

The function f is defined by

$$f: x \mapsto 3 + 2x - x^2, \quad x \in \mathbb{R}, \quad x > 1.$$

(a) Find $f^{-1}(x)$ and state the domain of f^{-1} . [3]

(b) On the same axes, sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = f^{-1}f(x)$. [3]

(c) Find the exact value of x that satisfies the equation $f(x) = f^{-1}(x)$. [2]

Another function g is defined by

$$g: x \mapsto \sqrt{4-x}, \quad x \in \mathbb{R}, \quad x \leq 4.$$

(d) Explain why the composite function gf exists and find $gf(x)$, simplifying your answer. [3]

14 [2023/MI/PU2/Promo/8]

Transformations

The functions f and g are defined by

$$f : x \mapsto \sqrt{x-2}, \quad x \in \mathbb{R}, x \geq 2,$$

$$g : x \mapsto 3x^2 - 6x + 5, \quad x \in \mathbb{R}, 0 \leq x \leq 4.$$

(i) Sketch the graph of $y = f(x)$ and state the range of f . [2]

(ii) Find $f^{-1}(x)$ and write down the domain of f^{-1} . [2]

(iii) Show that the composite function fg exists. [2]

(iv) Find $fg(x)$ and find the range of fg . [3]

The domain of g is now defined to be $x \in \mathbb{R}$.

(v) Describe fully a sequence of transformations which transforms the curve $y = g(x)$ onto the curve $y = x^2 - 1$. [3]

15 [2023/MI/PU1/Promo/8]

The functions f and g are defined by

$$f : x \mapsto x^2 - 2x + 5 \quad \text{for } x \in \mathbb{R},$$

$$g : x \mapsto \frac{1}{x-1} \quad \text{for } x \in \mathbb{R}, x \neq 1.$$

(i) Show that the composite function fg exists. [2]

(ii) Find an expression for $fg(x)$ and state its domain. (There is no need to express $fg(x)$ as a single algebraic fraction.) [2]

(iii) Find the range of fg . [2]

For the rest of the question, the domain of f is now restricted to $(-\infty, 1]$.

(iv) Find $f^{-1}(x)$ and state the domain of f^{-1} . [3]

(v) Find the range of values of x that satisfies the equation $ff^{-1}(x) = x$. [1]

(vi) It is given that $g^{-1}(a) = b$, where a and b are constants and $a \neq 0, b \neq 1$. Without finding an expression for $g^{-1}(x)$, find a in terms of b . [2]

6 [2023/RI/Promo/9]

It is given that

$$f: x \mapsto \frac{1}{x-4}, \text{ where } x \in \mathbb{R}, x \neq 4,$$

$$g: x \mapsto \ln(x+2), \text{ where } x \in \mathbb{R}, x > -2.$$

- (a) Explain why the composite function gf exists and find gf in a similar form. [3]
- (b) Find the range of gf . [1]
- (c) Explain why f does not have an inverse. [1]
- (d) If the domain of f is further restricted to $x < k$, state the maximum value of k such that the function f^{-1} exist. [1]

In the rest of this question, the domain of f is $x \in \mathbb{R}, x < 3$.

- (e) Find $f^{-1}(x)$ and state the domain of f^{-1} . [3]
- (f) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram and state the geometrical relation between the two graphs. [3]

17 [2023/NJC/Promo/7]

A function f is defined by $f: x \mapsto \frac{2x+3}{x-2}, x \in \mathbb{R}, x \neq 2$.

- (i) Find the domain of f^{-1} . [2]

A function h is said to be self-inverse if h and h^{-1} are the same function.

- (ii) Show that f is self-inverse. [2]
- (iii) State the value of $f^{2023}(6)$. [1]

Another function g has range $(k, k+10]$, where k is a constant.

- (iv) Find the range of values of k such that the composite function fg exists. [3]

Assume for the remainder of this question that fg exists and $k \neq 2$.

- (v) Find the range of fg in terms of k . [3]
- (vi) Given that $fg(x) = \frac{2x+2}{1-6x}$, find $g(x)$ in terms of x . [3]

Answers

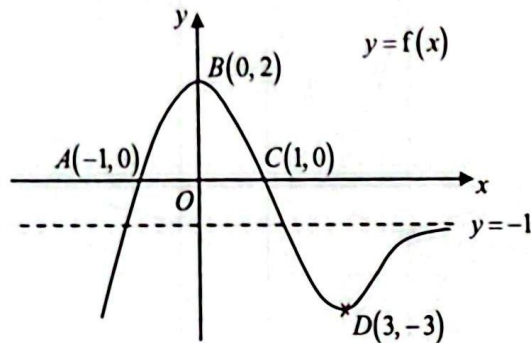
1	—	2	(b) $\lambda = -\sqrt{3}$ (c) $h^{-1}(x) = -\sqrt{\frac{1}{3}e^x + 3}; D_{h^{-1}} = (-\infty, \ln 18]$ (d) $\{x \in \mathbb{R} : -3 \leq x < -\sqrt{3}\}$
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3	(b) 0 (c) $g^{-1}(x) = -\sqrt{\frac{2}{x}} - 1$, $D_{g^{-1}} = (0, 2]$	4	(a) $f^{-1}(x) = 2 - \sqrt{x+1}$; $D_{f^{-1}} = [0, \infty)$ (b) $x = \frac{5 - \sqrt{13}}{2}$; fg exists
5	(a)(i) $D_{f^{-1}} = R_f = (-a, e^7 - a]$ (a)(ii) $a = -34$ (b)(i) $h(9) = \frac{3}{2}$	6	(ii) $fg: x \mapsto \frac{2x^2 - 4x + 5}{x^2 - 2x - 3}$, $x \in \mathbb{R}$, $0 < x < 3$, $R_{fg} = \left(-\infty, -\frac{3}{4}\right]$
7	(ii) $f^{-1}: x \mapsto \frac{1}{x-5} + 2$, $x > 5$ (iii) $ff: x \mapsto \frac{x-2}{3x-5} + 5$, $x > 2$; $R_{ff} = \left(5, \frac{16}{3}\right)$	8	$k = -1$ (ii) $f^{-1}(x) = \frac{-x-3}{x+1}$; $f^2(x) = ff^{-1}(x) = x$ (iii) $R_{fg} = (-3, -1)$
9	(a) $\frac{1}{2}$ (c)(i) $hg(x) = 1 + \left(x - \frac{1}{2}\right)^2$; $D_{hg} = (-1, 0)$ (c)(ii) $\frac{1 - \sqrt{2}}{2}$	10	(i) 1 (iii) $R_{fg} = (-2, -1] \cup [0, 9]$ (iv) $\{x \in \mathbb{R}, -2 < x < 0\}$
11	(ii) 0 (iii) $f^{-1}: x \mapsto -\sqrt{9-x^2}$, $0 < x \leq 3$ (v) $gf(x) = \frac{1}{\sqrt{9-x^2}}$, $-3 < x \leq 0$; $R_{gf} = \left[\frac{1}{3}, \infty\right)$	12	(a)(i) $f^{-1}: x \mapsto \frac{1}{x} - a$, $x \in \mathbb{R}$, $x \neq 0$ (a)(ii) $R_{gf} = (2, \infty)$
13	(a) $f^{-1}(x) = 1 + \sqrt{4-x}$; $D_{f^{-1}} = (-\infty, 4)$ (c) $x = \frac{1 + \sqrt{13}}{2}$ (d) $gf(x) = x - 1$	14	(i) $R_f = [0, \infty)$ (ii) $f^{-1}(x) = x^2 + 2$; Domain of $D_{f^{-1}} = [0, \infty)$ (iv) $fg(x) = \sqrt{3x^2 - 6x + 3}$, $R_{fg} = [0, \sqrt{27}]$
15	(ii) $\frac{1}{(x-1)^2} - \frac{2}{x-1} + 5$; $(-\infty, 1) \cup (1, \infty)$ (iii) $[4, \infty)$ (iv) $1 - \sqrt{x-4}$; $[4, \infty)$ (v) $x \geq 4$ (vi) $a = \frac{1}{b-1}$	16	(a) $gf: x \mapsto \ln\left(\left \frac{1}{x-4}\right + 2\right)$, where $x \in \mathbb{R}, x \neq 4$ (b) $R_{gf} = (\ln 2, \infty)$ (d) 4 (e) $f^{-1}(x) = 4 - \frac{1}{x}$, $D_{f^{-1}} = (0, 1)$
17	(i) $D_{f^{-1}} = (-\infty, 2) \cup (2, \infty)$ (iii) $\frac{15}{4}$ (iv) $k < -8$ or $k \geq 2$ (v) $\left[\frac{2k+23}{k+8}, \frac{2k+3}{k-2}\right)$ (vi) $g(x) = \frac{1-2x}{2x}$ (or $\frac{1}{2x} - 1$)		

Transformations

1 [2023/VJC/Promo/2]

The diagram shows the curve $y = f(x)$. The curve passes through the points $A(-1, 0)$, $B(0, 2)$, $C(1, 0)$ and $D(3, -3)$.



On separate clearly labelled diagrams, sketch the graphs of

(a) $y = -\frac{1}{2}f(x)$, [2]

(b) $y = f(1-x)$. [2]

2 [2023/VJC/Promo/5]

Conics

The curve C has equation $x^2 - 4y^2 = 4$.

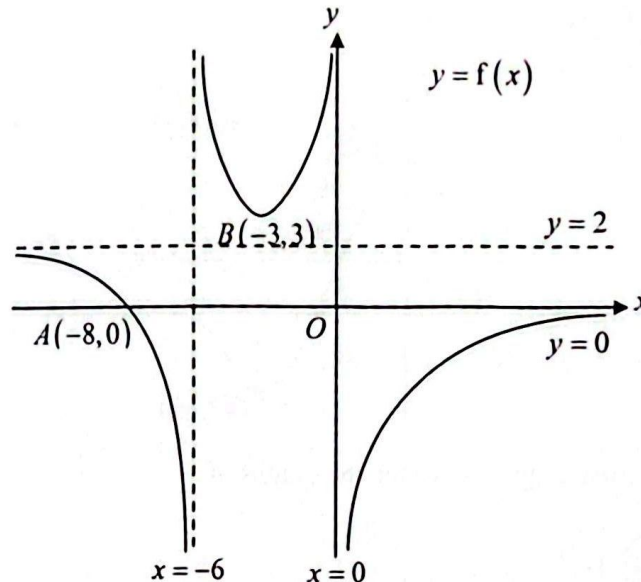
(a) Sketch C , stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

(b) Describe a pair of transformations which transforms the graph of C onto the graph of $(x-2)^2 - y^2 = 4$. [2]



3 [2023/TMJC/Promo/3]

The diagram shows the graph of $y = f(x)$ with asymptotes $x = -6$, $x = 0$, $y = 0$ and $y = 2$. The curve cuts the x -axis at $A(-8, 0)$ and has a turning point at $B(-3, 3)$.



Sketch, on separate diagrams, the graph of the following, showing clearly the corresponding coordinates of points A and B and the equations of the asymptotes.

- (i) $y = |f(x)|$ [2]
- (ii) $y = f(2x - 3)$ [3]

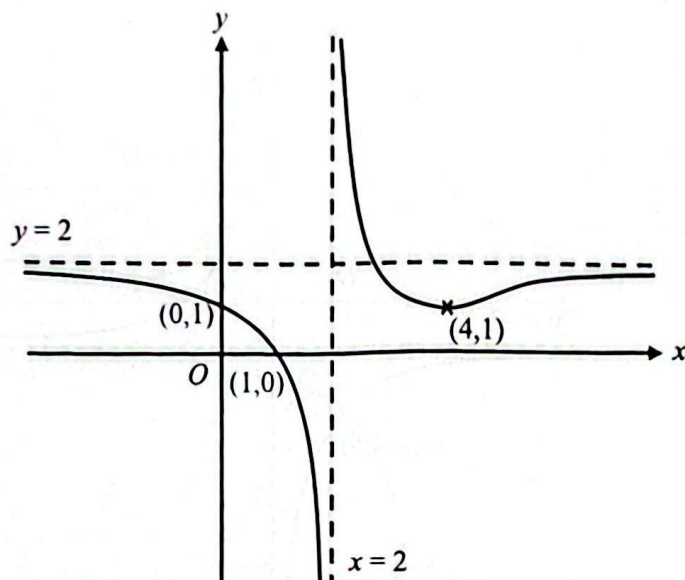
4 [2023/MI/Promo/3]

Conics

- (i) Sketch the curve C with equation $x^2 - (y + 1)^2 = 1$, indicating clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes. [3]
- (ii) The curve C undergoes a sequence of three transformations as follows:
 - I: A scaling parallel to x -axis with a scale factor of 4,
 - II: A scaling parallel to y -axis with a scale factor of $\frac{1}{2}$,
 - III: A translation of 2 units in the positive y -direction.Find the equation of the resulting curve. [3]

5 [2023/II/Promo/4]

The diagram below shows the graph of $y = f(x)$. The asymptotes of the graph are $x = 2$ and $y = 2$. The graph crosses the x -axis at $(1, 0)$ and the y -axis at $(0, 1)$, and has a turning point at $(4, 1)$.



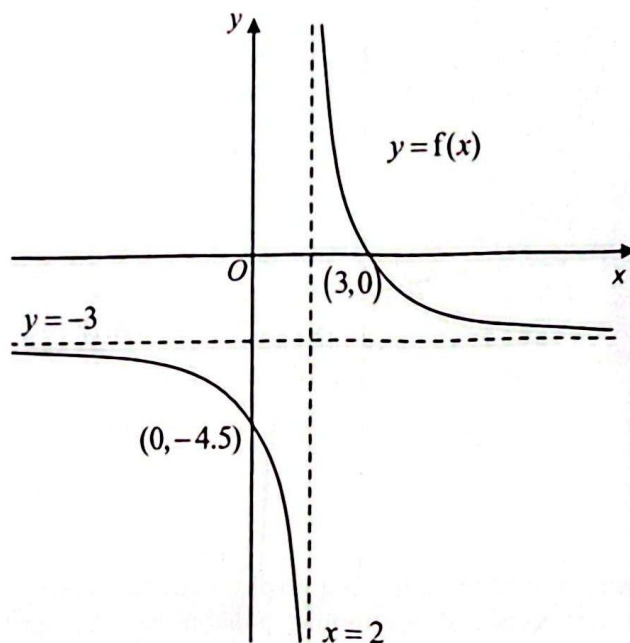
On separate diagrams, sketch the following graphs, indicating clearly the coordinates of any points of intersections with both axes and any turning point(s), and the equations of any asymptotes where possible.

(i) $y = f(|x|) + 1$, [3]

(ii) $y = \frac{1}{f(x)}$. [3]

6 [2023/MI/Promo/3] **Derivative Graph**

The diagram below shows the graph of $y = f(x)$. The graph intersects the x -axis at $(3, 0)$ and the y -axis at $(0, -4.5)$. It has asymptotes $x = 2$ and $y = -3$.



Sketch the following graphs and state, where applicable, the axial intercepts and the equations of any asymptotes.

- (i) $y = f(|x| - 2)$, [3]
- (ii) $y = \frac{1}{f(x)}$, [3]
- (iii) $y = f'(x)$. [2]

[2023/RI/Promo/7]

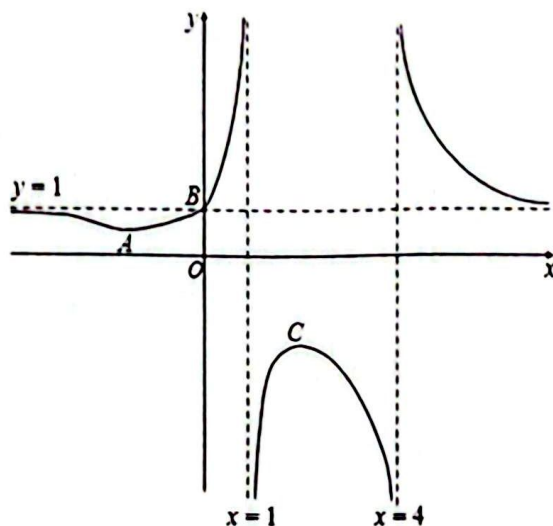
Differentiation

- (a) A curve C has equation $y = \frac{ax-3}{(x-3)(x-1)}$ where a is a real constant such that $a \neq 1, 3$.

Determine the range of values of a for which the curve C has no turning points.

[4]

(b)



The diagram above shows the graph of $y = f(x)$. It has asymptotes $y = 1$, $x = 1$ and $x = 4$.

The curve cuts the y -axis at the point $B(0, 1)$, has a minimum at the point $A\left(-2, \frac{2}{3}\right)$ and a maximum at the point $C(2, -2)$.

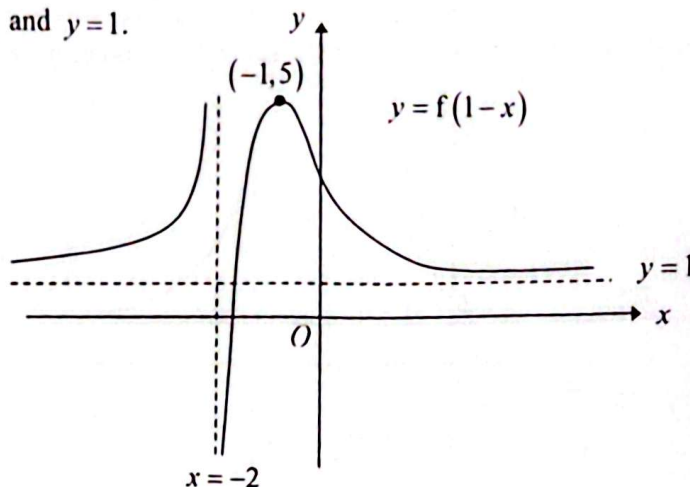
By showing clearly the equations of asymptotes and the coordinates of the points corresponding to A , B and C where possible, sketch, on separate diagrams, the graphs of

(i) $y = 3f(x+1)$, [4]

(ii) $y = \frac{1}{f(x)}$. [4]

8 [2023/DHS/Promo/3] **Derivative Graph**

The diagram shows the curve $y = f(1-x)$. The curve has a maximum point at $(-1, 5)$ and two asymptotes $x = -2$ and $y = 1$.

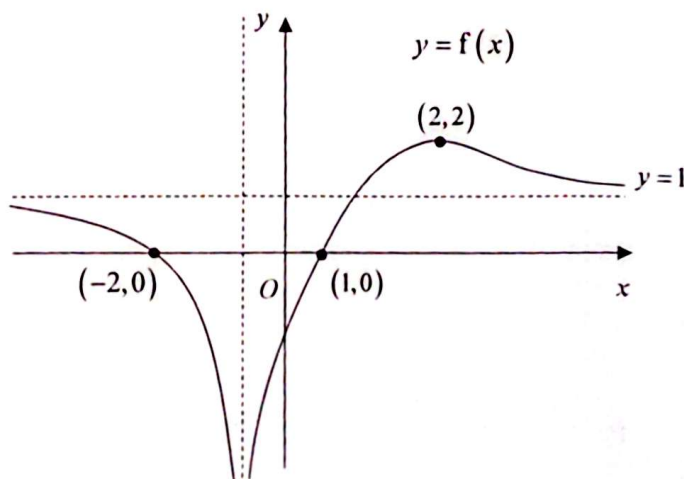


On separate diagrams, sketch the graphs of

- (a) $y = f(x)$, [2]
(b) $y = g'(x)$, where $g(x) = f(1-x)$. [2]

9 [2023/EJC/Promo/2] **Derivative Graph**

The diagram below shows the graph of $y = f(x)$. The curve passes through the x -axis at $(-2, 0)$ and $(1, 0)$, and has a maximum point with coordinates $(2, 2)$. The lines $x = -1$ and $y = 1$ are asymptotes to the graph.



Stating the equations of any asymptotes and the coordinates of any points of intersection with the axes and stationary points, where possible, sketch the graphs of

- (a) $y = 3f(x+a)$, where a is a positive constant such that $1 < a < 2$, [3]
(b) $y = f'(x)$. [3]

10 [2023/NJC/Promo/2] **Conics**

A curve D has equation $\frac{(x+2)^2}{2^2} - (y-1)^2 = 1$.

- Another curve E has equation $x^2 - y^2 = 1$. Describe a sequence of transformations that transforms the graph of E onto D . [3]
- Find the equations of the asymptotes of D . [2]
- Sketch D , stating the equations of the asymptotes and the coordinates of the vertices. [3]
- Given an ellipse with equation $\left(\frac{x+2}{c}\right)^2 + (y-1)^2 = 1$, state the possible values of c such that the ellipse does not intersect D . [1]

11 [2023/ACJC/Promo/3] **Conics**

- (a) A curve with equation $\frac{(x-2)^2}{4} + y^2 = 1$ undergoes, in succession, the following transformations:

A: A stretch with scale factor $\frac{1}{2}$ parallel to the y -axis.

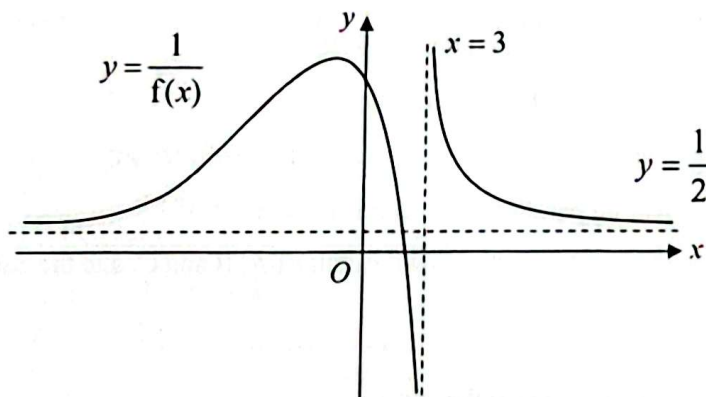
B: A translation of 5 units in the positive x -direction.

C: A reflection about y -axis.

Find the equation of the resulting curve. [3]

- (b) The diagram shows a sketch of the curve $y = \frac{1}{f(x)}$. The curve cuts the axes at $(2, 0)$ and $(0, 3)$.

It has a stationary point at $(-1, 4)$ and equations of asymptotes $x = 3$ and $y = \frac{1}{2}$.



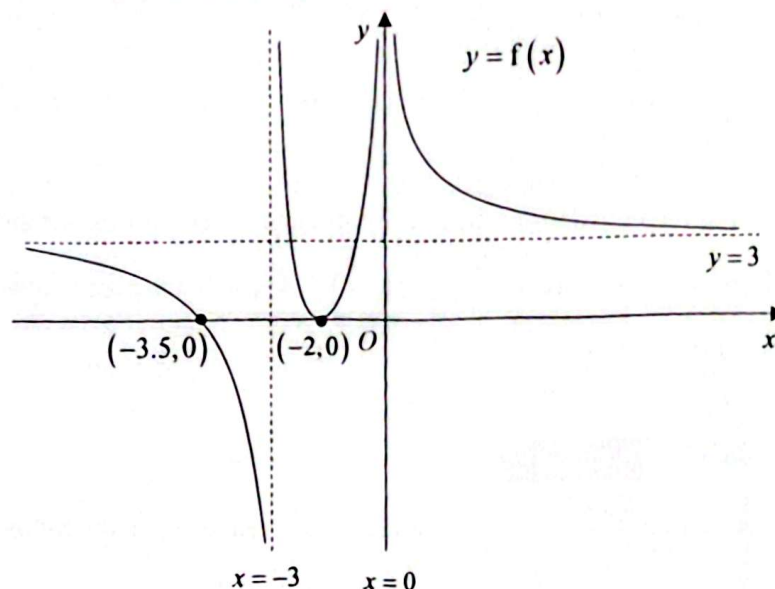
Sketch the following curves and state the equations of the asymptotes, the coordinates of the turning points and of points where the curve crosses the axes.

- (i) $y = f(x)$, [3]

- (ii) $y = -\frac{1}{f(x)} + 4$. [3]



12 [2023/SAJC/Promo/6] **Derivative Graph**
(a)



The diagram above shows the graph of $y = f(x)$. The curve crosses the x -axis at $(-3.5, 0)$ and has a minimum point at $(-2, 0)$. The lines $x = -3$, $x = 0$ and $y = 3$ are asymptotes of the curve.

On separate diagrams, sketch the graphs in (i) and (ii), stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes and of any turning points in exact form, if possible.

(i) $y = f(|x|)$, and

(ii) $y = f'(x)$.

(b) The transformations A, B and C are given as follows:

A: Reflection about the x -axis;

B: Translation of 4 units in the positive x -direction;

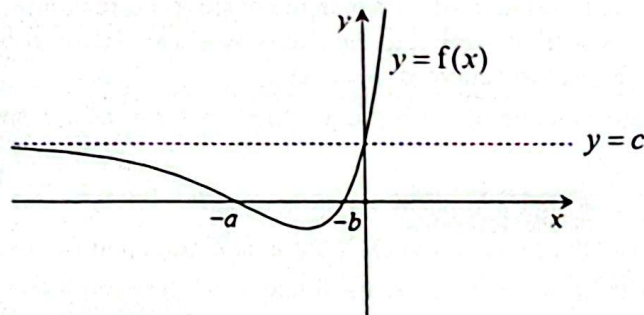
C: Scaling parallel to the x -axis by a factor of 2.

A curve undergoes in succession, the transformations A, B and C, and the equation of the resulting curve is $y = \frac{1}{3}(x+1)^2$.

Determine the original equation of the curve.

3 [2023/YJC/Promo/8] **Derivative Graph**

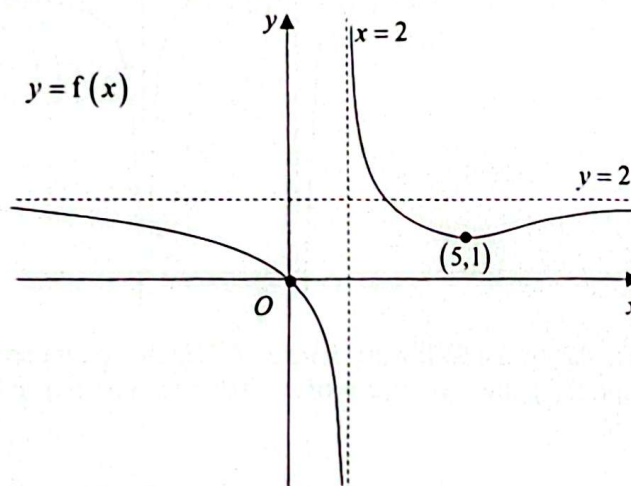
- (a) The curve with equation $y = f(x)$ is transformed by a translation of 2 units in the negative x -direction, followed by a scaling with scale factor 3 parallel to the x -axis, and followed by a translation of 4 units in the positive y -direction. The equation of the resulting curve is $y = 2e^{x^2-36}$. Find the equation of the curve $y = f(x)$. [3]
- (b) The diagram shows the graph of $y = f(x)$ with a horizontal asymptote $y = c$ and the curve meets the axes at $(-a, 0)$, $(-b, 0)$ and $(0, c)$.



- (i) Sketch the graph of $y = 2f(-x)$. [2]
- (ii) Sketch the graph of $y = \frac{1}{f(x)}$. [2]
- (iii) Sketch the graph of $y = f'(x)$. [2]

14 [2023/JPJC/Promo/10]

- (a) The diagram below shows the graph of $y = f(x)$. The curve passes through the origin and has a minimum point at $(5, 1)$. The lines $x = 2$ and $y = 2$ are asymptotes of the curve.



On separate diagrams sketch the graphs of

- (i) $y = f(2x) - 3$, [2]
- (ii) $y = f(|x|)$. [2]



(b) The equation of a curve is $y = g(x)$, where $g(x) = (x+a)^2 + b$ and that a and b are positive constants.

- (i) By sketching the curve $y = g(x)$, sketch the curve $y = \frac{1}{g(x)}$. State, in terms of a and b , the coordinates of any points where $y = \frac{1}{g(x)}$ crosses the axes and of any turning points. [3]

Another curve $y = h(x)$ undergoes in succession, the following transformations:

A: A translation of a units in the positive x direction;

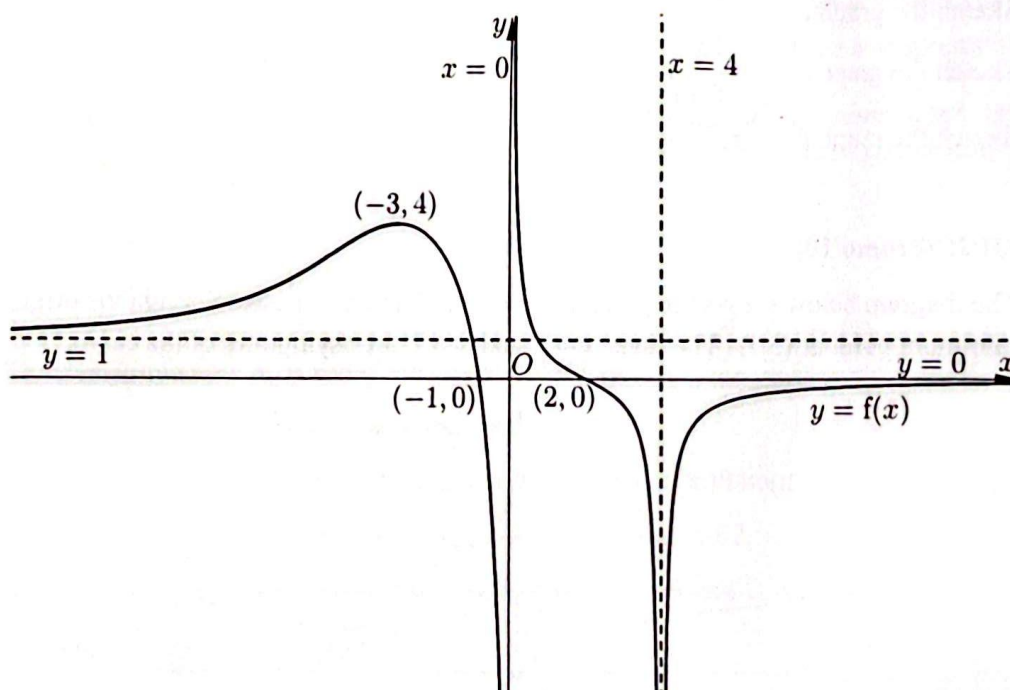
B: A scaling parallel to the y -axis by a scale factor of b ;

The equation of the resulting curve is $y = g(x)$.

- (ii) Determine the equation of the curve $y = h(x)$, in terms of a , b and x . [2]

15 [2023/NJC/Promo/4(a)] **Derivative Graph**

The diagram below shows the curve $y = f(x)$. It has a maximum point at $(-3, 4)$ and crosses the x -axis at $(-1, 0)$ and $(2, 0)$. The lines $x = 0$, $x = 4$, $y = 0$ and $y = 1$ are asymptotes of the curve.



On separate diagrams, sketch the following graphs. Label clearly, where they can be determined, the equations of the asymptote(s), the coordinates of the stationary point(s) and the point(s) where the curves cross the x - and y -axes.

- (i) $y = f(2 - |x|)$, [3]
- (ii) $y = \frac{1}{f(x)}$, [4]
- (iii) $y = f'(x)$, [3]

16 [2023/ASRJC/Promo/9] **Conics** **Inequalities**

- (i) State a sequence of three transformations which transform the graph with equation $x^2 + y^2 = 1$ to the graph with equation $\frac{(x+1)^2}{4} + \frac{y^2}{9} = 1$. [3]
- (ii) Sketch, on the same diagram, the graphs of $y = |2x - 1|$ and $9(x+1)^2 + 4y^2 = 36$ for $y \geq 0$. State clearly the axial intercepts of both the graphs and the coordinates of the intersections between the two graphs. [5]
- (iii) Hence solve the inequality $\sqrt{36 - 9(x+1)^2} \leq 2|2x - 1|$. [2]
- (iv) Using your answer in (iii), find the exact range of values of x for which $\sqrt{36 - 9\left(\frac{1}{x} + 1\right)^2} \leq 2\left|\frac{2}{x} - 1\right|$. [2]

17 [2023/HCI/Promo/2] **Functions**

The function $y = f(x)$ is a one-to-one function.

The graphs of $y = f(|x|)$ and $y = |f(x)|$ are shown below in Figure 1 and 2 respectively.

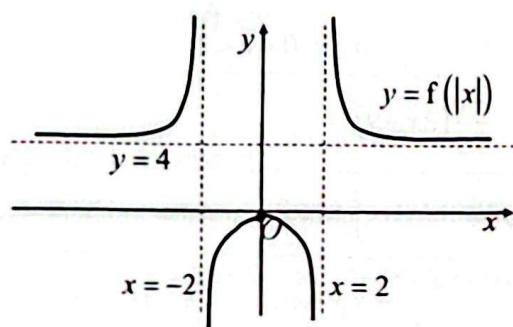


Figure 1

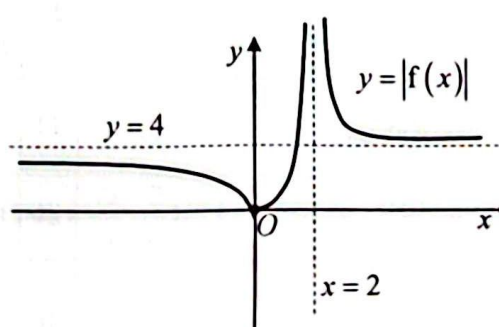


Figure 2

- (a) State the range of values of x for which $f(x) \leq 0$. [1]
- (b) State the equation(s) of the asymptote(s) of the graph of $y = \frac{1}{f(x)}$. [2]

18 [2023/NYJC/Promo/4] **Conics** **Functions**

(a) Describe fully a sequence of transformations which transforms the curve $y^2 = x$ onto the curve

$$\frac{y^2}{4} = 1 - x. \quad [2]$$

(b) A curve has equation $y = f(x)$, where

$$f(x) = \begin{cases} 2\sqrt{1-x} & \text{for } 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Sketch the curve of $y = f(x)$ for $-2 \leq x \leq 2$, stating the coordinates of the end points and the coordinates of any points where the curve crosses the axes. [3]

(ii) Sketch the curve with equation $y = |f(x) - 1|$ for $-2 \leq x \leq 2$, stating the coordinates of the end points and the coordinates of any points where the curve crosses the axes. [3]

Answers

1	—	2	—
3	—	4	(ii) $\left(\frac{x}{4}\right)^2 - (2y-3)^2 = 1$
5	—	6	—
7	(a) $1 < a < 3$	8	—
9	—	10	(ii) $y = \frac{1}{2}x + 2$ and $y = -\frac{1}{2}x$ (iii) $-2 < c < 0$ or $0 < c < 2$
11	(a) $\frac{(x+7)^2}{4} + 4y^2 = 1$	12	$y = -\frac{1}{3}(2x+9)^2$
13	(a) $f(x) = 2e^{9x(x-4)} - 4$	14	(b)(ii) $h(x) = \frac{1}{b}(x+2a)^2 + 1$
15	—	16	(iii) $-3 \leq x \leq -1$ or $0.92 \leq x \leq 1$ (iv) $-1 \leq x \leq -\frac{1}{3}$ or $1 \leq x \leq \frac{25}{23}$
17	(a) $0 \leq x < 2$ (b) $y = \frac{1}{4}$ and $x = 0$	18	—

Inequalities

1 [2023/HCI/Promo/1]

Without using a calculator, solve the inequality

$$\frac{4x^2 + 6x - 12}{2x - 1} \leq 3. \quad [3]$$

Deduce the solution of the inequality $\frac{4e^{2x} + 6e^x - 12}{2e^x - 1} \leq 3$ in exact form. [2]

2 [2023/JPJC/Promo/2]

Without using a calculator, solve the inequality $\frac{x}{2x-1} \leq \frac{1}{3x+2}$. [3]

Hence find the exact values of x for which $\frac{x^2}{2x^2-1} \leq \frac{1}{3x^2+2}$. [2]

3 [2023/NYJC/Promo/1]

Graphing Techniques

(a) Sketch, on the same axes, the graphs of $y = 5 - \frac{1}{x-1}$ and $y = \frac{1}{(x-5)^2}$. [2]

(b) Solve the inequality $5 - \frac{1}{x-1} < \frac{1}{(x-5)^2}$. [3]

4 [2023/YIJC/Promo/3]

(a) Without using a calculator, solve the inequality $\frac{7x-2}{x+1} \geq 2x$. [4]

(b) Hence solve the inequality $\frac{-7\sqrt{x}-2}{1-\sqrt{x}} \geq -2\sqrt{x}$. [2]

5 [2023/EJC/Promo/3]

Graphing Techniques

(a) Sketch, on the same diagram, the curves with equations $y = \left| \frac{x+1}{x-2} \right|$ and $y = \ln\left(1 - \frac{x}{2}\right) + \frac{1}{2}$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. Label the two curves clearly. [5]

(b) Hence solve the inequality $\left| \frac{x+1}{x-2} \right| \geq \ln\left(1 - \frac{x}{2}\right) + \frac{1}{2}$. [1]



6 [2023/RI/Promo/2]

Do not use a calculator in answering this question.

Solve the inequality $x^2 + 6x + 5 \leq \frac{x+5}{3x+1}$.

[4]

Hence solve

(a) $x^4 + 6x^2 + 5 \leq \frac{x^2 + 5}{3x^2 + 1}$,

[2]

(b) $(\ln x)^2 + 6 \ln x + 5 \leq \frac{5 + \ln x}{1 + 3 \ln x}$.

[2]

7 [2023/YIJC/Promo/6]

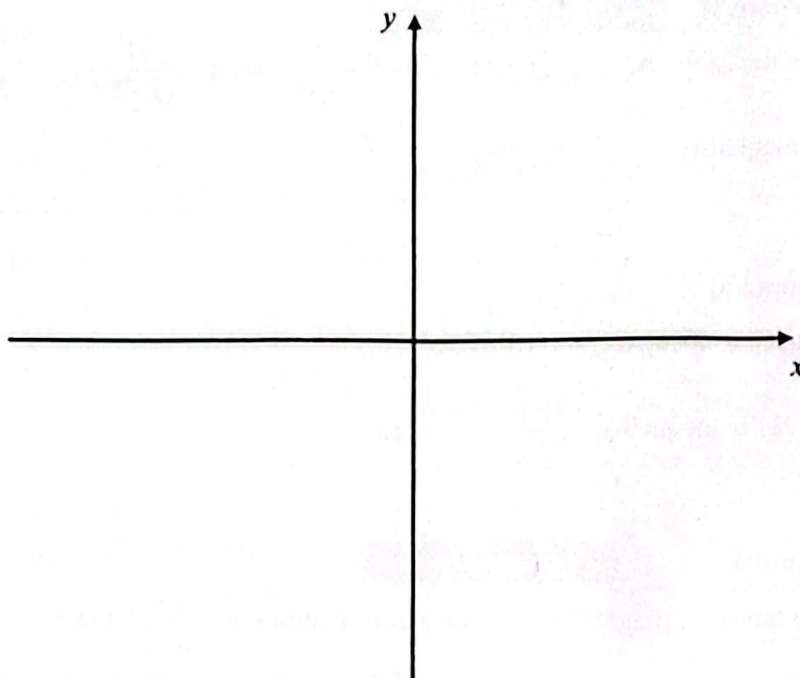
Graphing Techniques

The curve C has equation $y = \frac{x^2 - 4x + 15}{x + 3}$.

(a) Use an algebraic method to find the range of values of y .

[3]

(b) Sketch C on the axes. You should state the equations of the asymptotes and the coordinates of any point(s) of intersection of C with the axes. [3]



(c) On the same axes in part (b), sketch the graph of $y = -|10 - 2x|$. Hence, solve the inequality

$$\frac{x^2 - 4x + 15}{x + 3} \leq -|10 - 2x|.$$

[3]

8 [2023/DHS/Promo/2]

Given that $x < 0$, solve the inequality $\frac{2}{x} < |x| - a$, where a is constant such that $a > 3$. [4]

9 [2023/MI/PU2/P1/Promo/3]

(a) Using an algebraic method, solve the inequality $x - 3 < \frac{8}{x-1}$. [3]

(b) Hence, solve

(i) $e^{2x} - 3 < \frac{8}{e^{2x} - 1}$, [2]

(ii) $x + 3 > \frac{8}{x+1}$. [2]

10 [2023/TJC/Promo/3]

(a) Show algebraically that $x^2 - 4x + 5$ is positive for all real values of x . [1]

It is given that k is a constant where $0 < k < \frac{1}{2}$.

(b) Solve the inequality $\frac{x^2 - 4x + 5}{(kx - 1)(x - 2)} < 0$, giving your answer in terms of k . [3]

(c) Hence deduce the solution to the inequality $\frac{x^2 - 4|x| + 5}{(k|x| - 1)(|x| - 2)} < 0$. [3]

11 [2023/VJC/Promo/3]

Given that $a \in \mathbb{R}$ and $a > 1$, use an algebraic method to solve the inequality

$$\frac{ax - a + 2}{(x - a)(1 - x)} \leq 1. \quad [5]$$

Hence find the set of values of x such that $\frac{a|x| - a + 2}{(|x| - a)(1 - |x|)} \leq 1$. [2]

12 [2023/ACJC/Promo/2]

Graphing Techniques

(i) Sketch the graph of $y = \left| b + \frac{a}{x+b} \right|$, where $0 < a < b$, stating the equations of any asymptotes and the coordinates of the points that crosses the axes. [3]

(ii) Hence solve the inequality $-x \leq \left| b + \frac{a}{x+b} \right|$. [2]

(iii) Using part (ii), given that $a = 1$, $b = 2$, solve the inequality $e^x \leq \left| b + \frac{a}{b - e^x} \right|$. [2]

13 [2023/MI/PU1/Promo/6]

(i) Without using a calculator, solve the inequality

$$\frac{3x^2 + 9x - 8}{x - 1} < 2. \quad [4]$$

Hence solve the following inequalities

(ii) $\frac{3e^{2x} + 9e^x - 8}{e^x - 1} < 2, \quad [2]$

(iii) $\frac{3 + 9x - 8x^2}{x - x^2} < 2. \quad [3]$

14 [2023/SAJC/Promo/7]

Graphing Techniques

(i) Sketch, on the same diagram, the graphs $y = \frac{2x-1}{x-3}$ and $y = |\ln(1-x)|$, stating clearly the coordinates of axial intercept(s) and equation of asymptote(s).

Hence solve $\frac{2x-1}{x-3} = |\ln(1-x)|. \quad [5]$

(ii) Solve $\frac{2x-1}{x-3} \leq |\ln(1-x)|. \quad [1]$

(iii) Hence, solve $\frac{2x+1}{x-2} \leq |\ln(-x)|. \quad [3]$

Answers

1	$x \leq -\frac{3}{2}$ or $\frac{1}{2} < x \leq \frac{3}{2}; \ln \frac{1}{2} < x \leq \ln \frac{3}{2}$	2	$-\frac{2}{3} < x < \frac{1}{2}, -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$
3	(b) $1 < x < 1.20$ or $4.54 < x < 5.46, x \neq 5$	4	(a) $x < -1$ or $0.5 \leq x \leq 2$ (b) $x > 1$
5	(b) $0 \leq x < 2$	6	$-5 \leq x \leq -\frac{4}{3}$ or $-\frac{1}{3} < x \leq 0$ (a) $x = 0$ (b) $e^{-5} \leq x \leq e^{-4/3}$ or $e^{-1/3} < x \leq 1$
7	(a) $y \leq -22$ or $y \geq 2$ (c) $-6.71 \leq x < -3$	8	$-\frac{a}{2} + \sqrt{\frac{a^2-8}{4}} < x < 0$ or $x < -\frac{a}{2} - \sqrt{\frac{a^2-8}{4}}$
9	(a) $x < -1$ or $1 < x < 5$ (b)(i) $0 < x < 0.805$ (b)(ii) $x > 1$ or $-5 < x < -1$	10	(b) $2 < x < \frac{1}{k}$ (c) $-\frac{1}{k} < x < -2$ or $2 < x < \frac{1}{k}$
11	$x < 1$ or $x > a$; $\{x \in \mathbb{R} : x < -a \text{ or } -1 < x < 1 \text{ or } x > a\}$	12	(ii) $-\sqrt{a+b^2} \leq x < -b$ or $x > -b$ (iii) $\ln 2 < x \leq \ln \sqrt{5}$ or $x < \ln 2$
13	(i) $x < -3$ or $\frac{2}{3} < x < 1$ (ii) $\ln \frac{2}{3} < x < 0$ (iii) $-\frac{1}{3} < x < 0$ or $1 < x < \frac{3}{2}$	14	(i) $x = -1.33$ (to 3 sf) or 0.195 (to 3 sf) (ii) $x \leq -1.33$ or $0.195 \leq x < 1$ (iii) $x \leq -2.33$ or $-0.805 \leq x < 0$

System of Linear Equations

1 [2023/MI/PU2/Promo/1]

A curve C has equation $y = ax^3 + bx^2 + cx - 3$, where a , b and c are constants. It is given that C passes through the points $(-2, -21)$ and $(4, 165)$ and intersects the x -axis at $x = 1.5$. Find the equation of C . [4]

2 [2023/JPJC/Promo/1]

The equation of a curve C is $y = f(x)$, where $f(x)$ is a cubic polynomial in x . C has a turning point at $(-1, 45)$ and crosses the x -axis at $x = 0$ and $x = 2$. Find the equation of C . [4]

3 [2023/EJC/Promo/1]

A function f is defined by $f(x) = x^3 + ax^2 + bx + c$, where a , b and c are constants. The graph of $y = f(x)$ passes through the points $(-2, 1)$ and $(2, -3)$. The point $(2, 1)$ lies on the graph of $y = f(x+1)$. Find the values of a , b and c . [4]

4 [2023/RI/Promo/1]

Sequences and Series

The first 3 terms of a sequence are given by $u_1 = 1823$, $u_2 = 200$ and $u_3 = 2023$. Given that u_n is a quadratic polynomial in n , find u_n in terms of n . [4]

5 [2023/TMJJC/Promo/1]

Alice, Ben, and Charlie buy three types of gifts, Gift A , Gift B and Gift C , from a gift store. The table below shows the number of gifts they bought and the total amount each of them paid.

	Gift A	Gift B	Gift C	Total amount paid (\$)
Alice	2	1	3	53.50
Ben	3	5	1	69.40
Charlie	4	1	2	50.60

Find the selling price of Gift A , Gift B and Gift C . [4]



6 [2023/YIJC/Promo/1]

In a mathematics competition, each participant is required to answer 50 questions. For each question, a score of a is awarded for a correct answer, a penalty of b is given for a wrong answer, and a penalty of c is given if the question is unattempted. The following table shows a breakdown of the answers submitted by five participants and some of their final scores.

	Number of correct answers	Number of wrong answers	Number of unattempted questions	Final score
Alexander	24	13	13	33
Isabella	18	12	20	14
Maximilian	17	10	?	?
Rosendale	12	20	18	-23
Seraphina	25	10	15	?

Find the final scores obtained by Maximilian and Seraphina.

[4]

7 [2023/MI/PU1/Promo/2]

A coffee vendor sells three types of coffee beverages: Latte, Cappuccino and Mocha. The cups of coffee come in three different sizes: Short, Tall and Grande. The number of cups for each type of coffee sold in a particular day is given in the following table.

	Short	Tall	Grande
Latte	3	12	8
Cappuccino	4	8	7
Mocha	2	5	4

The price of a cup of coffee for each size is the same regardless of the type of coffee. The total amount collected on that day from the sale of Latte, Cappuccino and Mocha is \$147.90, \$120.10 and \$70 respectively.

- (i) Find the price of a short, tall and grande cup of coffee respectively. [3]
- (ii) The vendor offers a buy-one-get-one-free promotion if a customer buys a grande cup of Latte and gives a 10% discount if a customer buys a short or tall cup of Latte. Andrew wishes to have 1 short cup, 4 tall cups, and 4 grande cups of Latte. How much does Andrew need to pay in total? [2]

8 [2023/TJC/Promo/1]

To celebrate Senior Citizens' Week, Sports Hub is giving a 30% discount for every 5 Senior Citizen tickets purchased and a 10% discount for every 2 Senior Citizen tickets purchased. There is no discount for the Adult and Child tickets in that week. During that week, Anand, Ben and Charlie purchased tickets with the maximal discount on the Senior Citizen tickets. The number of tickets purchased and the total amount spent are shown in the table below.

	Number of tickets purchased			Total amount spent (\$)
	Senior Citizen	Adult	Child	
Anand	16	20	15	840
Ben	27	30	10	1093
Charlie	19	25	12	946

Find the original price of a Senior Citizen ticket.

[4]

9 [2023/RVHS/Promo/1]

Conics

An ellipse C has the equation $ax^2 + by^2 + cy = -3$ where a , b and c are constants. It is given that C passes through the points with coordinates $(0, 1.5)$, $(0.6, 0.6)$ and $(-1, 1)$.

(i) Find the values of a , b and c .

[3]

(ii) Identify the centre of C .

[2]

Answers

1	$y = 3x^3 - \frac{1}{2}x^2 - 4x - 3$	2	$y = 20x^3 - 5x^2 - 70x$
3	$a = -2, b = -5, c = 7$	4	$u_n = 1723n^2 - 6792n + 6892$
5	$a = 4.5, b = 8.8, c = 11.9$	6	Maximilian's final score = 14.5 Seraphina's final score = 42.5
7	(i) $x = 4.50, y = 6.20, z = 7.50$ (ii) \$41.37	8	\$10
9	(i) $a = 1, b = 4, c = -8$ (ii) $(0, 1)$		



Module II Sequences and Series

Recurrence Relation

1 [2015/SRJC/Promo/7(part)]

A sequence $u_1, u_2, u_3, \dots$ is given by

$$u_1 = 3 \quad \text{and} \quad u_{n+1} = \frac{3u_n - 1}{6} \quad \text{for } n \geq 1.$$

(i) Find the exact value of u_2 . [1]

(ii) It is given that $u_n \rightarrow l$ as $n \rightarrow \infty$. Show that the exact value of $l = -\frac{1}{3}$. [1]

2 [2015/TJC/Promo/6]

A sequence $\{u_n\}$ is defined by the recurrence relation

$$u_{n+1} = \frac{2}{2 - u_n} \quad \text{for } n \in \mathbb{Z}^+ \quad \text{and } u_1 \neq 0, 1 \text{ or } 2.$$

(a) Prove that the sequence does not converge. [3]

(b) Given that $u_1 = 4$,

(i) write down the next 4 terms of the sequence, [1]

(ii) prove algebraically that $u_{n+4} = u_n$. [3]

3 [2015/AJC/Promo/11(b)]

A sequence of real numbers $u_1, u_2, u_3, \dots$ satisfies the recurrence relation

$$u_n = u_{n-1} + n + 2^n \quad \text{for } n \geq 2.$$

Given that $u_1 = 4$, show that $u_n = \frac{(n+a)(n+b)}{2} + 2^{n+1}$, where a, b are integers to be determined.

Hence determine, with reasons, the behaviour of the sequence. [5]

4 [2015/IJC/Promo/4]

A sequence of positive numbers $x_1, x_2, x_3, \dots$ is given by

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \frac{3x_n + 2}{x_n + 5} \quad \text{for } n \geq 1.$$

(i) Write down the exact values of x_2 and x_3 . [2]

(ii) It is given that $x_n \rightarrow l$ as $n \rightarrow \infty$. Showing your working, find the exact value of l . [3]

(iii) By considering $x_{n+1} - x_n$, prove that $x_n < x_{n+1}$ if $x_n < l$. [3]

5 [2015/MI/PU1/Promo/3]

A sequence of negative numbers $u_1, u_2, u_3, \dots$, is given by

$$u_1 = -2 \text{ and } u_{n+1} = -\sqrt{2u_n + 5} \text{ for } n = 1, 2, 3, \dots,$$

- (i) Given that $u_n \rightarrow l$ as $n \rightarrow \infty$, find the exact value of l . [2]
(ii) Sketch the graph of $y = x + \sqrt{2x + 5}$. Hence, by considering $u_n - u_{n+1}$, show that $u_{n+1} < u_n$ if $u_n > l$. [3]

6 [2015/SRJC/Promo/8]

A sequence of positive real numbers $x_1, x_2, x_3, \dots$ satisfies the recurrence relation $x_{n+1} = \sqrt{4x_n + 5}$ for $n \geq 1$.

- (i) If $x_1 = 1$, find the value of x_6 . [1]
(ii) Determine the behaviour of the sequence when $x_1 = 1$. [1]
(iii) Sketch the graph of $y = \sqrt{4x + 5} - x$, indicating clearly the end point and the x -intercept. [2]
(iv) Hence by considering $x_{n+1} - x_n$, show that if $x_n > 5$, $x_{n+1} < x_n$. [2]

7 [2015/DHS/Promo/8]

A certain type of cell is grown for research. The number of cells, in millions, at the end of n days is denoted by u_n and the proportion of cells that died over a day is denoted by m , where $0 < m \leq 1$.

The sequence $u_0, u_1, u_2, \dots$ is given by

$$u_{n+1} = (1 - m)u_n + 5.$$

It is known that $u_0 = 10$ and $u_2 = 15.4$.

- (i) Find the value of m . [3]
(ii) If $u_n < \frac{5}{m}$, show algebraically that $u_n < u_{n+1} < \frac{5}{m}$ for $n \geq 0$. [2]

Hence or otherwise, explain algebraically why the number of cells will not grow beyond 25 million. [2]

8 [2015/MJC/Promo/4]

Michael planted a bonsai tree in his backyard. In order to keep the tree at a certain height, he plans to trim its height by 15% at the start of each year. By the end of each year, the bonsai tree is expected to grow by 0.5 metres in height. Let u_n denote the height of the bonsai tree in metres at the end of the n^{th}

year. It is given that $u_0 = \frac{2}{3}$, where u_0 represents the initial height of the bonsai tree.

- (i) Write down a recurrence relation for u_n . [1]
(ii) Show that $u_n = \frac{10}{3} - 0.85^n \left(\frac{8}{3} \right)$ for $n \geq 0$. [3]
(iii) By sketching the graph of $y = \frac{10}{3} - 0.85^x \left(\frac{8}{3} \right)$, comment on the height of the bonsai tree in the long run. [3]
(iv) If Michael does not want the height of the bonsai tree to grow beyond two metres in the long run, what is the minimum percentage that the height of the bonsai tree should be trimmed each year? [2]

Answers

1	(i) $\frac{4}{3}$	2	(b)(i) $u_2 = -1, u_3 = \frac{2}{3}, u_4 = \frac{3}{2}, u_5 = 4$
3	—	4	(i) $x_2 = \frac{5}{6}, x_3 = \frac{27}{35}$ (ii) $l = \sqrt{3} - 1$
5	(i) $1 - \sqrt{6}$	6	(i) 4.94
7	(i) $m = \frac{1}{5}$	8	(i) $u_n = 0.85u_{n-1} + 0.5, u_0 = \frac{2}{3}, n \geq 1$ (iv) 25%

Sigma Notation

1 [2023/SAJC/Promo/3(a)]

Given that $\sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$, find $\sum_{r=1}^n 6r(r+1)$, in terms of n . [2]

2 [2023/MI/PU2/P1/Promo/5(modified)]

Given that $\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{1}{2} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right)$,

(a) deduce $\sum_{r=1}^n \frac{1}{r(r+2)}$. [1]

(b) find, in terms of n , the series

$$\frac{1}{(6)(8)} + \frac{1}{(7)(9)} + \frac{1}{(8)(10)} + \dots + \frac{1}{(n+1)(n+3)}. \quad [3]$$

3 [2023/RI/Promo/4(modified)]

Given that $\sum_{r=1}^n \frac{1}{r^2 + r} = 1 - \frac{1}{n+1}$, find

(a) $\sum_{r=n+1}^{2n} \frac{1}{r^2 + r}$, [2]

(b) $\frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots$. [2]

4 [2023/EJC/Promo/5(modified)]

Given that $\sum_{r=1}^n \frac{-r^2 - 3r + 1}{(r+1)!} = \frac{3}{(n+1)!} + \frac{1}{n!} - 4$, find

$$\sum_{r=3}^n \frac{-r^2 - r + 3}{r!}. \quad [3]$$

5 [2023/NYJC/Promo/6(modified)]

Given that $\sum_{r=2}^N \ln \left(1 - \frac{1}{r^2} \right) = \ln \left(\frac{N+1}{2N} \right)$,

(a) explain why the series $\sum_{r=2}^{\infty} \ln \left(1 - \frac{1}{r^2} \right)$ converges and state the exact value of the sum to infinity of this series. [2]

(b) find an expression for the series

$$\ln \left(1 - \frac{1}{(2k+1)^2} \right) + \ln \left(1 - \frac{1}{(2k+2)^2} \right) + \ln \left(1 - \frac{1}{(2k+3)^2} \right) + \dots,$$

giving your answer as a single logarithmic function in terms of k . [3]



6 [2023/RVHS/Promo/6(modified)]

It is given that $\sum_{n=1}^N 2^n f(n) = \frac{2^{N+1}}{N+1} - 2$.

(a) Find $\sum_{n=2}^{2N} 2^{n-1} f(n-1)$. [2]

(b) Denote $S_N = \sum_{n=1}^N 2^n f(n)$. Find least integer k such that S_k is more than 5% of S_{10} . [2]

7 [2023/ACJC/Promo/10(modified)]

A sequence $a_1, a_2, a_3, \dots$ is such that $a_n > 0$, for $n \in \mathbb{Z}^+$, $n \geq 1$.

It is known that this sequences is defined by $\frac{a_{n+1}}{a_n} = \frac{1}{4}$.

Given that $\sum_{n=1}^N (a_n - a_{n+1})^2 = \frac{k}{16} \sum_{n=1}^N (a_n)^2$, find the value of k . [2]

Hence

(i) show that $\sum_{n=1}^N (a_n - a_{n+1})^2 = \frac{3}{5} a_1^2 \left(1 - \frac{1}{16^N}\right)$, for $N \in \mathbb{Z}^+$, $N \geq 1$, [3]

(ii) find $\sum_{n=2}^{\infty} (a_n - a_{n+1})^2$, leaving your answers in terms of a_1 . [3]

8 [2023/DHS/Promo/7]

(a) It is given that $\sum_{r=1}^n r^2 = \frac{1}{6} n(n+1)(2n+1)$.

(i) Find $\sum_{r=1}^n (2^{r+1} + 3r - r^2)$ in the form $A(2^n - 1) + f(n)$, where A is a constant and $f(n)$ is in fully factorised form. [3]

(ii) Using your answer in part a(i), find $\sum_{r=4}^N (2^r + 3r - (r-1)^2)$, leaving your answer in the form $B(2^N) + C[N(N-1)(5-N)] + DN + E$ where B, C, D and E are constants to be determined. [4]

(b) A sequence is such that $v_1 = p$, where p is a constant, and

$$v_{n+1} = 3v_n - 2, \text{ for } n \geq 1.$$

Describe how the sequence behaves when

(i) $p = 5$, [1]

(ii) $p = 1$. [1]

Answers

1	$u = 2n(n+1)(n+2)$	2	(a) $\frac{3}{4}$ (b) $\frac{13}{84} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$
3	(a) $\frac{1}{n+1} - \frac{1}{2n+1}$ (b) $\frac{1}{2}$	4	(b) $\frac{3}{(n+1)!} + \frac{1}{n!} - 4$ (c) $\left(\frac{3}{n!} + \frac{1}{(n-1)!}\right) - \frac{5}{2}$
5	(a) $\ln\left(\frac{1}{2}\right)$ (b) $\ln\left(\frac{2k}{2k+1}\right)$	6	(a) $\frac{2^{2N-1}}{N} - 2$ (b) 6
7	$k = 9$ (b)(ii) $\frac{3}{5} a_1^2 \left[1 - \frac{1}{16^N}\right]$ (b)(iii) $\sum_{n=1}^{\infty} (a_n - a_{n+1})^2 = \frac{3}{5} a_1^2; \frac{3}{80} a_1^2$	8	(a)(i) $4(2^n - 1) + \frac{1}{3} n(n+1)(4-n)$ (a)(ii) $2(2^N) + \frac{1}{3} N(N-1)(5-N) + 3N - 29$

Arithmetic Progression & Geometric Progression

1 [2023/MI/PU1/Promo/1]

The sum, S_n , of the first n terms of the sequence $u_1, u_2, u_3, \dots$ is given by

$$S_n = n^2 + 4n.$$

- (i) Find u_n in terms of n . [2]
- (ii) Hence show that the sequence is an arithmetic progression. [2]

2 [2023/ASRJC/Promo/1]

The sum of the first n terms, S_n , of a sequence $\{u_n\}$ is given by

$$S_n = -2n^2 + n.$$

Prove that the sequence $\{u_n\}$ is an arithmetic progression. [3]

3 [2023/ACJC/Promo/8]

- (a) (i) The sum of the first n terms of a sequence $u_1, u_2, u_3, \dots$ is given by $S_n = e^2 n^2 - 3n$.

Show that $u_n = e^2(2n-1) - 3$. [2]

- (ii) Hence show that the sequence is an arithmetic progression. [2]

- (iii) State the first 3 odd-numbered terms, u_1, u_3 and u_5 . Hence find the least value of n such that the sum of the first n odd-numbered terms is more than 500. [3]

- (b) (i) Find the range of values of x for which the following geometric series converges,

$$e - e^{\frac{1}{x}+1} + e^{\frac{2}{x}+1} - e^{\frac{3}{x}+1} + \dots$$
 [2]

- (ii) Given that $x = -1$, find the sum of all the terms of this geometric series after, and including, the 8th term. [3]

4 [2023/SAJC/Promo/4]

An arithmetic sequence $u_1, u_2, u_3, \dots, u_n, \dots$ is known to have a common difference of $\ln 3$.

The r th term, w_r , of another sequence $w_1, w_2, w_3, \dots$, is given by $w_r = e^{-u_r}$.

- (i) Show that the sequence $w_1, w_2, w_3, \dots, w_n, \dots$, is a geometric progression with common ratio $\frac{1}{3}$. [2]

- (ii) Explain why $\sum_{r=1}^{\infty} w_r$ converges. [1]

- (iii) Given that $u_1 = \ln 3$, find the smallest possible value of n such that the sum of the first n terms of the geometric progression given by $w_1, w_2, w_3, \dots$ is within 0.5% of the sum to infinity of the geometric series. [3]

5 [2023/RVHS/Promo/8]

Let S_n denote the sum of the first n terms of the sequence of numbers $u_1, u_2, u_3, \dots$. It is given that $S_n = 2n^2 + kn$, where k is a non-zero constant.

- (i) Show that the sequence of numbers $u_1, u_2, u_3, \dots$ is an arithmetic progression. [3]

A geometric progression has non-zero first term a and common ratio r . Given that the sum of the first 5 terms of the progression is $\frac{32}{33}$ of the sum of the first 10 terms of the progression.

- (ii) Show that $r = \frac{1}{2}$ and find the infinite sum of the geometric progression in terms of a . [4]

- (iii) Given further that u_{62}, u_{30}, u_t are three consecutive terms of the given geometric progression, find the value of t . [3]

6 [2023/EJC/Promo/6]

- (a) The first n terms of a series are given by $\log_a 3 + \log_a 27 + \log_a 243 + \dots + \log_a 3^{2^n - 1}$, where a is a positive constant.

- (i) Show that the series is an arithmetic series. [2]

- (ii) Given that sum of the first 30 terms of the series is 300, find the value of a . [2]

- (b) A geometric series has first term c and common ratio r , where c and r are non-zero. An arithmetic series has first term b and common difference d , where b and d are non-zero. It is given that the 5th, 8th and 10th terms of the arithmetic series are equal to the 2nd, 3rd and 4th of the geometric series respectively. Show that r satisfies the equation $3r^2 - 5r + 2 = 0$ and hence find the sum to infinity in terms of c . [4]

7 [2024/HCI/Promo/8]

- (a) It is given that

$$1 - a + a^2 - a^3 + \dots$$

is a convergent geometric series, where a is a real constant.

Find the range of values of a such that its sum to infinity is not more than 5. [4]

- (b) It is given that

U_n is the n^{th} term of the geometric sequence $\{1, -a, a^2, -a^3, \dots\}$,

T_n is the n^{th} term of the geometric sequence $\{1, a, a^2, a^3, \dots\}$ and

$$W_n = U_n + T_n.$$

A student commented that W_n is a geometric progression with first term 2 and common ratio a^2 . State, with justification, if the student is correct. [1]

- (c) Express W_{2n+1} in the form pa^{qn} , where p and q are real constants. [2]

8 [2023/MI/PU2/P2/Promo/6]

- (a) Sofia wants to save \$2000 to buy a coffee machine. She saves \$50 by the end of the first week, and on each subsequent week, she saves \$0.50 less than on the previous week. She will continue to save as long as the weekly saving amount is positive.
- Find an expression for the total amount Sofia saves by the end of n th week. [2]
 - Hence find the minimum number of weeks required for Sofia to save \$2000. [2]
 - Determine if she will save enough to buy a coffee machine if it costs \$2550. [2]
- (b) An arithmetic progression has first term a and common difference d , where a and d are non-zero. A geometric progression has first term g and common ratio r , where g and r are non-zero. It is given that first three terms of geometric progression are equal to the 13th, 4th and first term of arithmetic progression respectively.
- Show that $3r^2 - 4r + 1 = 0$. [3]
 - Hence find the exact value of r . Show that common ratio r cannot be 1 even though $r = 1$ is a root of this equation. [2]
 - Find the sum to infinity of the odd-numbered terms of geometric progression if $a = 4$. [2]

9 [2023/MI/PU1/Promo/9]

In a computer simulation, each player uses an identical pail to pour water repeatedly into his assigned tank, which is initially empty. Two players, Chris and Elliot, participate in this simulation and pour water into their assigned tanks as follows.

- Chris pours 5 litres of water into the tank using the pail for the first time. Subsequently, he pours 0.3 litres less than the preceding volume of water poured.
- Elliot pours 5 litres of water into the tank using the pail for the first time. Subsequently, he pours 85% of the preceding volume of water poured.

The simulation stops when the amount of water that can be poured from the pail by either player is no longer positive.

- Determine the number of times Chris can pour water into his tank before the simulation stops. Hence find the volume of water in the tank at this instant. [4]

The simulation also stops when any player reaches or exceeds the total target volume of water in their respective tanks. The total target volume of water in each tank is given to be 33.5 litres.

- Elliot comments that the target is unfair because he will not reach it. Explain if Elliot's comment is justified. [2]

The total target volume of water in each tank is now changed to 30 litres.

Chris and Elliot start the simulation at the same time. The simulation is designed such that all players pour water into their assigned tanks at the same time each time.

- Determine which player reaches or exceeds the total target volume of water in the tank first, showing your workings clearly. [4]
- Suppose that Gary is a new player. He pours 5 litres into his assigned tank using the pail the first time. Subsequently, he pours $r\%$ of the preceding volume of water poured. Find the smallest value of r such that Gary reaches or exceeds the total target volume of water of 30 litres in the tank after pouring for 7 times. Leave your answer to the nearest integer. [2]

10 [2023/RI/Promo/10]

In long-track speed skating, races are run counter-clockwise on a 400-metre two-lane oval rink. A full 400-metre round the rink is known as a lap. The current world records for the 10000 metres men's race and 500 metres men's race are 12 minutes 30.74 seconds (11 Feb 2022) and 33.61 seconds (9 Mar 2019) respectively.

Abel and Caine are 2 skaters training for the 10000 metres men's race. During a particular training session for Abel, he completes the first lap in b seconds and he takes 10% longer to complete each succeeding lap than he does in the previous lap.

- (a) Write down an expression for the time taken by Abel to complete n laps, in terms of b and n . Hence find the value of b that will enable Abel to complete 10000 metres in 13 minutes. [3]

- (b) Comment on the feasibility of this value of b in the context of the question. [1]

After training for 6 months, Abel and Caine both entered a 10000 metres men's race. During the race, Abel completes the first lap in k seconds and then he takes 1% longer to complete each succeeding lap than he does in the previous lap. Caine also completes the first lap in k seconds and on each subsequent lap he spends d seconds more than he spent on the previous lap. They arrive at the finish line at the same time.

- (c) Find the value of $\frac{d}{k}$, correct to 5 decimal places. [4]

- (d) Given that Abel completes his 25th lap in p seconds and Caine completes his 25th lap in q seconds, evaluate $\frac{q}{p}$, giving your answer correct to 3 decimal places.

Hence determine the skater who has the faster time for the first 24 laps. [4]

11 [2023/NYJC/Promo/10]

A radio game show called the Inverted Tower of Ionah is broadcasted live everyday. Each day, a participant is invited to play the game. The participant has to stack up discs of the same thickness but increasing cross-sectional area to build a tower. Each level of the tower consists of one disc. The base of the tower is a disc with base area 6 cm^2 . For each successive level, a disc with base area 8 cm^2 larger than the preceding disc is placed on the tower. The game will stop if the tower collapses and the participant loses the game.

Alan is a participant who participated in the game on Day 1.

- (a) Alan claims that one of the discs he used has base area 140 cm^2 . Explain if this is possible. [2]

- (b) Alan's completed tower consists of 22 discs with a total volume of 4950 cm^3 . Assuming that the discs have uniform cross-sectional areas with the same thickness of $h \text{ cm}$, find the value of h . [2]

The participant wins the prize money for the day if he is able to build a tower with 30 levels within 2 minutes. Ben will participate in the game on Day 2. He wants to win the game and decides to practise before the game. He uses 3 minutes to complete the tower in his first attempt and the time for each subsequent attempt is 5% less than the time for the previous attempt.

- (c) Find the minimum number of attempts he needs to build the tower within 2 minutes. [2]

On Day 1 of the radio game show, the prize money is \$200. For each day of the game, if the prize money is not won, the radio station will donate 20% of the prize money for that day to charity before topping up the balance prize money by \$150 for the next day. The prize money is increased in the same way every day until it is won.

- (d) If there are no winners since Day 1, show that the prize money at the start of Day n is $s[750 - 550(0.8)^{n-1}]$. [3]

If the prize money is won, it will start from \$200 again on the next day. It is now given that the first and second winner of the game won the prize money on Day k and Day 24 respectively.

- (e) By using part (d), find the set of possible values of k such that the first winner won a prize money of at least \$100 more than the second winner. [3]

12 [2023/YIJC/Promo/12]

- (a) Given that the sum of the first n terms of a series is S_n , where $S_n = 3n^2 - n$, state the value of its first term and write down an expression for S_{n-1} for $n \geq 2$. Hence show that the terms in the series form an arithmetic progression. [4]
- (b) The first term of an infinite geometric progression is a and the common ratio is r , where $|r| < 1$. Given that the sum of all the terms after the n th term is equal to the n th term of the progression, show that $r = \frac{1}{2}$ and hence find the sum to infinity of the progression in terms of a . [4]
- (c) A bank has an account for investors. Compound interest is added to the account at the end of each year at a fixed rate of 3% of the amount in the account at the beginning of that year. Mr Ang decides to invest \$2 000 on 1 January 2023 and a further \$2 000 on 1 January of each subsequent year.
- (i) How much compound interest has his original \$2 000 earned at the end of 31 December 2030? [2]
- (ii) By considering the total amount in the account at the end of the n th year, find the year in which the total in the account will first exceed \$40 000. Explain whether this occurs at the beginning or the end of the year. [6]

13 [2023/ASRJC/Promo/10]

Customers who open a savings account with SRF bank will earn an interest rate of 0.2% per month based on the amount in the account on the last day of each month. The interest is added to the account at the end of each month.

- (a) Peter has a savings account with the SRF bank. He puts \$2000 into his savings account at the beginning of each month starting from 1st January 2023. Find the amount he has in his savings account on 31st December 2027 after the interest has been added if no withdrawal is made. Show your workings clearly. [4]
- (b) Sandy, another customer of SRF bank, has retired from work starting 1st January 2023. Since then, she will no longer do any deposit of money into her own savings account with the bank. Instead, she will be using the money saved in that account for her monthly spending. She has \$750 000 in her SRF bank savings account on 1st January 2023.

- (i) As she still has some money with her for her daily spending in January 2023, she will only commence her monthly withdrawal of \$ k from her savings account in the middle of each month, starting from February 2023 onwards. Show that the amount of money in her savings account, at the end of n complete months from 1st January 2023, after the interest has been added, is given by $1.002^n (\$750000) - \$501k(1.002^{n-1}) + \$501k$. [4]
- (ii) Taking k to be 3000, find the amount of money left in her savings account immediately after her last complete withdrawal of \$3000. [4]

Answers

1	(i) $2n+3$	2	—
3	(a)(iii) 7 (b)(i) $x < 0$ (b)(ii) -0.00181	4	(iii) n
5	(ii) $2a$ (iii) 14	6	(a)(ii) $a = 27$ (b) $3c$
7	(a) $-\frac{4}{5} \leq a < 1$ (c) $W_{2n+1} = 2a^{2n}$	8	(a)(i) $S_n = 50.25n - 0.25n^2$ (a)(ii) 55 (b)(ii) $r = \frac{1}{3}$ (b)(iii) $\frac{81}{2}$
9	(i) 44.2 (iv) 95	10	(a) $10b(1.1^n - 1)$, $b = 7.93$ (c) 0.01081 (d) 0.992, Abel
11	(b) 2.5 (c) 9 (c) $\{k \in \mathbb{Z}^+ : 17 \leq k \leq 23\}$	12	(a) $u_1 = 2$, $S_{n-1} = 3n^2 - 7n + 4$ (b) $2a$ (c)(i) \$533.54 (c)(ii) 2038
13	(a) \$127616.46 (b)(ii) \$2754.86		

Module III Vectors

Vectors (Basic)

1 [2023/MI/PU1/Promo/5]

With reference to the origin O , the position vectors of points A , B and C are expressed as $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively such that

$$\overrightarrow{OA} = (1 - \lambda)\overrightarrow{OB} + \lambda\overrightarrow{OC},$$

where λ is a non-zero constant.

(i) Show that A , B , C are collinear. [2]

It is given that $\lambda = \frac{1}{6}$, $\overrightarrow{OA} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

(ii) Find the position vector of the point D that lies on the line segment AC such that $AD : DC = 1 : 4$. [3]

It is given that the point E has coordinates $(1, -1, 2)$.

(iii) Find the exact area of the triangle ABE . [3]

(iv) Find the length of projection of $\overrightarrow{OE}$ onto $\overrightarrow{AB}$. [2]

2 [2023/TMJJC/Promo/6]

(a) Referred to the origin, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors and $(\mathbf{a} + 2\mathbf{b}) \cdot (\mathbf{a} - 2\mathbf{b}) = 0$, find $OA : OB$. [3]

(b) Points C and D have coordinates $(m, n, 1)$ and $(1, 2, 2)$ respectively, for some real numbers m and n , such that OC is perpendicular to OD .

(i) Find a linear equation satisfied by m and n . [2]

(ii) Given further that triangle OCD is an isosceles triangle, find the possible values of m and n . [4]

3 [2023/DHS/Promo/1]

The parallelogram $OABC$ has points F and G that divide AB and BC in the ratio of 2:1 and 4:1 respectively. The lines OF and AG intersect at the point E . By letting $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$, find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. Hence find the ratio $OE : EF$. [5]

4 [2023/MI/PU2/P1/Promo/6]

Relative to the origin O , the position vectors of the points A , B and C are

$$\mathbf{a} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{c} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

respectively.

(i) Given that the point D lies on AB , between A and B , such that $\overrightarrow{OD} = \mathbf{i} + t\mathbf{k}$. Find the value of t and write down the ratio $AD : AB$. [3]

(ii) Find the area of the triangle ABC . [3]

(iii) Find angle BAC . [2]

(iv) Given that the vector $\mathbf{p}$ is a unit vector in the direction of $\overrightarrow{OD}$, give a geometrical meaning of $|\mathbf{p} \cdot \mathbf{a}|$. [1]

5 [2023/RI/Promo/3]

Vectors $\mathbf{a}$ and $\mathbf{b}$ are such that the magnitude of $\mathbf{a}$ is 2 and $\mathbf{b}$ is a unit vector perpendicular to $\mathbf{a}$.

- (a) Find the area of the parallelogram with adjacent sides formed by the vectors $\mathbf{a} + 3\mathbf{b}$ and $5\mathbf{a} - 4\mathbf{b}$. [3]

A vector $\mathbf{c}$ is such that $\mathbf{b} \times \mathbf{c} = 21(\mathbf{a} \times \mathbf{b})$.

- (b) Show that $\mathbf{c} = \lambda\mathbf{b} - 21\mathbf{a}$, where λ is a constant. [2]
(c) Give the geometrical meaning of $|\mathbf{b} \cdot \mathbf{c}|$ and find the possible values of λ if $|\mathbf{b} \cdot \mathbf{c}| = 5$. [3]

6 [2023/NYJC/Promo/3]

With reference to the origin O , the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.

- (a) A point C has position vector $\mathbf{a} + \mathbf{b}$. State the geometric shape formed by O, A, B and C . [1]

Point M lies on AC , between A and C , such that $AM : MC = 1 : 2$.

- (b) Show that the area of triangle OAM can be written as $\alpha |\mathbf{a} \times \mathbf{b}|$, where α is a constant to be found. [2]

It is given that $|\mathbf{b}| = 5$ and the shortest distance from A to OB is $\sqrt{3}$.

- (c) Using your result from part (b), find the exact value of the area of triangle OAM . [2]
(d) Point N lies on OM , between O and M , such that $ON : NM = 3 : 1$. Show that A, N and B are collinear and hence find the ratio $AN : NB$. [3]

7 [2023/ASRJC/Promo/6]

Referred to the origin O , the points P and Q are such that $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are not parallel and non-zero vectors. The point R lies on OQ such that $\overrightarrow{OR} = k\overrightarrow{OQ}$, where k is a constant. S lies on line segment PR such that

$$PS : SR = 2 : 3 \text{ and } T \text{ is on } PQ \text{ such that } PT : TQ = 1 : 3.$$

- (i) Find $\overrightarrow{OS}$ in terms of $\mathbf{p}, \mathbf{q}$ and k . [1]
(ii) Given that O, S and T are collinear, find k . [3]
(iii) If $OQ = 5$, show that the shortest distance from T to OQ can be expressed $\beta |\mathbf{p} \times \mathbf{q}|$, where β is a constant to be found. [3]

8 [2023/ACJC/Promo/6]

Referred to the origin O , the points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, where vector $\mathbf{p}$ and vector $\mathbf{q}$ are not parallel. Point A lies between P and Q , such that $PA : AQ = \lambda : (1 - \lambda)$, where $0 < \lambda < 1$.

- (i) Express $\overrightarrow{OA}$ in terms of $\mathbf{p}, \mathbf{q}$ and λ . [1]
(ii) The point B lies between O and Q , such that $OB : OQ = 1 : 3$. Given that $\overrightarrow{AB}$ is parallel to $(\mathbf{q} - 4\mathbf{p})$, find the value of λ . [4]
(iii) It is now given that $\mathbf{p}$ is a unit vector which is perpendicular to $\mathbf{q}$ and $|\mathbf{q}| = 2$, find the area of triangle OAQ , in terms of λ . [4]

9 [2023/EJC/Promo/4]

Referred to the origin O , let P , Q and R be distinct points with position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively.

(a) Show that $(\mathbf{r}-\mathbf{p}) \times (\mathbf{r}-\mathbf{q}) = \mathbf{p} \times \mathbf{q} + \mathbf{q} \times \mathbf{r} + \mathbf{r} \times \mathbf{p}$. [2]

(b) Give the geometrical meaning of $\frac{1}{2} |\mathbf{p} \times \mathbf{q} + \mathbf{q} \times \mathbf{r} + \mathbf{r} \times \mathbf{p}|$. [2]

(c) Given that $\mathbf{p} \times \mathbf{q} + \mathbf{q} \times \mathbf{r} + \mathbf{r} \times \mathbf{p} = \mathbf{0}$, $PR = 3QR$, and that $PQ > PR$, express $\mathbf{r}$ in terms of $\mathbf{p}$ and $\mathbf{q}$. [3]

10 [2023/HCI/Promo/5]

The position vectors of points P and R with respect to the origin O are $\mathbf{p}$ and $\mathbf{r}$ respectively, and $\mathbf{q}$ is the position vector of the point Q on PR such that $2\overrightarrow{PQ} = \overrightarrow{QR}$.

(a) Write down $\mathbf{q}$ in terms of $\mathbf{p}$ and $\mathbf{r}$. [1]

(b) It is given that $OR = 3OP$ and the angle between $\mathbf{p}$ and $\mathbf{r}$ is θ where $\cos \theta = \frac{4}{5}$. F is the foot of the perpendicular from Q to the line OP .

(i) By using a suitable scalar product, find the exact value of OF in terms of $|\mathbf{p}|$. [4]

(ii) Hence determine if F lies within the line segment OP , justifying your answer. [1]

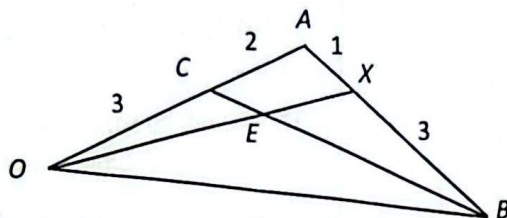
Answers

1	(ii) $\begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix}$ (iii) $\sqrt{21}$ (iv) $\frac{3\sqrt{14}}{7}$	2	(a) $OA:OB = 2:1$ (b)(i) $m+2n+2=0$ (b)(ii) $n=\frac{2}{5}$, $m=-\frac{14}{5}$ and $n=-2$, $m=2$
3	$\overrightarrow{OE} = \frac{15}{23}\mathbf{a} + \frac{10}{23}\mathbf{c}$, $OE:EF = 15:8$	4	(i) $t=2$, $AD:AB = 1:2$ (ii) 14 (iii) 77.9°
5	(a) 38 (c) ± 5	6	(a) Parallelogram (b) $\frac{1}{6}$ (c) $\frac{5\sqrt{3}}{6}$ (d) 1:3
7	(i) $\overrightarrow{OS} = \frac{3\mathbf{p}+2k\mathbf{q}}{5}$ (ii) $k = \frac{1}{2}$	8	(i) $\overrightarrow{OA} = \lambda\mathbf{q} + (1-\lambda)\mathbf{p}$ (ii) $\lambda = \frac{1}{9}$ (iii) $1-\lambda$ (since $0 < \lambda < 1$)
9	(c) $\mathbf{r} = \frac{\mathbf{p}+3\mathbf{q}}{4}$	10	(a) $\overrightarrow{OQ} = \frac{2}{3}\mathbf{p} + \frac{1}{3}\mathbf{r}$ (b)(i) $\frac{22}{15} \mathbf{p} $

Vectors (Lines)

1 [2023/TJC/Promo/6]

Referred to the origin O , points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. Point C lies between O and A such that $OC : CA = 3 : 2$. Point X lies between A and B such that $AX : XB = 1 : 3$. The lines BC and OX meet at the point E .



- Show that the vector equation of the line BC can be written as $\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ where λ is a parameter. [1]
- By finding a vector equation of the line OX and using the result in (a), show that the position vector of E is $\frac{1}{2}\mathbf{a} + \frac{1}{6}\mathbf{b}$. [4]
- It is given that angle AOB is 30° and $\mathbf{b}$ is a unit vector. Find the exact perpendicular distance from E to the line OA . [3]

2 [2023/SAJC/Promo/5]

The position vectors of the points A and B relative to the origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively.

- Point M on AB is such that $AM : MB = 1 : 2$. Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. If the area of triangle OBM is 4 units², find $|\mathbf{a} \times \mathbf{b}|$. [5]
- Another point P has position vector $\mathbf{p}$ and $\mathbf{p} \neq \mathbf{0}$. Given $(\mathbf{p} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a}) = \mathbf{0}$, what can you deduce about the relationship between the vectors $\overrightarrow{AP}$ and $\overrightarrow{AB}$? Hence find the vector equation of line l that passes through points A and P in terms of $\mathbf{b}$ and $\mathbf{a}$. [2]
- Given instead that $\mathbf{a} = (4\mathbf{a} \cdot \mathbf{b})\mathbf{b}$, describe in words, the relation between $\mathbf{a}$ and $\mathbf{b}$ and find $|\mathbf{b}|$. [3]

3 [2023/JPJC/Promo/7]

The points A and B have coordinates $(3, 1, 1)$ and $(1, -1, 2)$ respectively.

- Find a vector equation of the line AB . [1]
- The point C has coordinates $(3, -16, 3)$. Find the position vector of the foot of perpendicular from C to the line AB . [3]
- Find a vector equation of the line which is a reflection of the line AC in the line AB . [3]



Answers

1	(c) $\frac{1}{12}$	2	(i) $\overline{OM} = \frac{2\vec{a} + \vec{b}}{3}; \vec{a} \times \vec{b} = 12$ (ii) $l_{AP}: \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \lambda \in \mathbb{R}$ (iii) $ \vec{b} = \frac{1}{2}$
3	(i) $\vec{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$ (ii) $\begin{pmatrix} -5 \\ -7 \\ 5 \end{pmatrix}$ (iii) $\vec{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -16 \\ 1 \\ 6 \end{pmatrix}, \mu \in \mathbb{R}$		

Vectors (Planes)

1 [2023/TJC/Promo/10]

The line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ where λ is a parameter. The point A has coordinates $(2, -2, 1)$.

(a) Find the position vector of the foot of perpendicular from A to l . [3]
The plane p_1 contains A and l .

(b) Find, in scalar product form, a vector equation of p_1 . [3]

It is given that l also lies on a second plane p_2 with equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = -1$ where a and b are constants.

(c) Show that $b = -\frac{1}{3}$ and find the value of a . [3]

(d) Hence find the acute angle between p_1 and p_2 . [2]

2 [JPJC/Promo/11]

The plane Π_1 has equation $\mathbf{r} \cdot \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} = 30$, where α and β are positive constants, and contains the point A with coordinates $(0, 10, 5)$.

(i) Given that the perpendicular distance from the origin O to the plane Π_1 is 6 units, find α and β . [5]

Another plane Π_2 has equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 4$.

(ii) Find the acute angle between the line OA and the plane Π_2 . [2]

(iii) Find a Cartesian equation of the plane Π_3 which contains the line OA and is perpendicular to the plane Π_2 . [2]



3 [2023/MI/PU1/Promo/7]

Relative to the origin O , the point A has position vector $3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$. The plane π contains A and is parallel to the vector $-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and the line $\mathbf{r} = \alpha(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, where $\alpha \in \mathbb{R}$.

(i) Show that the cartesian equation of plane π can be expressed as $4x + 5y - 2z = 45$. [3]

The point B has coordinates $(2, 0, 4)$.

(ii) Find the position vector of the point F , the foot of the perpendicular from B to the plane π . [3]

(iii) Hence find the position vector of the point B' , the reflection of B in the plane π . [2]

(iv) Find angle BAB' . [3]

4 [2023/NJC/Promo/1]

The equations of three planes p_1 , p_2 and p_3 are

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} = 3, \quad \mathbf{r} \cdot \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} = -5 \quad \text{and} \quad \mathbf{r} \cdot \begin{pmatrix} 17 \\ 5 \\ a \end{pmatrix} = b$$

respectively, where a and b are constants.

If $a = -20.9$ and $b = 16.6$, find the coordinates of the point at which these planes meet. [2]

The planes p_1 and p_2 intersect in a line l .

(i) Find a vector equation of l . [2]

(ii) Given that all three planes meet in the line l , find the values of a and b . [3]

5 [2023/MI/PU2/Promo/5]

The equations of the line l_1 and the plane π_1 are given by

$$l_1: x - 2 = \frac{y - 4}{2} = \frac{z - 5}{-2},$$

$$\pi_1: x + 2y + 3z = 24.$$

(i) Given that both l_1 and π_1 intersect at the point A , find the position vector of A . [3]

(ii) Find the acute angle between l_1 and π_1 . [2]

(iii) Show that the point B with coordinates $(1, 2, 7)$ lies on l_1 . [1]

(iv) Hence or otherwise, find the shortest distance from B to π_1 . [3]

The equation of the plane π_2 is given by

$$\pi_2: ax + by - 3z = 15, \quad \text{where } a \text{ and } b \text{ are constants.}$$

It is also given that both π_1 and π_2 intersect at a line l_2 with equation

$$l_2: \mathbf{r} = \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

(v) Find the values of a and b . [3]

6 [2023/TMJJC/Promo/8]

The plane π contains the point $A(1, -3, 1)$ and the line $l_1: \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}.$

- (i) Show that the cartesian equation of plane π is $x + y - z = -3$. [3]
- (ii) Find the position vector of the foot of the perpendicular from the point $B(0, 1, 0)$ to the plane π . [3]
- (iii) Hence find the position vector of the point B' which is the reflected point of B in the plane π . [2]
- (iv) The line l_2 is parallel to vector $\begin{pmatrix} \alpha \\ 1 \\ \beta \end{pmatrix}$ where α and β are real constants. Find the relationship between α and β such that l_2 meets π at a unique point. [2]
- (v) Given that $\alpha = 4$ and $\beta = 6$, find the acute angle between l_2 and π . [2]

7 [2023/NYJC/Promo/9]

Referred to the origin O , the points A , B and C have position vectors $3\mathbf{i} - \mathbf{j}$, $5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $2\mathbf{i} + \alpha\mathbf{j} + \sqrt{2}\mathbf{k}$ respectively, where α is a non-zero constant.

- (a) Given that the acute angle between the x -axis and the line passing through A and C is 60° , find the value of α . [3]

The line l passes through A and B .

- (b) A point Q lies on l such that AQ is 3 units. Find the two possible position vectors of Q . [4]

The plane π containing A has equation $x + 2z = 3$.

- (c) Find the position vector of the point F on π that is closest to B . [3]
- (d) Hence find a vector equation of the line of reflection of l in π . [3]

8 [2023/SAJC/Promo/10]

The line l_1 and the plane π_1 have equations $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -5 \\ p \end{pmatrix} = 4$

respectively, where p is a real constant.

- (i) Given that the line l_1 and the plane π_1 intersect at the point $A(-5, -7, 7)$, show that $p = -3$. [2]
- (ii) Find the acute angle between the line l_1 and the plane π_1 . [2]
- (iii) B is the point on l_1 where $\lambda = 1$. Find the position vector of the foot of perpendicular, F , from the point B to π_1 . [4]
- (iv) Find the equation of the line of reflection of l_1 in the plane π_1 . [3]

Another line l_2 has the following properties.

- l_2 passes through point B ,
 - l_2 is perpendicular to l_1 , and
 - l_2 is parallel to π_1 .
- (v) Find, in vector form, the equation of l_2 . [2]

9 [2023/RI/Promo/11]

The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 10 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

respectively, where λ and μ are parameters.

- (a) Without the use of a calculator, show that l_1 and l_2 are skew lines. [3]
- (b) Find a vector, $\mathbf{n}$, that is perpendicular to both l_1 and l_2 . [1]

Referred to the origin O , points P and Q have position vectors $\begin{pmatrix} 4 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ -1 \\ 2 \end{pmatrix}$ respectively.

- (c) Find the exact length of projection of $\overrightarrow{PQ}$ onto $\mathbf{n}$. [2]

A plane π has equation $3x + z = 11$.

- (d) Find, in degrees, the acute angles between l_1 and π , and between l_2 and π . Hence, or otherwise, determine which one of l_1 or l_2 is not intersecting π . [4]
- (e) Find the exact distance between the line determined in part (d) and π . [2]

10 [2023/ASRJ/C/Promo/11]

With respect to the origin O , the position vectors of the points A and B are $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} - 4\mathbf{k}$ respectively. The equation of the planes p_1 and p_2 are given by

$$p_1: \mathbf{r} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 1 \text{ and } p_2: \mathbf{r} \cdot \begin{pmatrix} 0 \\ 7 \\ \alpha \end{pmatrix} = -4\alpha, \text{ where } \alpha \text{ is a real constant with } \alpha > 0.$$

- (i) Find the position vector of the foot of perpendicular from A to p_1 . [3]
(ii) Verify that B lies on both p_1 and p_2 . Show that the vector equation of the line of intersection, L , of p_1 and p_2 is of the form

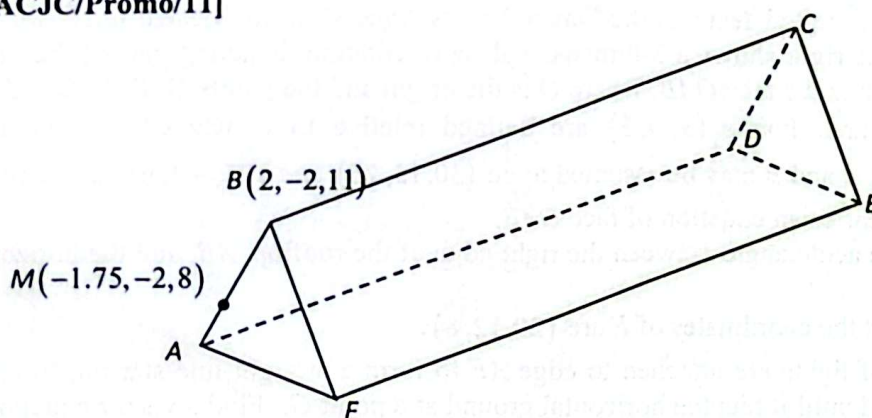
$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} + s \begin{pmatrix} 3\alpha \\ -\alpha \\ 7 \end{pmatrix}, s \in \mathbb{R}. \quad [2]$$

- (iii)(a) The point $C(-1, 1, -4)$ is equidistant from both p_1 and p_2 . Calculate the value of α for this case. Hence find in scalar product form the equation of the plane p_3 in which both the line L and the point C lie. [5]

- (b) The plane p_4 has equation $y - z = 2$.

Given that the angle between L and p_4 is $\frac{\pi}{6}$, find the value of α . [3]

11 [2023/ACJC/Promo/11]

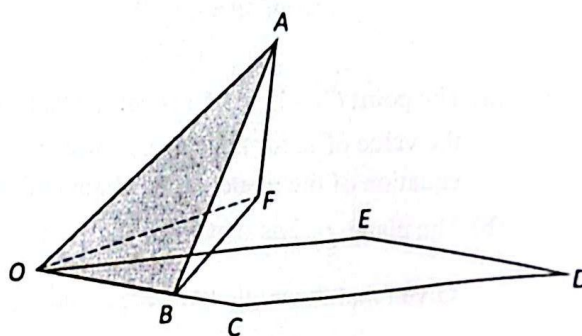
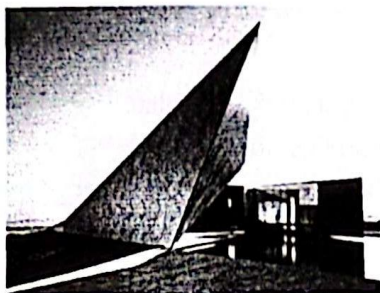


The diagram shows a roof structure modelled by a prism $ABCDEF$ with parallelogram sides. The triangular ends ABF and DCE are similar and parallel to the xz plane. The origin O is a point on the horizontal ground which is parallel to the xy plane. The coordinates of point B and M are $(2, -2, 11)$ and $(-1.75, -2, 8)$ respectively. It is given that plane $ABCD$ has equation $48x + 5y - 60z = -574$ and plane $BCEF$ has equation $12x - y + 12z = 158$.

- (i) Find the acute angle between plane $ABCD$ and plane $BCEF$, giving your answer correct to 2 decimal places. [2]
(ii) Without the use of the calculator, show that the equation of line BC is $x = 2, \frac{y+2}{12} = z - 11$. [2]

- (iii) It is now given that the vertical height of point C from the horizontal ground is 11.25 metres. State the coordinates of point C . [2]
- (iv) Find the acute angle between the line BC and the horizontal ground, giving your answer correct to 2 decimal places. [2]
- (v) Given that point N lies on BF such that MN is parallel to the x -axis, find the length of MN . [3]
- (vi) To strengthen the roof structure and minimise material cost, a steel beam MW is constructed such that point W lies on plane $BCEF$ and is closest to point M . Find the length of the steel beam MW . [3]

12 [2023/YIJC/Promo/11]



Source of picture: <https://eventcomm.com/projects/oman-across-ages-museum>

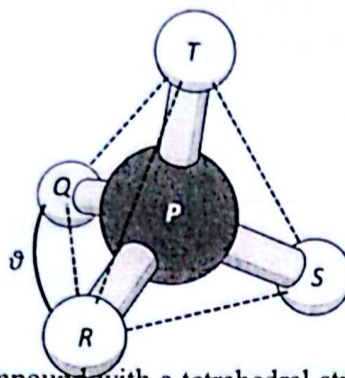
The picture on the left features the Oman Across Ages Museum located in Nizwa, Oman. The diagram on the right shows a 3-dimensional representation depicting part of the museum. The slanted rooftop is the face OAB , where O is the origin and the points O, B, C, D and E lie on the horizontal ground. Points (x, y, z) are defined relative to O where units are metres. The coordinates of A and B may be assumed to be $(30, 15, 25)$ and $(25, -4, 0)$ respectively.

- (a) Find a cartesian equation of face OAB . [3]
- (b) Find the acute angle between the right edge of the rooftop, AB , and the horizontal ground. [2]

It is given that the coordinates of F are $(20, 12, 8)$.

- (c) Tubes of lights are attached to edge AF to form a straight line starting from A to F and extended until it hits the horizontal ground at a point G . Find a vector equation of the line AF and the exact coordinates of G . [4]
- (d) Find, correct to the nearest metre, the shortest distance from F to face OAB . [3]

13 [EJC Promo 9758/2023/12]



Methane (CH_4) is a chemical compound with a tetrahedral structure. The 4 hydrogen (H) atoms form a regular tetrahedron, and the carbon (C) atom is in the centre.

Let the centre of the C-atom be the point P , and the centres of the 4 H-atoms be the points Q , R , S and T . The coordinates of P , Q , R and S are $(-1, 0, 2)$, $(-2, -1, 1)$, $(-2, 1, 3)$ and $(0, -1, a)$ respectively.

The angle θ subtended by any two C-H bonds at the C-atom, such as angle QPR , is known as the H-C-H bond angle (see diagram above).

- Find the bond angle, correct to 2 decimal places. [3]
- By using the fact that $QS = RS$, show that $a = 3$. [2]
- Find a cartesian equation for plane π , which contains the points P , Q and R . [3]
- F is the point on π that is closest to the point S .
 - State a vector equation for the line SF . [1]
 - Hence, show that the coordinates of F are $(0, 0, 2)$. [3]
 - Given that the point T is the mirror image of the point S in π , find the position vector of T . [2]

14 [2023/HCI/Promo/12]

Points (x, y, z) are defined relative to an origin $(0, 0, 0)$ on a horizontal ground, where units are measured in metres. Engineers are installing tension wires on a construction site. The tension wires with negligible thickness are constructed in straight lines. A slanted platform Π is built on top of the horizontal ground of the construction site. The equation of Π is $5x - y - 11z + 38 = 0$. It is given that points $A(6, 2, 6)$ and $B(1, 3, 17)$ lie on tension wire R .

- Find a vector equation of tension wire R . What can you say about tension wire R and the slanted platform Π ? [3]

The cartesian equation of tension wire Q is given by

$$21(x-1) = 15(y-3) = 7(z-17).$$

- Find the acute angle between tension wires Q and R , giving your answer in degrees. [3]
- Find the coordinates of the point C at which tension wire Q meets the slanted platform Π . [3]

Shining a light source on the tension wires can help improve visibility, making it easier to inspect wires, especially in dimly lit area.

- (d) A fixed light source at point L located above the horizontal ground with coordinates $(-6, -1, 5)$ is to be shone on tension wire Q . Find the shortest distance from L to tension wire Q . [4]

Answers

1	(a) $\begin{pmatrix} 23/10 \\ -1 \\ 9/10 \end{pmatrix}$ (b) $\vec{r} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 7$ (c) $a = 3$ (d) 88.2° or 1.54 radians	2	(i) $\alpha = 4, \beta = 3$ (ii) 46.9 (iii) $3x - y + 2z = 0$
3	(ii) $\begin{pmatrix} 6 \\ 5 \\ 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 10 \\ 10 \\ 0 \end{pmatrix}$ (iv) 121.7°	4	(i) $\left(\frac{7}{11}, -\frac{4}{11}, -\frac{4}{11}\right)$ (ii) $\vec{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ (iii) $a = -22, b = 17$
5	(i) $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$ (ii) 5.1° (iv) 0.535 (v) $a = 2, b = 1$	6	(ii) $\frac{1}{3} \begin{pmatrix} -4 \\ -1 \\ 4 \end{pmatrix}$ (iii) $-\frac{1}{3} \begin{pmatrix} 8 \\ 5 \\ -8 \end{pmatrix}$ (iv) $\alpha \neq \beta - 1$ (v) 4.5° or 0.0794 rad
7	(a) $\alpha = -2$ (b) $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$ (c) $\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$ (d) $\vec{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, t \in \mathbb{R}$	8	(i) $p = -3$ (ii) $\theta = 0.0788$ rad or 4.5° (iii) $\overrightarrow{OF} = \begin{pmatrix} 70/19 \\ -4/19 \\ 28/19 \end{pmatrix}$ (iv) $\vec{r} = \begin{pmatrix} -5 \\ -7 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 159 \\ 144 \\ -96 \end{pmatrix}, \mu \in \mathbb{R}$ (v) $\vec{r} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -16 \\ 5 \\ -19 \end{pmatrix}, s \in \mathbb{R}$

9	(b) $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ (c) $\frac{1}{\sqrt{6}}$ (d) $72.5^\circ, 0^\circ$ (e) $\frac{3\sqrt{10}}{5}$	10	(i) $\begin{pmatrix} 2 \\ 5 \\ \frac{1}{5} \\ 1 \end{pmatrix}$ (iii)(a) $\vec{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 13$ (iii)(b) $\alpha = \frac{7}{4} + \sqrt{\frac{147}{16}}$
11	(i) 83.5° (iii) $C(2, 1, 11.25)$ (iv) 4.76° (v) $\overline{MN} = 6.75$	12	(a) $20x + 125y - 99z = 0$ (b) 51.8° (c) $\vec{r} = \begin{pmatrix} 30 \\ 15 \\ 25 \end{pmatrix} + \lambda \begin{pmatrix} 10 \\ 3 \\ 17 \end{pmatrix}, \lambda \in \mathbb{R};$ $\left(\frac{260}{17}, \frac{180}{17}, 0\right)$ (d) $7m$
13	(a) $\hat{QPR} = 109.47^\circ$ (c) $\pi: y - z = -2$ (d)(i) $\vec{r} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}.$ (d)(iii) $\overline{OT} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	14	(a) $\vec{r} = \begin{pmatrix} 1 \\ 3 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \\ 11 \end{pmatrix}, \lambda \in \mathbb{R}$ (b) 45.5° (c) $C(-4, -4, 2)$ (d) 3.39 m



Module IV Differentiation and its Applications

Techniques of Differentiation

1 [2023/ACJC/Promo/1]

Differentiate each of the following expressions with respect to x .

(i) $\tan^{-1} \sqrt{x+1}$ [2]

(ii) $\ln \left(\frac{x+e^x}{3^x} \right)$ [3]

2 [2023/NYJC/Promo/2]

(a) Find $f'(x)$ where $f(x) = \ln(x + \sqrt{1+x^2})$, giving your answer in its simplest form. [3]

(b) Find $g'(x)$ where $g(x) = \tan^{-1} \left(\frac{2+x}{1-2x} \right)$, giving your answer in its simplest form. [3]

3 [2023/VJC/Promo/1]

(a) Find an expression for $\frac{dy}{dx}$ in terms of x if $y = \tan^{-1} \sqrt{x^2-1}$. [3]

(b) It is given that $y = x^{\cos x}$ for $x > 0$. By taking logarithm first, find an expression for $\frac{dy}{dx}$ in terms of x . [3]

4 [2023/SAJC/Promo/2]

(a) Differentiate $\frac{\sin^{-1}(2x)}{1-4x^2}$, for $-\frac{1}{2} < x < \frac{1}{2}$, with respect to x , simplifying your answer as much as possible as a single fraction. [3]

(b) It is given that $y^2 = 3e^{4x} + 4$. By repeated differentiation, show that

$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 - 8y^2 = k$, where k is a constant to be determined. [4]

Answers

1	(i) $\frac{1}{2(x+2)(\sqrt{x+1})}$ (ii) $\frac{1+e^x}{x+e^x} - \ln 3$	2	(a) $\frac{1}{\sqrt{1+x^2}}$ (b) $\frac{1}{x^2+1}$
3	(a) $\frac{1}{x\sqrt{x^2-1}}$ (b) $x^{\cos x} \left(\frac{\cos x}{x} - \sin x \ln x \right)$	4	(a) $\frac{2\sqrt{1-4x^2} + 8x \sin^{-1}(2x)}{(1-4x^2)^2}$ (b) $k = -32$

Applications of Differentiation

STATIONARY POINTS

1 [2023/ACJC/Promo/4]

Do not use a calculator in answering this question.

A curve C has equation $y^2 + 3xy = 13 + x^2$.

- Find the coordinates of the stationary points of C . [4]
- For the stationary point with $x > 0$, determine whether it is a maximum or minimum point. [3]

2 [2023/EJC/Promo/8]

The curve C has equation

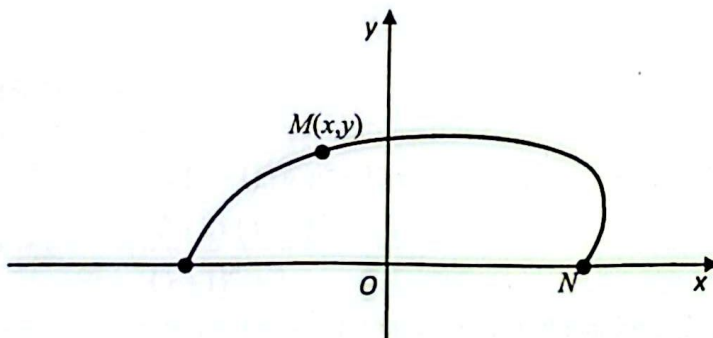
$$x + y = (x - y)^2.$$

It is given that C has only one turning point.

- Show that $1 - \frac{dy}{dx} = \frac{2}{1 + 2x - 2y}$. [4]
- Hence, or otherwise, show that $\frac{d^2y}{dx^2} = \left(1 - \frac{dy}{dx}\right)^3$. [3]
- Hence state, with a reason, whether the turning point is a minimum or a maximum. [2]

3 [2023/HCI/Promo/11]

The diagram below shows the curve C with equation $x^2 - xy + y^3 = 16$. N is a fixed point where C intersects the positive x -axis and M is a variable point (x, y) on C where $y > 0$.



- Show that $(3y^2 - x)\frac{dy}{dx} - y + 2x = 0$. [2]
- Show that $(3y^2 - x)\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 2 = 0$. [2]
- Show that A , the area of triangle OMN , is given by $2y$. [2]
- Hence find the value of x for which A has a stationary value. Determine the nature of this stationary value. [5]

Answers

1	(i) (3, 2) (ii) minimum point	2	(c) minimum
3	(d) $x = 1.30$ maximum		

TANGENT AND NORMALS

1 [2023/ASRJC/Promo/3]

A curve C has equation $4yx^2 + xy^2 = 5$.

- (i) Find the equation of the normal at point A with coordinates $(1, 1)$. [3]
- (ii) The normal at point A meets the x and y axes at the points Q and R respectively. Find the exact area of OQR . [2]

2 [2023/RI/Promo/6]

- (a) Let $y = a^x$, where a is a positive constant. Show that $\frac{dy}{dx} = a^x \ln a$. [2]

- (b) A curve has equation

$$y(3^x) + 2^{y-1} = 2.$$

Find the equation of the normal to the curve at the point $(0, 1)$. Give your answer in the form $y = Ax + B$, where A and B are constants in the exact form. [6]

3 [2023/JPJC/Promo/3]

The curve C has equation

$$x^2 + y^2 - 2xy + 3x = 4.$$

- (i) Find $\frac{dy}{dx}$ in terms of x and y . [2]
- (ii) Find the equation of the tangent to C which is parallel to the y -axis. [3]

4 [2023/YIJC/Promo/2]

Find the equations of the tangents to the curve $3x^2 - 2xy - 4y + y^2 = 2$ that are parallel to the y -axis. [4]

5 [2023/RVHS/Promo/9]

A curve C has equation

$$y(x+1)^2 + y^2 - 12x = 0, \text{ where } x, y \geq 0.$$

- (i) Show that $\frac{dy}{dx} = \frac{12 - 2y(x+1)}{(x+1)^2 + 2y}$. [2]
- (ii) Find the equation of the normal L to the curve at the point $P(3, 2)$. [3]
- (iii) Find the area of the triangle bounded by L and the axes. [1]
- (iv) The curve C is transformed by a series of transformations resulting in a curve D , with equation

$$3y(2x+1)^2 + 9y^2 - 24x = 0,$$

with point Q on D corresponding to the point P on C .

The normal L undergoes the same sequence of transformations as curve C , resulting in the line l .

Describe a possible sequence of transformations that maps C to D and hence deduce the exact area of the triangle bounded by l and the axes. [3]

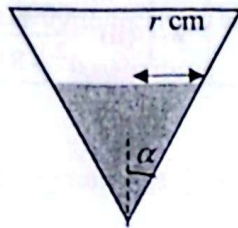


Answers

1	(i) $y = \frac{2}{3}x + \frac{1}{3}$	(ii) $\frac{1}{12} \text{ units}^2$	2	(b) $y = \left(\frac{1 + \ln 2}{\ln 3} \right)x + 1$
3	(i) $\frac{dy}{dx} = \frac{2y - 2x - 3}{2y - 2x}$	(ii) $x = \frac{4}{3}$	4	$x = -1$ or $x = 3$
5	(ii) $y = 5x - 13$	(iii) $\frac{169}{10} \text{ units}^2$		
	(iv) $\frac{169}{60} \text{ units}^2$			

CONNECTED RATE OF CHANGE

1 [2023/JPJC/Promo/4]

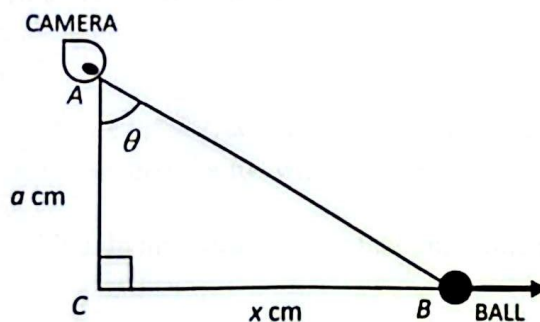


Water leaks out at a rate of 2 cm^3 per second from a container in the form of an open cone. The semi-vertical angle of the cone is α , where $\tan \alpha = \sqrt{2}$. At time t seconds after the start, the radius of the water surface is $r \text{ cm}$ (see diagram). Find the rate of change of the depth of water when the volume of water in the container is 1000 cm^3 . [5]

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3} \pi r^2 h$.]

2 [2023/RVHS/Promo/11]

Fixed vertical cameras have to rotate quickly to keep track on moving objects. A camera at A is attached on a vertical wall AC with fixed height $a \text{ cm}$. A ball B is moving horizontally away from the wall AC . At time t seconds, the ball is $x \text{ cm}$ from the wall and the camera would have complete a rotation of θ radians from the wall to keep track of the ball. At this instant, the camera is rotating at a speed of $\frac{d\theta}{dt}$ radians per second. Assume that the triangle ABC lies on a vertical plane containing the path of the ball.



- (i) By expressing $\tan \theta$ in terms of x and a , show that $\frac{d\theta}{dt} = \left(\frac{a}{a^2 + x^2} \right) \frac{dx}{dt}$. [4]
- (ii) When the ball is 9 cm from the wall, it is travelling at 6 m per second . At this instant, find the speed the camera is rotating at in radians per second, leaving your answer in terms of a . [1]
- (iii) Given that $a = 4$ and that the ball is travelling such that $x = t^2 \text{ cm}$, find the exact maximum value of the rotating speed of the camera in radians per second. [7]

Answers

1	-0.00521	2	(ii) $\frac{600a}{a^2+81}$ (iii) $\frac{3^{3/4}}{4}$
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MAXIMIZATION AND MINIMIZATION

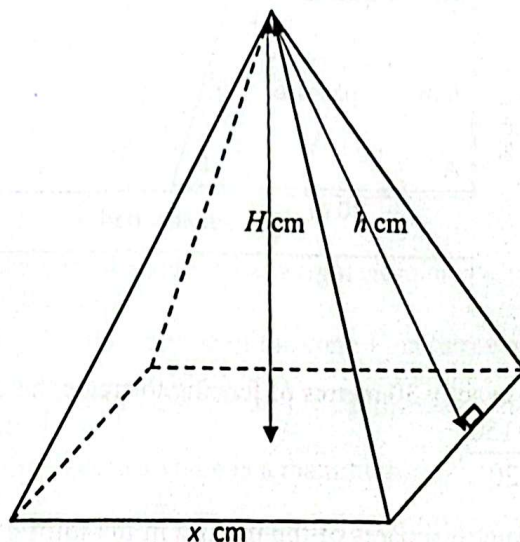
1 [2023/JPJC/Promo/5]

A solid cylinder has radius r cm, height h cm and total surface area 1200 cm^2 . Find the exact value of the maximum volume of the cylinder. Find also the ratio $r : h$ that gives this maximum volume.

[7]

2 [2023/ACJC/Promo/9]

[It is given that the volume of a pyramid is $\frac{1}{3} \times \text{base area} \times \text{height}$.]



A container in the shape of a right pyramid with vertical height H cm consists of a square base of side length x cm and four isosceles triangles, each with base x cm and perpendicular height h cm (see diagram). The pyramid has a fixed surface area $A \text{ cm}^2$.

(a) Show that $h = \frac{A - x^2}{2x}$. [1]

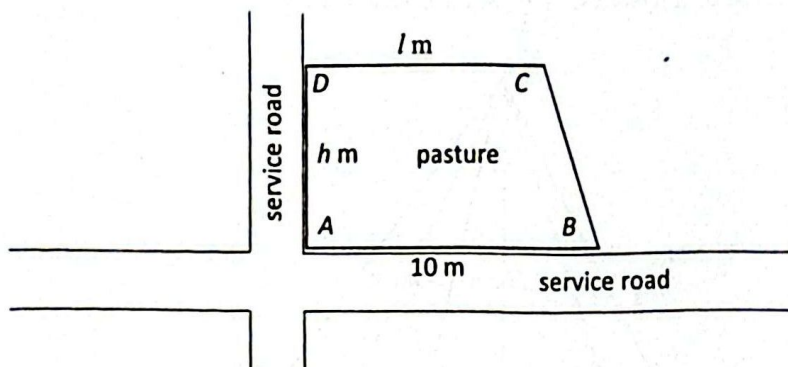
(b) Hence show that $36V^2 = A^2x^2 - 2Ax^4$, where V is the volume of the pyramid. [3]

(c) Find, in terms of A , the value of x for which V is maximum, and show that this maximum value of V is $\sqrt{\frac{A^3}{288}}$. You do not need to show that this value is a maximum. [3]

3 [2023/TJC/Promo/5]

[It is given that the area of a trapezium is $\frac{1}{2}(a+b)h$ where a and b are the lengths of the parallel sides and h is the perpendicular distance between the 2 parallel sides.]

At the junction of two perpendicular service roads, a farmer wishes to create a pasture in the shape of a trapezium $ABCD$ for his herd of goats. It is given that AB is perpendicular to AD , $AB = 10$ metres, $DC = l$ metres and $AD = h$ metres.

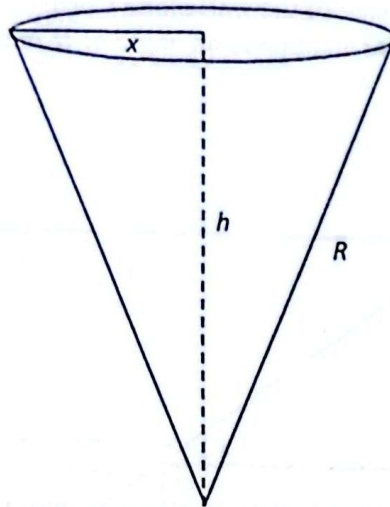


The farmer wishes to use exactly 30 metres of fencing to create the pasture.

- (a) Show that $h = \frac{10l - 150}{l - 20}$. [3]
- (b) Express the area S metre squares of the pasture in the form $5\left(l + c + \frac{d}{l - 20}\right)$ where c and d are constants to be determined. [2]
- (c) Use differentiation to find the exact value of l that maximises S . [3]

4 [22023/YIJC/Promo/10]

[It is given that the volume of a cone of base radius r and height h is $V = \frac{1}{3}\pi r^2 h$.]



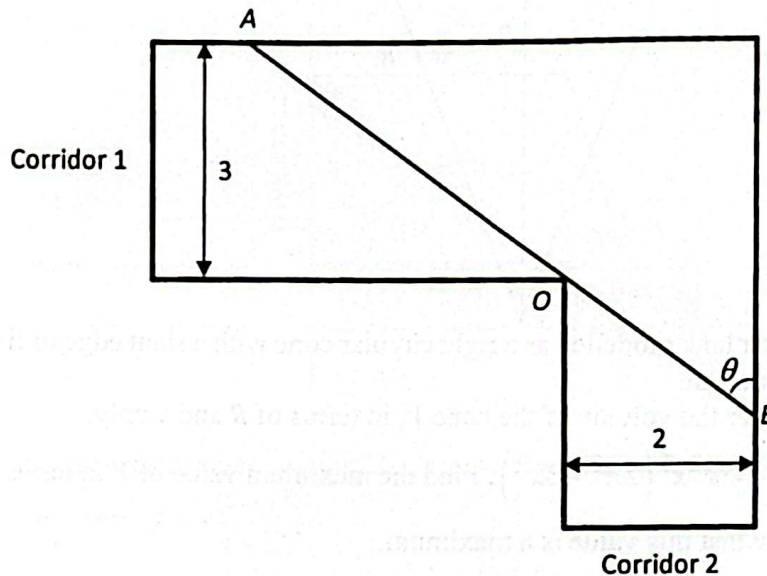
The diagram shows a water tank modelled as a right circular cone with a slant edge of fixed length R m, height h m and radius x m.

- (a) Find an expression for the volume of the cone V , in terms of R and x only. [1]
- (b) Show that $9V\left(\frac{dV}{dx}\right) = \pi^2 x^3 (2R^2 - 3x^2)$. Find the maximum value of V in terms of R . You do not need to show that this value is a maximum. [6]
- (c) It is known instead that $x = \frac{h}{2}$. The tank is being filled with water at the rate of $2\text{ m}^3\text{s}^{-1}$. Find the rate of increase of the depth of water at the instant when the volume of water in the tank is 8 m^3 . [4]



5 [2023/DHS/Promo/12]

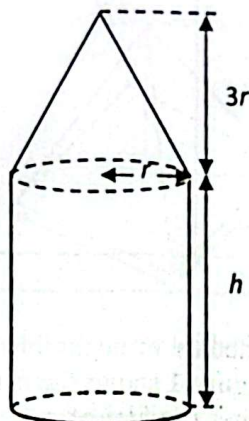
In a particular building, there is an L -shaped corridor (see diagram below). Corridor 1 has a fixed width of 3 m and Corridor 2, which is perpendicular to the first corridor, has a fixed width of 2 m. The points A and B are on one side of the wall along Corridor 1 and Corridor 2 respectively, such that line AB touches the corner O and makes an angle of θ radians with one of the walls as shown where $0 \leq \theta \leq \frac{\pi}{2}$.



- (a) Show that $AB = \frac{\alpha}{\cos \theta} + \frac{\beta}{\sin \theta}$, where α and β are constants to be determined. [1]
- (b) If θ is decreasing at 0.1 radians per second, find the rate at which AB is increasing at the instant when $OB = 4$ m. [4]
- (c) A pole of length p m is carried through this L -shaped corridor parallel to the ground. By using differentiation, show that the maximum possible value of p is $\left[(2)^k + (3)^k\right]^{\frac{1}{k}}$, where k is a constant to be determined exactly. [7]

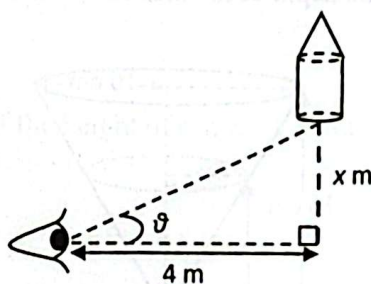
6 [2023/TMJC/Promo/10]

[It is given that for a cone of radius r and height h , the volume is $\frac{1}{3}\pi r^2 h$ and the curved surface area is $\pi r l$, where l is the slant height of the cone.]



A toy manufacturer designs a prototype for their remote-controlled toy rocket. The prototype of the toy rocket is completely hollow and consists of two portions, a cone and an open cylinder (i.e. the toy rocket has no base). The cone has radius r cm and height $3r$ cm. The cylinder has radius r cm and height h cm. The volume of the toy rocket is constant at 1000 cm^3 . It can be assumed that the material used to make the toy rocket has negligible thickness.

- (i) Show that the external surface area, $A \text{ cm}^2$, of the rocket is given by $\frac{2000}{r} + (\sqrt{10} - 2)\pi r^2$. [3]
- (ii) To save on the material cost to make the toy rocket, the toy manufacturer wishes to minimise the external surface area of the rocket. Using differentiation, find the value of r such that A is minimised. Hence, find the minimum value of A . [5]



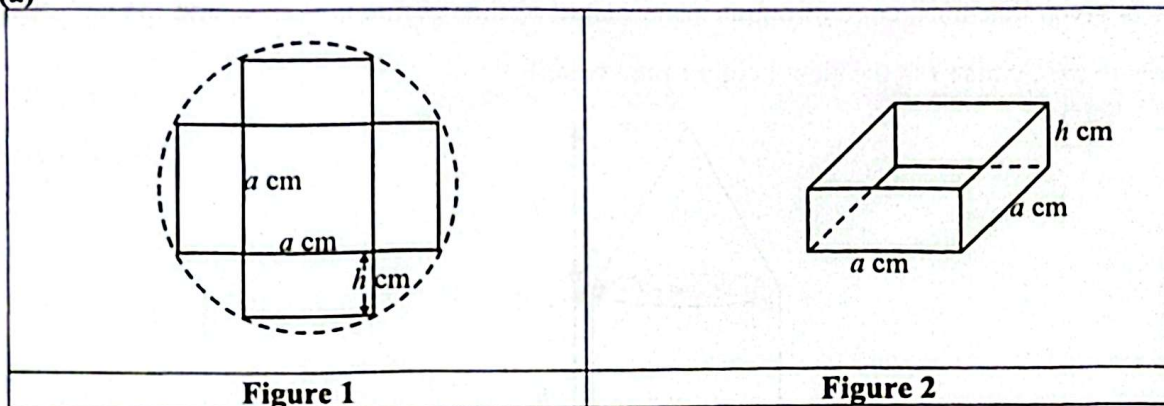
The toy rocket is flown vertically at a constant speed of 0.02 m s^{-1} . An observer watches the flying toy rocket from 4 m away as shown in the diagram above. The angle of elevation from the observer's eye level to the bottom of the toy rocket is denoted by θ and the height above the eye level of the observer is denoted as $x \text{ m}$.

- (iii) Find the rate of change of θ when $x = 3$. [4]



7 [2023/NYJC/Promo/11]

(a)



Jane bought a piece of circular cardboard of negligible thickness and radius 20 cm to make an open box with a square base. **Figure 1** shows the net of a square-based box that she cuts from the circular cardboard. The net consists of a square of side a cm and four identical rectangles, each with sides a cm by h cm. The net is folded to form a box which has a square base of side a cm and vertical height h cm, as shown in **Figure 2**.

- (i) Show that the volume of the box is $\left(a^2\sqrt{400 - 0.25a^2} - 0.5a^3\right)\text{cm}^3$. [2]
- (ii) By differentiation, find the values of a and h which give a maximum volume of the box, giving your answers correct to 1 decimal place. You do not need to show that these values give a maximum volume. [5]
- (b) Jane later bought an open metal cone to make a large candle. The inverted cone has height 16 cm and the circular opening at the top has radius 10 cm as shown in **Figure 3**. Melted wax is poured at a constant rate of 5 cm^3 per second into the inverted cone. At time t seconds after the start, the depth of the melted wax is x cm. Find the rate of increase of the depth of the melted wax when the depth of the melted wax is 12 cm. [5]

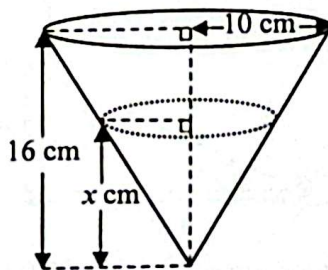
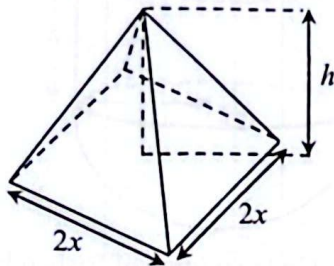


Figure 3

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]

8 [2023/ASRJC/Promo/12]

In an amusement park, the construction of a model Egyptian pyramid is done in 2 stages. In the first stage, a mould that consists of only 4 identical isosceles triangles as its slanted sides, is created using wooden boards with negligible thickness which are joined together as shown in the diagram below.



- The pyramid has an open square base with sides $2x$ m.
- The height of the pyramid is h m.

It is given that the volume of the model has a fixed value of 1 m^3 .

- (i) If A is the external surface area of the 4 slanted sides, show that

$$A = \frac{\sqrt{9+16x^6}}{x}. \quad [3]$$

[The volume of a square-based pyramid is $\frac{1}{3} \times \text{base area} \times \text{height}$]

- (ii) Using differentiation, find the exact value of x that gives the minimum A . [4]

It is now decided that $x = 1$. In the second stage, the mould is placed on the ground, and at time $t = 0$ seconds, cement is poured into the mould through a hole at the tip of the pyramid at a constant rate of $0.005 \text{ m}^3/\text{s}$.

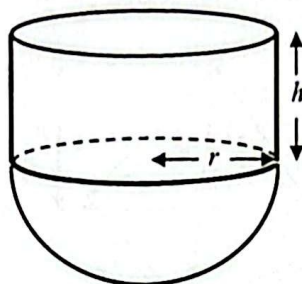
- (iii) If the volume and the height of cement in the mould at time t seconds are given by $V \text{ m}^3$ and $w \text{ m}$ respectively, show that the volume of the cement in the mould is given

$$\text{by } V = 1 - \frac{64}{27} \left(\frac{3}{4} - w \right)^3. \quad [2]$$

- (iv) Find the rate of change of the height of cement 1 minute after the cement has been poured. [5]

9 [2023/EJC/Promo/11]

[A sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



A water fountain is to be constructed in the middle of Bishan East Park. It consists of a hemisphere with radius r m joined to an open cylinder with radius r m and height h m (see diagram). The thickness of the fountain is negligible. It is given that the fountain, when filled to the brim, can hold a fixed volume k m³ of water.

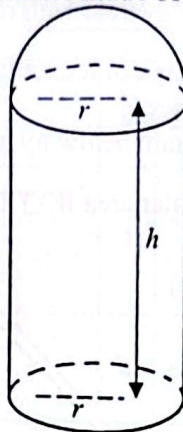
- (a) The interior of the fountain is to be painted with a layer of special reflecting paint. The cost of painting is \$3 per m² for the hemispherical surface and \$2.50 per m² for the cylindrical wall. Show that the total cost of painting, \$ C , is given by $\$ \left(\frac{8}{3}\pi r^2 + \frac{5k}{r} \right)$. [3]
- (b) Using differentiation, find the value of r , in terms of k , such that C is a minimum. [4]

Keeping C at a minimum, it is now given that $k = 50$.

- (c) Find the numerical values of r and h . [2]
- (d) When the fountain is filled to the brim, a leak develops at the joint between the cylinder and the hemisphere. Water leaks at a constant rate of 0.002 m³ per minute. Assuming that water is neither lost nor added to the fountain in any other way, find the rate at which the level of water is decreasing. [3]

10 [2023/VJC/Promo/11]

A company manufactures a closed container made of glass as shown in the figure below.



The closed container, of negligible thickness is made up of two components. The bottom component is a cylinder of base radius r cm and a height of h cm. The top component is a hemisphere of radius r cm. The company requires the volume of the container to be fixed at $108\pi \text{ cm}^3$.

The external surface area of the container is denoted by $A \text{ cm}^2$. The company wants the value of A to be as small as possible to reduce the cost of production.

[The volume of a sphere of radius r is given by $\frac{4}{3}\pi r^3$ and its surface area is given by $4\pi r^2$.]

- (a) Show that $A = \left(\frac{5r^2}{3} + \frac{216}{r} \right) \pi$. [3]
- (b) Using differentiation, find the exact value of r that gives the minimum value of A , proving that A is a minimum. [4]
- (c) Sketch the graph showing the external surface area of the container as the radius of the hemisphere varies, stating the coordinates of the end point of the graph. [3]

The company decides to produce the glass container with $r = 3$. To use the container as a decorative piece, the glass container is filled completely with a viscous liquid.

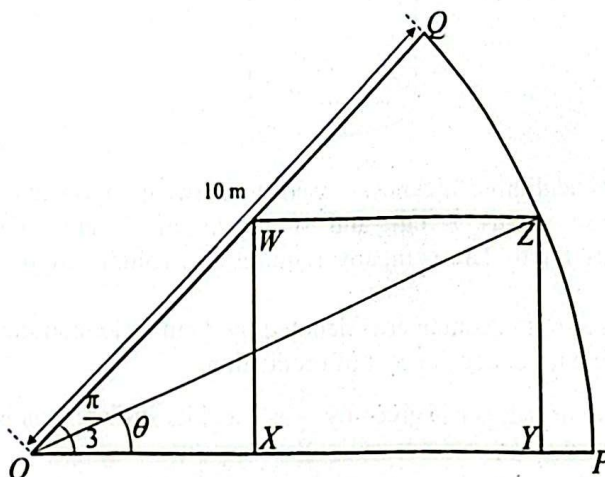
- (d) A small crack at the bottom of the container causes the viscous liquid to leak out of the container at a constant rate of 5 cm^3 per second. Find the rate of decrease of the height of the viscous liquid in the container 14 seconds after the container cracked. [3]

11 [2023/SAJC/Promo/11]

[It is given that the area of a sector of a circle with radius r and angle θ is given by $\frac{1}{2}r^2\theta$.]

A florist owns a greenhouse in the shape of a circular sector OPQ , with center O , and intends to enclose a rectangular area to grow roses.

The garden is represented in the diagram below by a fixed sector OPQ where $OP = OQ = 10$ m and $\angle POQ = \frac{\pi}{3}$ radians. The rectangular area $WXYZ$ is inscribed in sector OPQ such that $\angle ZOP = \theta$ radians.



- (i) By considering triangles OWX and OZY , or otherwise, show that the area of rectangle $WXYZ$, $A = 50 \left(\sin 2\theta - \frac{2\sqrt{3}}{3} \sin^2 \theta \right)$ [3]
- (ii) Using differentiation, find the maximum value of A in exact form. [6]
The florist subsequently decides to use remaining regions in the sector to grow marigolds. The cost of maintaining the plants each month, $\$C$, can be broken down into $\$5/\text{m}^2$ for roses and $\$4/\text{m}^2$ for marigolds.
- (iii) Sketch the graph of C as θ varies, indicating any turning point(s) and end point(s). [3]
- (iv) Hence determine the range of values of θ for the florist to have a maintenance cost of less than $\$220$. [1]

Answers

1	$400\sqrt{\frac{200}{\pi}}; 1:2$	2	2(a) $h = \frac{A-x^2}{2x}$ (c) $V = \sqrt{\frac{A^3}{288}}$
3	3(b) $S = 5\left(l + 15 + \frac{150}{l-20}\right)$ (c) $l = 20 - \sqrt{150}$	4	4(a) $V = \frac{1}{3}\pi x^2\sqrt{R^2 - x^2}$ (b) $\frac{2}{9\sqrt{3}}\pi R^3$ (c) 0.26053 m/s
5	5(b) 0.493m/s (c) $\left[(2)^{\frac{2}{3}} + (3)^{\frac{2}{3}}\right]^{\frac{3}{2}}$	6	6(ii) 6.49; 462 cm ² (iii) 0.0032
7	7(a)(ii) $a = 19.6; h = 7.6$ (b) 0.0283	8	8(ii) $\left(\frac{9}{32}\right)^{\frac{1}{6}}$ (iv) 0.00159 m/s
9	9(b) $r = 3\sqrt{\frac{15k}{16\pi}}$ (c) $r = 2.46, h = 0.985$ (d) -1.05×10^{-4} m/min	10	10(b) $r = 3\sqrt{\frac{324}{5}}$ (d) 0.177
11	11(ii) $\frac{50\sqrt{3}}{3}$ units ² (iv) $0 < \theta < 0.114$ or $0.933 < \theta < \frac{\pi}{3}$		



Maclaurin Series

1 [2023/MI/PU2/P2/Promo/1]

Given that x is small, show that

$$\tan x - \sqrt{2} \cos\left(\frac{\pi}{4} - 2x\right) \approx 2x^2 - x - 1. \quad [3]$$

2 [2023/RI/Promo/5]

In this question you may use expansions from the List of Formulae (MF27).

Find the first four non-zero terms of the Maclaurin series of $e^{-x}(1 + \cos 3x)$, in ascending

(a) powers of x . [4]

(b) It is given that the first three terms of this series are equal to the first three terms in the series expansion, in ascending powers of x , of $\frac{1}{a+bx} + cx^2$. Find the values of a , b and c . [4]

3 [2023/TJC/Promo/2]

Given that $2y = \cos^{-1}(x^2)$, show that $(\sin 2y) \frac{dy}{dx} = -x$. [1]

By further differentiation of this result, show that $\frac{d^3y}{dx^3} = 0$ when $x = 0$. [3]

Hence find the Maclaurin series of y in ascending powers of x , up to and including the term in x^2 . [2]

4 [2023/EJC/Promo/9]

It is given that $y = \ln(2 - e^{-2x})$.

(a) Show that $\frac{dy}{dx} = 4e^{-y} - 2$. [2]

(b) Hence find the Maclaurin series for y , up to and including the term in x^2 . [3]

(c) Using standard series from the List of Formulae (MF27), expand $\ln(2 - e^{-2x})$ as far as the term in x^2 , and use this expansion as a check on the correctness of the series found in part (b). [4]

5 [2023/TMJC/Promo/7]

It is given that $y = \tan^{-1}(2x)$.

- (i) Show that $(1 + 4x^2) \frac{dy}{dx} = k$, where k is a real constant to be determined. [2]
- (ii) Hence, by further differentiation, find the Maclaurin's series for y , up to and including the term in x^3 . [5]
- (iii) By substituting $x = \frac{1}{2}$ into your result in part (ii), show that $\pi \approx \frac{p}{q}$, where p and q are integers to be determined. [2]
- (iv) Suggest one recommendation to improve the accuracy of your estimation in part (iii). [1]

6 [2023/MI/PU2/P1/Promo/7]

- (i) Given that $y = \ln(1 - \sin x)$, show that

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = e^{-y} - 1.$$

By further differentiation of this result, find the Maclaurin Series for y in ascending powers of x , up to and including the term in x^3 . [7]

- (ii) Write down the first three terms in the series expansion for $(1 - x)^{-1}$. Hence find the Maclaurin series of $\frac{\ln(1 - \sin x)}{1 - x}$, up to and including the term in x^3 . [3]

7 [2023/DHS /Promo/4]

It is given that $e^y = 2 + e^x$.

- (a) Show that $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$. Hence find the Maclaurin series for y up to and including the term in x^2 , giving the coefficients in exact form. [4]
- (b) By using appropriate standard series expansions from the List of Formulae (MF27), verify the correctness of the series expansion found in part (a). [3]

8 [2023/NYJC/Promo/8]

It is given that $y = f(x)$, where $f(x) = \tan(\alpha + \beta x)$, for constants α and β .

- (a) Show that $f'(x) = \beta(1 + y^2)$. Hence find $f''(x)$ and $f'''(x)$ in terms of β and y . [5]
- (b) In the case where $\alpha = \frac{\pi}{4}$, use your results from part (a) to find the Maclaurin series for $f(x)$ in terms of β , up to and including the term in x^3 . [3]
- (c) Using your result from part (b), state the equation of the tangent to the curve $y = \tan\left(\frac{\pi}{4} + 3x\right)$ at $x = 0$. [1]
- (d) Explain why a Maclaurin series for $y = \cot 3x$ cannot be found. [1]

9 [2023/VJC/Promo/8]

It is given that $f(x) = \ln(1+2x^2)$.

- (a) A polynomial $p(x) = ax^2 + bx + c$ is used to approximate $f(x)$ for $1 \leq x \leq 2$. Given that $p(1) = f(1)$, $p(1.5) = f(1.5)$ and $p(2) = f(2)$, find the values of the coefficients a , b and c . [3]
- (b) Using standard series from the List of Formulae (MF27), find the Maclaurin series for $f(x)$ in ascending powers of x , up to and including the term in x^6 . [2]
- (c) Find the set of values of x for the expansion in part (b) to be valid. [2]
- (d) Use $x^2 = \frac{1}{6}$ in your series from part (b) to show that $\ln\left(\frac{4}{3}\right) \approx \frac{m}{162}$, where m is an integer to be determined. [2]
- (e) Using your series from part (b), find the series expansion for $\ln\left(\frac{1+2x^2}{1+2x+x^2}\right)$, up to and including the term in x^4 . Give the coefficients in exact form. [2]

Answers

1	—	2	(a) $2 - 2x - \frac{7}{2}x^2 + \frac{25}{6}x^3 + \dots$ (b) $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = -\frac{11}{2}$
3	$y = \frac{\pi}{4} - \frac{1}{2}x^2 + \dots$	4	(b) $y = 2x - 4x^2 + \dots$
5	(i) $k = 2$ (ii) $\tan^{-1} 2x \approx 2x - \frac{8}{3}x^3$ (iii) $p = 8$, $q = 3$	6	(i) $y = -x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \dots$ (ii) $-x - \frac{3}{2}x^2 - \frac{5}{3}x^3 - \dots$
7	(a) $y = \ln 3 + \frac{x}{3} + \frac{x^2}{9} + \dots$	8	(a) $f''(x) = 2\beta^2(y + y^3)$; $f'''(x) = 2\beta^3(3y^2 + 1)(1 + y^2)$ (b) $f(x) = 1 + 2\beta x + 2\beta^2 x^2 + \frac{8\beta^3}{3}x^3 + \dots$ (c) $y = 1 + 6x$
9	(a) $a = -0.227$, $b = 1.78$ and $c = -0.455$ (b) $2x^2 - 2x^4 + \frac{8}{3}x^6 + \dots$ (c) $\left\{x \in \mathbb{R} : -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}\right\}$ (d) $m = 47$ (e) $-2x + 3x^2 - \frac{2}{3}x^3 - \frac{3}{2}x^4 + \dots$		