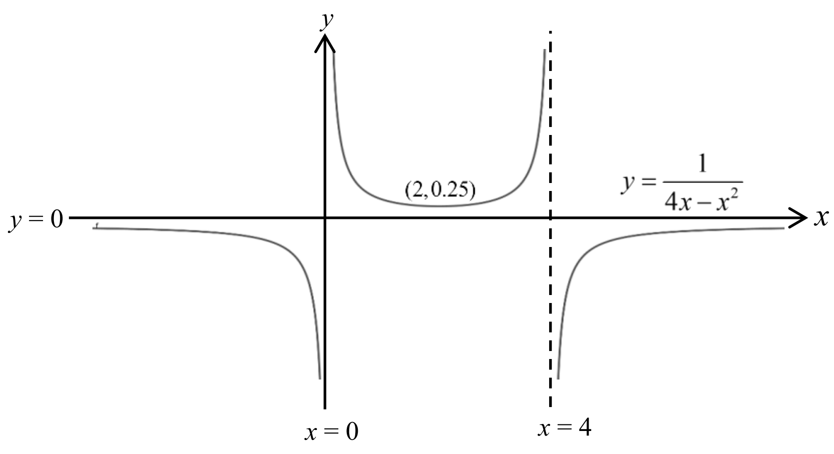
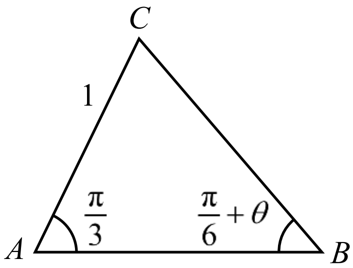
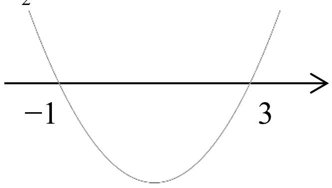
2025 TJC Prelim Exam H2 Math Paper 1 Solution

1(a)	$\dfrac{v_n}{v_{n-1}} = \dfrac{9}{u_n} \div \dfrac{9}{u_{n-1}} = \dfrac{9}{a\left(\frac{3}{2}\right)^{n-1}} \times \dfrac{a\left(\frac{3}{2}\right)^{n-2}}{9} = \dfrac{2}{3}$ (constant) for all positive integers n. Therefore $\{v_n\}$ is a geometric sequence.
1(b)	$v_2 = u_2 \quad \Rightarrow \quad \left(\dfrac{9}{a}\right)\left(\dfrac{2}{3}\right) = a\left(\dfrac{3}{2}\right)$ $\Rightarrow \quad a^2 = 4 \quad \Rightarrow \quad a = 2$ (rej -2 since a is positive)

2	Differentiate $xy = 1 + (x-y)^3$ w.r.t. x, we have $x\dfrac{dy}{dx} = 3(x-y)^2\left(1 - \dfrac{dy}{dx}\right)$ At $y = 0$, $x = -1$ Gradient of tangent at $A\,(-1, 0)$ is $-\dfrac{dy}{dx} = 3\left(1 - \dfrac{dy}{dx}\right) \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{3}{2}$

3(a)	$y = \dfrac{a}{2bx - x^2} = \dfrac{a}{b^2 - (x-b)^2}$ (Completing Square) $\downarrow$ replace x by $x + b$ $y = \dfrac{a}{b^2 - x^2}$ Translation of b units in the negative direction of the x-axis.
3(b)	

4(a)	Using Sine Rule, $$\frac{BC}{\sin\frac{\pi}{3}}=\frac{1}{\sin\left(\frac{\pi}{6}+\theta\right)}$$ $$\frac{BC}{\sin\frac{\pi}{3}}=\frac{1}{\sin\frac{\pi}{6}\cos\theta+\cos\frac{\pi}{6}\sin\theta}$$ $$\frac{BC}{\frac{\sqrt{3}}{2}}=\frac{1}{\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta}$$ $$BC=\frac{\sqrt{3}}{\cos\theta+\sqrt{3}\sin\theta}$$ 
4(b)	$$BC\approx\frac{\sqrt{3}}{\left(1-\frac{\theta^2}{2}\right)+\sqrt{3}\theta}\quad\text{(since }\theta\text{ is small)}$$ $$=\sqrt{3}\left[1+\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)\right]^{-1}$$ $$\approx\sqrt{3}\left[1-\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)+\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)^2\right]$$ $$\approx\sqrt{3}\left[1-\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)+3\theta^2\right]$$ $$=\sqrt{3}\left(1-\sqrt{3}\theta+\frac{7\theta^2}{2}\right)$$

5(a)	$\dfrac{2x-25}{x^2-2x-3} \le 2$ $\dfrac{2x-25-2\left(x^2-2x-3\right)}{x^2-2x-3} \le 0$ $\dfrac{-2x^2+6x-19}{x^2-2x-3} \le 0$ $\dfrac{-2\left(x-\frac{3}{2}\right)^2-\frac{29}{2}}{x^2-2x-3} \le 0$ --- (1) Since the numerator $-2\left(x-\frac{3}{2}\right)^2-\frac{29}{2}<0$ for all real values of x, $x^2-2x-3>0$. $(x-3)(x+1)>0$ $x<-1$ or $x>3$ 
5(b)	Replace x by $\lvert x+1\rvert$, from **(a)**, $\lvert x+1\rvert<-1$ or $\lvert x+1\rvert>3$ $\lvert x+1\rvert<-1$ has no solution $\lvert x+1\rvert>3$ => $x+1>3$ or $x+1<-3$ $x>2$ or $x<-4$

6(a)	Using ratio theorem, $\overrightarrow{OX}=\dfrac{3\mathbf{a}+\mathbf{b}}{4}$
6(b)	$\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\left(\dfrac{\pi}{6}\right)=\dfrac{\sqrt{3}}{2}\lambda^2$
6(c)	$\overrightarrow{OY}=\left(\overrightarrow{OX}\cdot\mathbf{b}\right)\mathbf{b}=\left(\dfrac{1}{4}(3\mathbf{a}+\mathbf{b})\cdot\dfrac{\mathbf{b}}{\lvert\mathbf{b}\rvert}\right)\dfrac{\mathbf{b}}{\lvert\mathbf{b}\rvert}$ $=\dfrac{1}{4}\left(\dfrac{\mathbf{b}}{\lvert\mathbf{b}\rvert^2}\right)(3\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b})$ $=\dfrac{1}{4}\left(\dfrac{\mathbf{b}}{\lvert\mathbf{b}\rvert^2}\right)\left(3\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\dfrac{\pi}{6}+\lvert\mathbf{b}\rvert^2\right)$ $=\dfrac{1}{4}\left(\dfrac{\mathbf{b}}{\lambda^2}\right)\left(3\lambda^2\left(\dfrac{\sqrt{3}}{2}\right)+\lambda^2\right)$ $=\dfrac{2+3\sqrt{3}}{8}\mathbf{b}$ (Shown) **<u>Alternative solution:</u>** $\overrightarrow{OY}=\mu\mathbf{b}\quad\Rightarrow\overrightarrow{XY}=\mu\mathbf{b}-\dfrac{3\mathbf{a}+\mathbf{b}}{4}=-\dfrac{3}{4}\mathbf{a}+\left(\mu-\dfrac{1}{4}\right)\mathbf{b}$ $\overrightarrow{XY}\bullet\overrightarrow{OB}=0$ $\left[-\dfrac{3}{4}\mathbf{a}+\left(\mu-\dfrac{1}{4}\right)\mathbf{b}\right]\bullet\mathbf{b}=0$ $-\dfrac{3}{4}\mathbf{a}\bullet\mathbf{b}+\left(\mu-\dfrac{1}{4}\right)\mathbf{b}\bullet\mathbf{b}=0$ $-\dfrac{3}{4}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\dfrac{\pi}{6}+\left(\mu-\dfrac{1}{4}\right)\lvert\mathbf{b}\rvert^2=0$ $-\dfrac{3}{4}\lvert\mathbf{b}\rvert^2\left(\dfrac{\sqrt{3}}{2}\right)+\left(\mu-\dfrac{1}{4}\right)\lvert\mathbf{b}\rvert^2=0$ $\mu-\dfrac{1}{4}=\dfrac{3\sqrt{3}}{8}\Rightarrow\mu=\dfrac{2+3\sqrt{3}}{8}$ $\therefore\overrightarrow{OY}=\dfrac{2+3\sqrt{3}}{8}\mathbf{b},\ (\text{Shown})$

6(d)	Area of $\Delta OXY = \frac{1}{2}\left|\overrightarrow{OX}\times\overrightarrow{OY}\right|$ $= \frac{1}{2}\left|\left(\frac{3\mathbf{a}+\mathbf{b}}{4}\right)\times\left(\frac{\left(2+3\sqrt{3}\right)}{8}\mathbf{b}\right)\right|$ $= \frac{\left(2+3\sqrt{3}\right)}{64}\left|\left(3\mathbf{a}+\mathbf{b}\right)\times\mathbf{b}\right|$ $= \frac{\left(2+3\sqrt{3}\right)}{64}\left|3\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{b}\right|$ $= \frac{3\left(2+3\sqrt{3}\right)}{64}\left|\mathbf{a}\times\mathbf{b}\right|$ where $\mathbf{b}\times\mathbf{b}=\mathbf{0}$ $= \frac{3\left(2+3\sqrt{3}\right)}{64}\left(\left|\mathbf{a}\right|\left|\mathbf{b}\right|\sin\frac{\pi}{6}\right)$ $= \frac{3\left(2+3\sqrt{3}\right)}{64}\left|\mathbf{b}\right|^2\left(\frac{1}{2}\right)$ $\left(\left|\mathbf{a}\right|=\left|\mathbf{b}\right|=\lambda=\text{radius of circle}\right)$ $= \frac{3\left(2+3\sqrt{3}\right)}{128}\lambda^2$ units^2

7(a)

For Mabel:

Amount of money Mabel saves in a year

$$= \$(101 + 102 + \ldots +112)$$

$$= \$\frac{12}{2}\left(101+112\right) = \$1278$$

Amount of money Mabel saves in 5 year $= \$1278 \times 5 = \$6390$

For Janice:

nth mth	Amount at start of month (\$)	Amount at end of month (\$)
1 Jan26	100	$100(1.003)$
2	$100+100(1.003)$ $=100(1+1.003)$	$100(1+1.003)(1.003)$ $=100\left(1.003+1.003^2\right)$
$\vdots$		$\vdots$
n		$100\left(1.003+1.003^2+\cdots+1.003^n\right)$ $=\frac{100(1.003)\left(1.003^n-1\right)}{1.003-1}$

At Dec 2030, n =60.

Therefore, Amount Janice saves in 5 years

$$=\frac{100(1.003)\left(1.003^{60}-1\right)}{1.003-1} = \$6582.85 \quad \text{(2 d.p.)} \quad [\text{A1}]$$

$\therefore$ Janice will have more money in her savings account than Mabel has in her piggy bank.

7(b)	At end of 4th year, Mabel will have \$1278 x 4 = \$5112 Janice will have $\dfrac{100(1.003)\left(1.003^{48}-1\right)}{1.003-1}=\$5169.97 \;\; > \;\; \$5112$ At end of 3rd year, Mabel will have \$1278 x 3 = \$3834 Janice will have $\dfrac{100(1.003)\left(1.003^{36}-1\right)}{1.003-1}=\$3806.97 \;\; < \;\; \$3834$ Therefore, the year when Janice's saving is more than Mabel's saving happen in 4th year (i.e. 2029). Let it be the kth month. Then set Janice's saving > Mabel's saving $\Rightarrow \dfrac{100(1.003)\left(1.003^{36+k}-1\right)}{1.003-1}>3834+\dfrac{k}{2}\left[2(100)+(k-1)(1)\right]$ $\Rightarrow \left(\dfrac{100003}{3}\right)\left(1.003^{36+k}-1\right)-3834-\dfrac{k}{2}\left[199+k\right]>0$ Using GC, we have $k \geq 4$. Therefore, the month and year is Apr 2029.

8(a)

$2z+|w|=2-4\text{i}$ --- (1)

$\text{i}z-w=2\text{i}$ --- (2)

From (1) : $z=\dfrac{2-4\text{i}-|w|}{2}$

Substitute into (2):

$\text{i}\left(\dfrac{2-4\text{i}-|w|}{2}\right)-w=2\text{i}$

$2\text{i}+4-\text{i}|w|-2w=4\text{i}$

Let $w=a+\text{i}b,\ \ a,b\in\mathbb{R}$,

$2\text{i}+4-\text{i}\sqrt{a^2+b^2}-2(a+\text{i}b)=4\text{i}$

$4-2a+\text{i}(2-\sqrt{a^2+b^2}-2b)=4\text{i}$

Compare real and imaginary parts:

$4-2a=0\ \Rightarrow\ a=2$ --- (3)

$2-\sqrt{a^2+b^2}-2b=4$ --- (4)

Substitute (3) into (4):

$-2-2b=\sqrt{4+b^2}$ --- (5)

$\Rightarrow(-2-2b)^2=4+b^2$

$4+8b+4b^2=4+b^2$

$b(3b+8)=0$

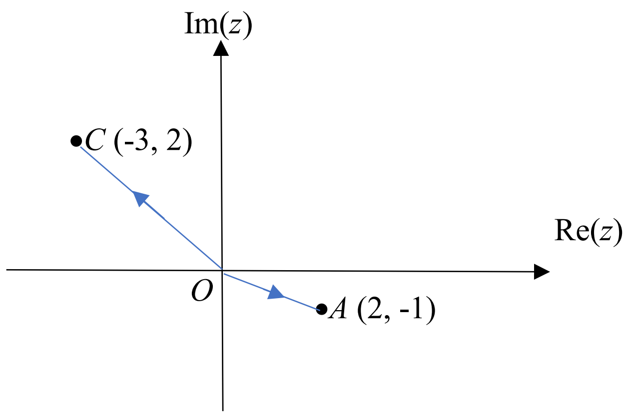
$\Rightarrow\ b=0$ (rejected since $\sqrt{4+b^2}\neq-2$ from (5))

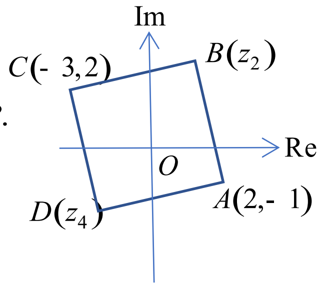
or $b=-\dfrac{8}{3}$

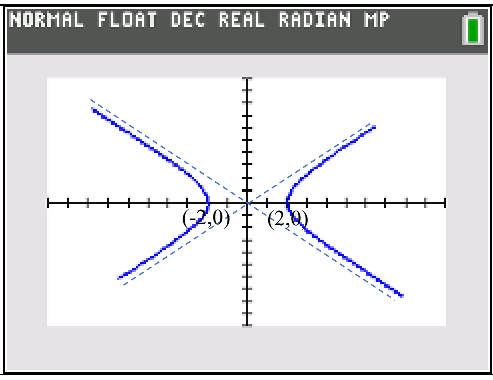
$\therefore\ w=2-\dfrac{8}{3}\text{i}$

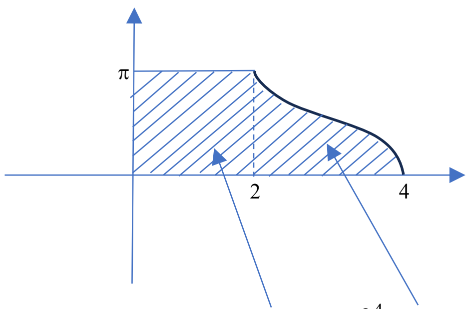
$z=\dfrac{2-4\text{i}-(-2-2b)}{2}=2-2\text{i}+\left(-\dfrac{8}{3}\right)=-\dfrac{2}{3}-2\text{i}$

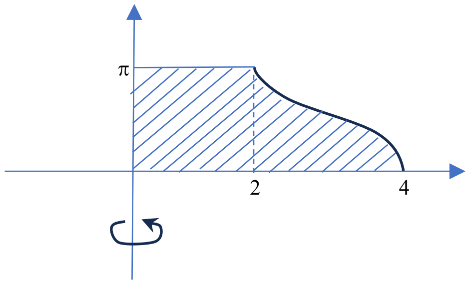
8(b)



	$\overrightarrow{BA}$ represents the complex number $2-\mathrm{i}-z_2$ and $\overrightarrow{BC}$ represents the complex number $-3+2\mathrm{i}-z_2$. Since $\angle ABC=\dfrac{\pi}{2}$, A is obtained by rotating C through 90° in anti-clockwise direction about B.  $\mathrm{i}(-3+2\mathrm{i}-z_2)=2-\mathrm{i}-z_2$ $-3\mathrm{i}-2-\mathrm{i}z_2=2-\mathrm{i}-z_2$ $(1-\mathrm{i})z_2=4+2\mathrm{i}$ $z_2=\dfrac{4+2\mathrm{i}}{1-\mathrm{i}}\times\dfrac{1+\mathrm{i}}{1+\mathrm{i}}=\dfrac{2+6\mathrm{i}}{2}=1+3\mathrm{i}$ $\overrightarrow{AB}=\overrightarrow{DC}$ $1+3\mathrm{i}-(2-\mathrm{i})=-3+2\mathrm{i}-z_4$ $z_4=-3+2\mathrm{i}+1-4\mathrm{i}=-2-2\mathrm{i}$

9(a)	
9(b)	$\left.\begin{array}{l} x = t + \frac{1}{t} \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \frac{1}{t^2} = \frac{t^2-1}{t^2} \\ y = t - \frac{1}{t} \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \frac{1}{t^2} = \frac{t^2+1}{t^2} \end{array}\right\\} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2+1}{t^2-1}$ Tangent parallel to y-axis $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined $\Rightarrow \quad t^2 - 1 = 0 \quad \Rightarrow \quad t = \pm 1$
(c)	At $t = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{3}$, $x = \frac{5}{2}$, $y = \frac{3}{2}$ Equation of normal at $t = 2$ is $y - \frac{3}{2} = -\frac{3}{5}\left(x - \frac{5}{2}\right)$ or $y = -\frac{3}{5}x + 3$
(d)	$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}$ ----- (1) $y = -\frac{3}{5}x + 3$ ----- (2) Sub (1) into (2): $t - \frac{1}{t} = -\frac{3}{5}\left(t + \frac{1}{t}\right) + 3$ $\Rightarrow \quad t^2 - 1 = -\frac{3}{5}t^2 + 3t - \frac{3}{5}$ $\Rightarrow \quad \frac{8}{5}t^2 - 3t - \frac{2}{5} = 0$ $\Rightarrow \quad t = -\frac{1}{8}$ or $t = 2$ (this is already used) So coordinates of Q are $\left(-\frac{65}{8}, \frac{63}{8}\right)$

10(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1-x^2} = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$ $\int_{-1}^{1} \cos^{-1} x \,\mathrm{d}x = \left[x\cos^{-1} x\right]_{-1}^{1} - \int_{-1}^{1} x\left[-\frac{1}{\sqrt{1-x^2}}\right]\mathrm{d}x$ $= 1(0)-(-1)(\pi) - \int_{-1}^{1} \frac{1}{2}\cdot\frac{-2x}{\sqrt{1-x^2}}\,\mathrm{d}x$ $= \pi - \left[\sqrt{1-x^2}\right]_{-1}^{1}$ $= \pi - 0$ $= \pi$
10(bi)	Translate the graph of $y=\cos^{-1}(x-3)$ three units in the negative direction of x-axis to obtain the graph of $y=\cos^{-1} x$
10(bii)	 Area of the region $R = (2)(\pi) + \int_{2}^{4} \cos^{-1}(x-3)\,\mathrm{d}x$ $= 2\pi + \int_{-1}^{1} \cos^{-1} u \,\mathrm{d}u$ $= 3\pi$ Alternative Method: Area of the region $R = \int_{0}^{\pi} 3+\cos y \,\mathrm{d}y$ $= [3x + \sin y]_{0}^{\pi}$ $= 3\pi$

10(c)	 Volume generated when rotated completely about y-axis $=\int_0^{\pi} \pi x^2 \, \mathrm{d}y$ $=\pi\int_0^{\pi} (\cos y+3)^2 \, \mathrm{d}y$ $=\pi\int_0^{\pi} (\cos^2 y+6\cos y+9) \, \mathrm{d}y$ $=\pi\int_0^{\pi} \left[\frac{1}{2}(\cos 2y+1)+6\cos y+9\right] \mathrm{d}y$ $=\pi\left[\frac{1}{4}\sin 2y+6\sin y+\frac{19}{2}y\right]_0^{\pi}$ $=\frac{19}{2}\pi^2$

11(a)	$\begin{pmatrix}1\\1\\\alpha\end{pmatrix}\times\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}1+\alpha\\-1+2\alpha\\-3\end{pmatrix}$
11(b)	xy-plane $\Rightarrow\ z=0$ $x+y=6\quad ...(1)$ $2x-y=4\quad ...(2)$ $(1)+(2): 3x=10\ \Rightarrow\ x=\frac{10}{3}$ $\therefore\ y=6-\frac{10}{3}=\frac{8}{3}$ $\therefore\ \overrightarrow{OC}=\frac{1}{3}\begin{pmatrix}10\\8\\0\end{pmatrix}$ Hence coordinates of C is $(\frac{10}{3},\frac{8}{3},0)$. Equation of the ridge line: $\underset{\sim}{r}=\frac{1}{3}\begin{pmatrix}10\\8\\0\end{pmatrix}+\mu\begin{pmatrix}1+\alpha\\-1+2\alpha\\-3\end{pmatrix},\mu\in\mathbf{R}$
11(c)	$\cos 60^\circ=\dfrac{\left\|\begin{pmatrix}1\\1\\\alpha\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}\right\|}{\sqrt{2+\alpha^2}\sqrt{6}}\quad\Rightarrow\quad\frac{1}{2}=\dfrac{\|1+\alpha\|}{\sqrt{2+\alpha^2}\sqrt{6}}$ $\sqrt{2+\alpha^2}\sqrt{6}=2\|1+\alpha\|$ $6\left(2+\alpha^2\right)=4\left(1+\alpha\right)^2$ $3\left(2+\alpha^2\right)=2\left(\alpha^2+2\alpha+1\right)$ $\alpha^2-4\alpha+4=0$ $\left(\alpha-2\right)^2=0$ $\Rightarrow\alpha=2$
11(d)	$\overrightarrow{OE}=\begin{pmatrix}2\\3\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$ for some $\lambda\in\mathbf{R}$ $\overrightarrow{OE}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=4$ $\begin{pmatrix}2+2\lambda\\3-\lambda\\1+\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=4$ $2\left(2+2\lambda\right)-\left(3-\lambda\right)+\left(1+\lambda\right)=4$ $\lambda=\frac{1}{3}$

	$\overrightarrow{OE}=\begin{pmatrix}\frac{8}{3}\\ \frac{8}{3}\\ \frac{4}{3}\end{pmatrix}$ i.e. $E\left(\frac{8}{3},\frac{8}{3},\frac{4}{3}\right)$

12(a)	Differentiate $w=x^2y$ with respect to x, $\frac{\mathrm{d}w}{\mathrm{d}x}=2xy+x^2\frac{\mathrm{d}y}{\mathrm{d}x}$ ---------- (1) Multiply x to given D.E., we have $2xy+x^2\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^5y^2}{\sqrt{x^2+1}}$ ------ (2) Subst. (1) into (2) and replace x^4y^2 by w^2 : $\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{w^2x}{\sqrt{x^2+1}}$ $\int\frac{1}{w^2}\,\mathrm{d}w=\int\frac{x}{\sqrt{x^2+1}}\mathrm{d}x$ $\int w^{-2}\,\mathrm{d}w=\frac{1}{2}\int 2x\left(x^2+1\right)^{-\frac{1}{2}}\mathrm{d}x$ $\frac{w^{-1}}{-1}=\frac{1}{2}\frac{\left(x^2+1\right)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+C$ where C is arb. constant $-\frac{1}{w}=\sqrt{x^2+1}+C$ $-\frac{1}{x^2y}=\sqrt{x^2+1}+C$ $y=-\frac{1}{x^2\left(\sqrt{x^2+1}+C\right)}$
(b)(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=kx\left(P-x\right)$ where $k>0$ Given that $x=4$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=P-4$, $P-4=4k\left(P-4\right)$ $k=\frac{1}{4}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{4}x\left(P-x\right)$

(ii)

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{4}x\left(P-x\right)$$

$$\int\frac{1}{x\left(P-x\right)}\ \mathrm{d}x=\int\frac{1}{4}\ \mathrm{d}x$$

$$\frac{1}{P}\int\frac{1}{x}+\frac{1}{P-x}\ \mathrm{d}x=\int\frac{1}{4}\ \mathrm{d}t$$

$\ln|x|-\ln|P-x|=\frac{P}{4}t+C$ where C is arbitrary constant

$\ln\frac{x}{P-x}=\frac{P}{4}t+C$ where $P-x>0$ and $x>0$

$\frac{x}{P-x}=A\mathrm{e}^{\frac{P}{4}t}$ where $A=\mathrm{e}^{C}$

$$\frac{P-x}{x}=\frac{1}{A}\mathrm{e}^{-\frac{P}{4}t}$$

$\frac{P}{x}-1=B\mathrm{e}^{-\frac{P}{4}t}$ where $B=\frac{1}{A}$

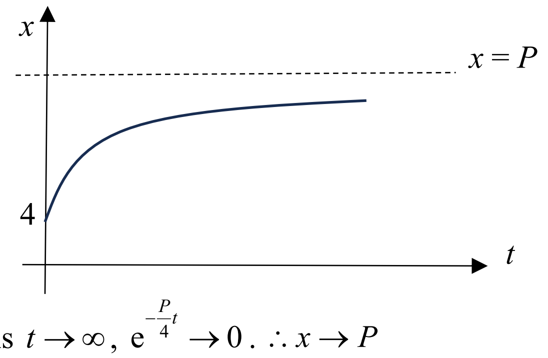
$$x=\frac{P}{1+B\mathrm{e}^{-\frac{P}{4}t}}$$

When $t=0$, $x=4$, $\frac{P}{4}-1=B$

$$x=\frac{P}{1+\left(\frac{P-4}{4}\right)\mathrm{e}^{-\frac{P}{4}t}}=\frac{4P}{4+\left(P-4\right)\mathrm{e}^{-\frac{P}{4}t}}$$

(iii)

$$x=\frac{4P}{4+\left(P-4\right)\mathrm{e}^{-\frac{P}{4}t}}$$



As $t\to\infty$, $\mathrm{e}^{-\frac{P}{4}t}\to 0$. $\therefore x\to P$

In the long term, all the durians in the plantation will be infected and become bad durians.