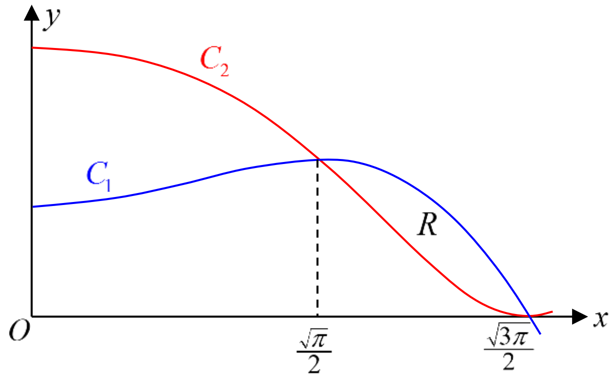
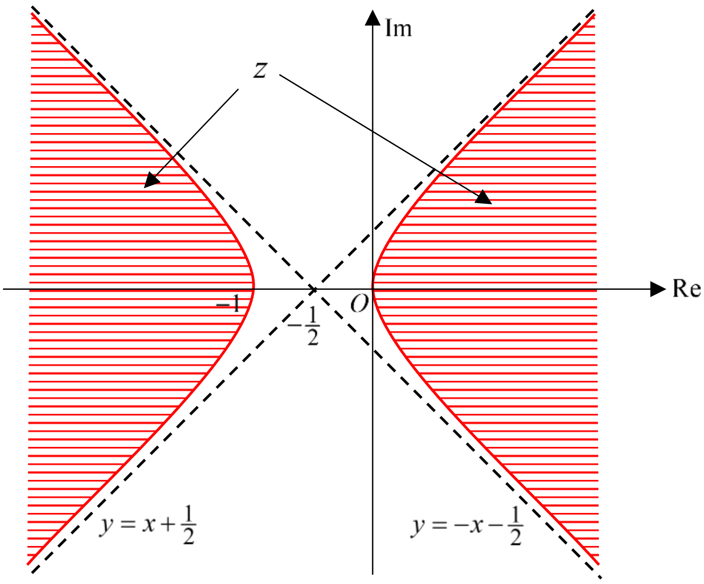
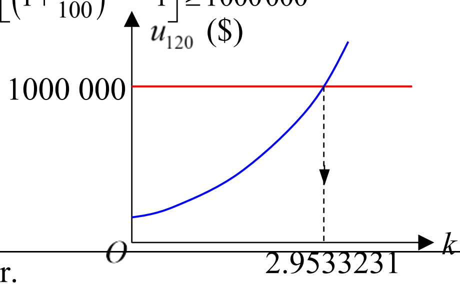
2022 Preliminary Examination H2 Further Mathematics Paper 2 (9649/02)
Solutions

s/n	Solution
1(i)	 Method 1: (R-formula) $\cos\left(x^2-\frac{\pi}{4}\right)=1-\sin\left(x^2-\frac{\pi}{4}\right)$ $\sin\left(x^2-\frac{\pi}{4}\right)+\cos\left(x^2-\frac{\pi}{4}\right)=1$ $\sqrt{2}\sin\left[\left(x^2-\frac{\pi}{4}\right)+\frac{\pi}{4}\right]=1$ $\sin(x^2)=\frac{1}{\sqrt{2}}$ $x^2=\frac{\pi}{4},\frac{3\pi}{4}$ since $0\le x^2\le\frac{\pi^2}{4}$ $\therefore\quad x=\frac{\sqrt{\pi}}{2},\frac{\sqrt{3\pi}}{2}$ since $x\ge 0$ Method 2: (sum and difference identities) $\cos\left(x^2-\frac{\pi}{4}\right)=1-\sin\left(x^2-\frac{\pi}{4}\right)$ $\cos(x^2)\cos\frac{\pi}{4}+\sin(x^2)\sin\frac{\pi}{4}=1-\sin(x^2)\cos\frac{\pi}{4}+\cos(x^2)\sin\frac{\pi}{4}$ $\cos(x^2)\frac{1}{\sqrt{2}}+\sin(x^2)\frac{1}{\sqrt{2}}=1-\sin(x^2)\frac{1}{\sqrt{2}}+\cos(x^2)\frac{1}{\sqrt{2}}$ $\sqrt{2}\sin(x^2)=1$ $\sin(x^2)=\frac{1}{\sqrt{2}}$ $x^2=\frac{\pi}{4},\frac{3\pi}{4}$ since $0\le x^2\le\frac{\pi^2}{4}$ $\therefore\quad x=\frac{\sqrt{\pi}}{2},\frac{\sqrt{3\pi}}{2}$ since $x\ge 0$
(ii)	Method 1: (standard form) $\int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} 2\pi x(y_1-y_2)\,\mathrm{d}x$ $=\int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} 2\pi x\left(\cos\left(x^2-\frac{\pi}{4}\right)-\left[1-\sin\left(x^2-\frac{\pi}{4}\right)\right]\right)\mathrm{d}x$ $=\pi\int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} 2x\cos\left(x^2-\frac{\pi}{4}\right)+2x\sin\left(x^2-\frac{\pi}{4}\right)-2x\,\mathrm{d}x$ $=\pi\left[\sin\left(x^2-\frac{\pi}{4}\right)-\cos\left(x^2-\frac{\pi}{4}\right)-x^2\right]_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}}$ $=\pi\left[\sin\frac{\pi}{2}-\cos\frac{\pi}{2}-\frac{3\pi}{4}\right]-\pi\left[\sin(0)-\cos(0)-\frac{\pi}{4}\right]$ $=\pi\left[0-(-1)-\frac{3\pi}{4}\right]-\pi\left[0-1-\frac{\pi}{4}\right]$ $=2\pi-\frac{\pi^2}{2}$ unit^3 Method 2: (R-formula)

s/n	Solution
	$\int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} 2\pi x(y_1 - y_2)\,\mathrm{d}x$ $= \int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} 2\pi x\left(\cos\left(x^2 - \frac{\pi}{4}\right) - \left[1 - \sin\left(\frac{\pi}{4} - x^2\right)\right]\right)\mathrm{d}x$ $= \int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} 2\pi x\left(\cos\left(x^2 - \frac{\pi}{4}\right) + \sin\left(x^2 - \frac{\pi}{4}\right) - 1\right)\mathrm{d}x$ $= \int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} 2\pi x\left(\sqrt{2}\sin(x^2) - 1\right)\mathrm{d}x$ $= \int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} 2\sqrt{2}\pi x\sin(x^2)\,\mathrm{d}x - \int_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} 2\pi x\,\mathrm{d}x$ $= \left[-\sqrt{2}\pi\cos(x^2)\right]_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} - \left[\pi x^2\right]_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}}$ $= -\sqrt{2}\pi\left[\cos\frac{3\pi}{4} - \cos\frac{\pi}{4}\right] - \pi\left[\frac{3\pi}{4} - \frac{\pi}{4}\right]$ $= 2\pi - \frac{\pi^2}{2}$ unit^3
2(a)	True. Let $p(\lambda) = 0$ be the characteristic equation of A. When $\lambda = 0$, $p(0) = \det\left(\mathrm{A} - \mathrm{I}(0)\right) = \det(\mathrm{A}) = 0$ Since $\det(\mathrm{A}) = 0$, A is not invertible.
(b)	True. Let I be the identity matrix. Then $\mathrm{A} = \mathrm{I}^{-1}\mathrm{A}\mathrm{I} \Rightarrow$ A is diagonalisable
(c)	False. <u>Method 1</u>: Let $\mathrm{A} = \begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & -1\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 1\end{pmatrix}$ $\therefore$ A is diagonalisable, but A is not diagonal. <u>Method 2</u>: Let $\mathrm{A} = \begin{pmatrix}1 & -1\\0 & 2\end{pmatrix}$ Then $\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & -1\\0 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$ $\therefore$ A is diagonalisable, but A is not diagonal. <u>Method 3</u>: Let $\mathrm{A} = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$ $\therefore$ characteristics equation is $\begin{vmatrix}\lambda - 1 & -1\\0 & \lambda\end{vmatrix} = 0$ $\lambda(\lambda - 1) = 0$

s/n	Solution
	$\therefore\ \lambda = 0$ or $\lambda = 1$ When $\lambda = 0$, $\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix} = k_1\begin{pmatrix}1\\ -1\end{pmatrix}$ When $\lambda = 1$, $\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix} = k_2\begin{pmatrix}1\\ 0\end{pmatrix}$ Then $\mathrm{A} = \begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}^{-1}$ is diagonalisable, but A is not diagonal.
(d)	False. Consider the identity matrix I. Its eigenvalue 1 is repeated, but I is diagonalisable since $\mathrm{I} = \mathrm{I}^{-1}(\mathrm{I})\mathrm{I}$.
3(i)	Let $z = x + y\mathrm{i}$ $\mathrm{Re}(z + z^2) \geq 0 \;\Rightarrow\; \mathrm{Re}(x + y\mathrm{i} + (x + y\mathrm{i})^2) \geq 0$ $\mathrm{Re}(x + y\mathrm{i} + x^2 + 2xy\mathrm{i} - y^2) \geq 0$ $x^2 + x - y^2 \geq 0$ $\therefore\ \left(x + \frac{1}{2}\right)^2 - y^2 \geq \frac{1}{4} \Rightarrow \frac{\left(x+\frac{1}{2}\right)^2}{\frac{1}{4}} - \frac{y^2}{\frac{1}{4}} \geq 1$ 
(ii)	$\left\lvert z + \frac{1+\sqrt{2}}{2}\right\rvert - \left\lvert z + \frac{1-\sqrt{2}}{2}\right\rvert = \left\lvert z - \frac{-1-\sqrt{2}}{2}\right\rvert - \left\lvert z - \frac{-1+\sqrt{2}}{2}\right\rvert$ Using relationship $c^2 = a^2 + b^2$ for hyperbola, $\therefore\ c^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \Rightarrow c = \frac{\sqrt{2}}{2}$ Hence foci of hyperbola are at $\left(-\frac{1}{2} - \frac{\sqrt{2}}{2}, 0\right)$ and $\left(-\frac{1}{2} + \frac{\sqrt{2}}{2}, 0\right)$ $\therefore\ \left\lvert z - \frac{-1-\sqrt{2}}{2}\right\rvert - \left\lvert z - \frac{-1+\sqrt{2}}{2}\right\rvert = \pm 2a = \pm 2(\frac{1}{2}) = \pm 1$
4(i)	

s/n	Solution
	Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ $\therefore\ AB = \left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \mid b_{12}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right)$ Hence $T = \left\{ p\left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) + q\left(b_{12}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) \mid p, q \in \mathbb{R} \right\}$ $\subseteq \left\{ \lambda\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + \mu\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \mid \lambda, \mu \in \mathbb{R} \right\}$ $= \text{colsp(A)}$ Since p and q can both be 0, the zero column belongs to T. Hence $T \neq \varnothing$. Let $p_1\left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) + q_1\left(b_{12}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right)$, $p_2\left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) + q_2\left(b_{12}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right)$ be two vectors from T. Since $p_1\left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) + q_1\left(b_{12}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right)$ $+ p_2\left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) + q_2\left(b_{12}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right)$ $= (p_1 + p_2)\left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) + (q_1 + q_2)\left(b_{12}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right)$ T is closed under vector addition Let k be a constant. Since $k\left[p\left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) + q\left(b_{12}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) \right]$ $= kp\left(b_{11}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right) + kq\left(b_{12}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \right)$, T is closed under scalar multiplication. $\therefore\ T$ is a subspace of colsp(A). (shown) Alternative Solution (To show that T is a subset of colsp(A)) If $\mathbf{v} \in T$ then there exist $\mathbf{u} \in \mathbb{R}^2$ such that $\mathbf{v} = (AB)\mathbf{u} = A(B\mathbf{u}) \in \text{colsp}(A)$ Therefore $T \subseteq \text{colsp}(A)$
(ii)	From (ii), colsp(AB) is a subspace of colsp(A).

s/n	Solution
	$\therefore$ by part (i), $\dim(\text{colsp}(\text{AB})) \le \dim(\text{colsp}(\text{A}))$. Hence $\text{rank}(\text{AB}) = \dim(\text{colsp}(\text{AB})) \le \dim(\text{colsp}(\text{A})) = \text{rank}(\text{A})$ (shown)
(iii)	If B is invertible, then $\text{rank}(\text{A}) = \text{rank}(\text{ABB}^{-1}) \le \text{rank}(\text{AB}) \le \text{rank}(\text{A})$ Hence $\text{rank}(\text{AB}) = \text{rank}(\text{A})$ (shown)
5(i)	$u_1 = \left(1+\frac{k}{100}\right)900 = 900 + 9k$ $u_{n+1} = \left(1+\frac{k}{100}\right)(u_n + 900)$ $= \left(1+\frac{k}{100}\right)u_n + 900 + 9k$
(ii)	$u_n = \left(1+\frac{k}{100}\right)u_{n-1} + (900+9k)$ $= \left(1+\frac{k}{100}\right)\left[\left(1+\frac{k}{100}\right)u_{n-2} + (900+9k)\right] + (900+9k)$ $= \left(1+\frac{k}{100}\right)\left[\left(1+\frac{k}{100}\right)\left[\left(1+\frac{k}{100}\right)u_{n-3} + (900+9k)\right] + (900+9k)\right] + (900+9k)$ $= \left(1+\frac{k}{100}\right)^3 u_{n-3} + \left(1+\frac{k}{100}\right)^2(900+9k) + \left(1+\frac{k}{100}\right)(900+9k) + (900+9k)$ $= \qquad \vdots$ $= \left(1+\frac{k}{100}\right)^{n-1} u_{n-(n-1)} + (900+9k)\left[\frac{\left(1+\frac{k}{100}\right)^{n-1} - 1}{\left(1+\frac{k}{100}\right) - 1}\right]$ $= \left(1+\frac{k}{100}\right)^{n-1} u_1 + \frac{100(900+9k)}{k}\left[\left(1+\frac{k}{100}\right)^{n-1} - 1\right]$ $= \left(1+\frac{k}{100}\right)^{n-1}(900+9k) + \frac{100(900+9k)}{k}\left[\left(1+\frac{k}{100}\right)^{n-1} - 1\right]$
(iii)	10 years $\Rightarrow n = 10(12) = 120$ $\therefore \left(1+\frac{k}{100}\right)^{119}(900+9k) + \frac{100(900+9k)}{k}\left[\left(1+\frac{k}{100}\right)^{119} - 1\right] \ge 1000000$ Using GC graph, $k \ge 2.9533231$ $\therefore$ minimum value of k is 2.96. (3 s.f.) 
(iv)	Adnan should not take up the offer. An interest of 2.96% per month is already very attractive, which makes the monthly 2.96% compounded interest too good to be true. To invest an amount of $\$1000 \times 12 \times 10 = \$120000$ and turn it into over a million dollars in 10 years is rather incredulous. This is likely a scam.
6(a)	$\frac{dy}{dx} = p \Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}$ $\therefore \frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = 1$ $\frac{dp}{dx} - \frac{1}{x}p = 1$ ...(1) Integrating factor: $e^{\int -\frac{1}{x}dx} = \frac{1}{x}$

s/n	Solution
	Multiply (1) throughout by $\frac{1}{x}$: $\frac{1}{x}\frac{\mathrm{d}p}{\mathrm{d}x}-\frac{1}{x^2}p=\frac{1}{x}$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{x}\right)=\frac{1}{x}$ ...(2) Integrating (2) w.r.t. x : $\frac{p}{x}=\ln x+C$ (since $x>0$) $p=x\ln x+Cx$ $\frac{\mathrm{d}y}{\mathrm{d}x}=x\ln x+Cx$ ...(3) Integrating (3) w.r.t. x : $y=\int x\ln x+Cx\ \mathrm{d}x$ $=\frac{x^2}{2}\ln x-\int\frac{x}{2}\ \mathrm{d}x+\frac{Cx^2}{2}$ $=\frac{x^2}{2}\ln x-\frac{x^2}{4}+\frac{Cx^2}{2}+E$ $=\frac{x^2}{2}\ln x+Dx^2+E$
(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=p \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}p}{\mathrm{d}x}=\left(\frac{\mathrm{d}p}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}p}{\mathrm{d}y}p$ $\therefore \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 \Rightarrow p\frac{\mathrm{d}p}{\mathrm{d}y}=\mathrm{e}^y p^3$ $\Rightarrow \frac{1}{p^2}\frac{\mathrm{d}p}{\mathrm{d}y}=\mathrm{e}^y$ (shown) …(*) Integrating (*): $\int\frac{1}{p^2}\ \mathrm{d}p=\int \mathrm{e}^y\ \mathrm{d}y$ $-\frac{1}{p}=\mathrm{e}^y+C$ $p=-\frac{1}{\mathrm{e}^y+C}$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\mathrm{e}^y+C}$ When $x=0$, $y=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$. $\therefore -1=-\frac{1}{\mathrm{e}^0+C} \Rightarrow C=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\mathrm{e}^y}$

s/n	Solution
	$\int e^y \, dy = -\int 1 \, dx$ $e^y = -x + D$ $y = \ln\lvert -x + D \rvert$ When $x = 0$, $y = 0$. $\therefore\ 0 = \ln\lvert 0 + D \rvert \Rightarrow D = 1$ $\therefore\ y = \ln\lvert 1 - x \rvert$ Hence $y = \ln(1 - x)$ (since $0 \le x < 1$)

7(a)

Let m_d be the difference between the rating before the HBL and after the HBL

$H_0 : m_d = 0$

$H_1 : m_d > 0$

Student	A	B	C	D	E	F	G
Before HBL	4	8	7	7	1	9	9
After HBL	6	9	3	4	a	1	2
Before – After	–2	–1	4	3	$1-a$	8	7
$\lvert$Before – After$\rvert$	2	1	4	3	$a-1$	8	7
Rank	2	1	4	3	λ	α	β
Sign	–	–	+	+	–	+	+

Since E has different rating before and after HBL, $\therefore\ a \ne 1$.

Since A, B, C, D have ranks 1 to 4 and no tied ranks among students,

$\therefore\ a \ne 2, 3, 4, 5, 8, 9$.

Hence $a = 6, 7, 10$

Thus $\lambda > 0$

Using Wilcoxon matched-pair signed rank test,

$t_- =$ sum of negative ranks $= 3 + \lambda > 3$

$t_+ =$ sum of positive ranks $= \text{sum}_{\text{total}} - t_-$

$$= \tfrac{7}{2}(1+7) - (3+\lambda)$$
$$= 25 - \lambda$$

$n = 7$

Under H_0, test statistic: $T = \min(t_-, t_+) = t_- = 3 + \lambda > 3$

Level of significance: 5%

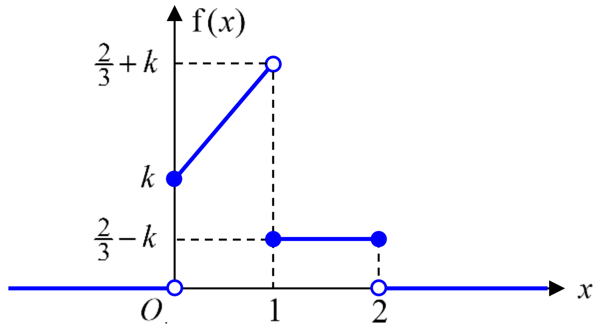
Reject H_0 if $T \le 3$

Since $t = 3 + \lambda > 3$, we do not reject H_0 and conclude that at the 5% level of significance, there is insufficient evidence that the median quality of learning experiences is lower after the implementation of HBL.

Assumption: The 7 students must be randomly selected from the student population.

(b)

For a paired-sample t-test, it is necessary to assume the differences between the paired ratings follow a normal distribution. Since the sample size of 7 students is very small, the differences between the paired ratings may not be normally distributed. Hence a Wilcoxon matched-pairs signed rank test is more appropriate in this context since it is not necessary to assume that the differences between the paired ratings follow a normal distribution. Further, having a larger sample size will improve the Wilcoxon matched-pair signed rank test.

s/n	Solution
8(i)	$\mathrm{f}(x)=\begin{cases}\frac{2}{3}x+k\ , & 0\le x<1;\\ \frac{2}{3}-k\ , & 1\le x\le 2;\\ 0\ , & \text{otherwise}.\end{cases}$  $\frac{2}{3}+k\ge 0$ and $k\ge 0$ and $\frac{2}{3}-k\ge 0$ $k\ge -\frac{2}{3}$ and $k\ge 0$ and $k\le \frac{2}{3}$ Hence $0\le k\le \frac{2}{3}$
(ii) (a)	$\mathrm{F}(1)=\frac{1}{3}+k>\frac{1}{4}$ $\therefore$ 1st quartile will lie in the interval $0\le x<1$. Let q be 1st quartile. For $0\le x<1$, $\frac{1}{3}q^2+kq=\frac{1}{4}$ $4q^2+12kq-3=0$ $q=\frac{-12k\pm\sqrt{144k^2+48}}{8}=\frac{-3k\pm\sqrt{9k^2+3}}{2}$ $\therefore\ q=\frac{-3k+\sqrt{9k^2+3}}{2}$ (reject the other root as $q>0$)
(b)	$\mathrm{P}(Y\le 4)=\mathrm{P}(11X-6X^2\le 4)$ $=\mathrm{P}(6X^2-11X+4\ge 0)$ $=\mathrm{P}\big((3X-4)(2X-1)\ge 0\big)$ $=\mathrm{P}\left(X\le\frac{1}{2}\text{ or }X\ge\frac{4}{3}\right)$ $=1-\left(\mathrm{F}\left(\frac{4}{3}\right)-\mathrm{F}\left(\frac{1}{2}\right)\right)$ $=1-\frac{1}{3}\left(\frac{5}{3}\right)-\frac{2}{3}k+\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{2}k$ $=\frac{19}{36}-\frac{k}{6}$

9(i)	Let X g be mass of a sachet of coffee mix.

Using mid-interval values of x (see table below) and GC,
An unbiased estimate of population mean, $\bar{x} = 19.725$
An unbiased estimate of population variance, $s^2 = 2.3453$

H_0 : $X \sim N(19.725,\ 2.3453)$
H_1 : X does not follow $N(19.725,\ 2.3453)$

x (g)	16-18	18-19	19-20	20-21	21-22	22-25
Mid-interval value of x (g)	17	18.5	19.5	20.5	21.5	23.5
Observed frequency O	9	25	27	16	19	4
Range of x, K	$X \le 18$	$18 < X \le 19$	$19 < X \le 20$	$20 < X \le 21$	$21 < X \le 22$	$X > 22$
$p = P(K)$	0.13000	0.18796	0.25329	0.22620	0.13385	0.06870
Expected frequency $E = 100p$	13.000	18.796	25.329	22.620	13.385	6.870

$\sum O = 100$, $\sum E = 100$

No. of classes $= 6$
No. of restrictions $= 3$ ($\sum E = 100$, estimated population mean & variance)
$\therefore$ degrees of freedom $\nu = 6 - 3 = 3$

Under H_0, using a χ^2 goodness of fit test,

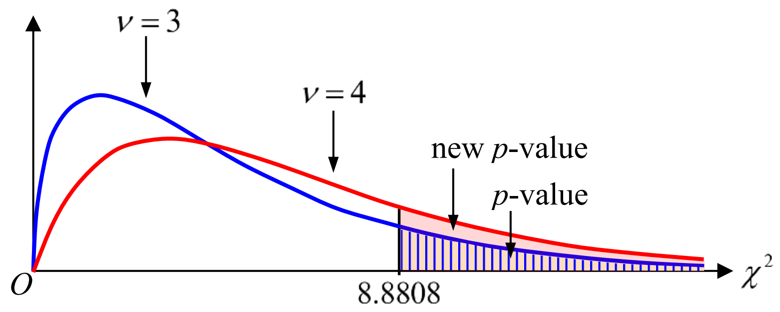
Test statistic $X^2 = \sum \frac{(O-E)^2}{E} \sim \chi^2(3)$
Level of significance: 5%
Reject H_0 if p-value < 0.05 [or if $X^2 > \chi^2_{0.05}(3) = 7.815$]
Assuming H_0 is true, using GC,

$X^2 = \sum \frac{(O-E)^2}{E} = 8.8808$
p-value $= 0.030918$
Since p-value $= 0.030918 < 0.05$ [or $X^2 = 8.8808 > 7.815$], we reject H_0 and conclude that at the 5% level of significance, there is sufficient evidence that the data are not consistent with $N(19.725,\ 2.3453)$.
Since p-value is 0.030918, we reject the claim that the data is consistent with a normal distribution at the 5% level, but not at the, say, 1% level. Hence there is sufficient, but not very strong, evidence that the normal distribution is not an appropriate model for the mass of sachets.

(ii)

If population mean mass of sachets is known, the number of restrictions would be reduced to two, namely $\sum E = 100$ and estimated population variance. Hence degrees of freedom is $\nu = 6 - 2 = 4$.

Since sample mean mass is equal to population mean mass, the test statistic X^2 remains the same at 8.8808.



From graph, the χ^2 curve with $\nu = 4$ has a lower maximum point compared to the χ^2 curve with $\nu = 3$. Since total area under each curve $=$ total probability $= 1$, the p-value with $\nu = 4$ must be larger than the p-value with $\nu = 3$ at the same test statistic $X^2 = 8.8808$.

10 (i)

Let X_A, X_B and X_C be no. of 4-page additional booklets requested by classes A, B and C respectively in 1st hour.

$X_A \sim \text{Po}(0.2)$, $X_B \sim \text{Po}(0.3)$, $X_C \sim \text{Po}(\lambda)$

$\therefore$ total no. of 4-page additional booklets required by all 3 classes in 1st hour is $X_A + X_B + X_C$, where

$X_A + X_B + X_C \sim \text{Po}(0.5 + \lambda)$

Given $\text{P}(X_A + X_B + X_C = 0) = \dfrac{1}{2}$

$$\therefore \quad e^{-(0.5+\lambda)} = \frac{1}{2}$$

$$-(0.5 + \lambda) = \ln\left(\frac{1}{2}\right)$$

$$0.5 + \lambda = \ln 2$$

$$\therefore \quad \lambda = \ln 2 - \frac{1}{2} \quad \text{where } k = 2 \quad \text{(shown)}$$

(ii)

P(at least 1 of 3 classes has no requests for booklets)

$= 1 - \text{P(none of 3 classes has no requests for booklets)}$

$= 1 - \text{P}(X_A \geq 1)\text{P}(X_B \geq 1)\text{P}(X_C \geq 1)$

$= 1 - \left[1 - \text{P}(X_A = 0)\right]\left[1 - \text{P}(X_B = 0)\right]\left[1 - \text{P}(X_C = 0)\right]$

$= 1 - (0.181269)(0.259182)(0.175639)$

$= 0.9917481665$

$= 0.992 \quad \text{(3 s.f.)}$

(iii)	$P(M>1)$ $= P(X_A>1)P(X_B>1)P(X_C>1)$ $= [1-P(X_A \le 1)][1-P(X_B \le 1)][1-P(X_C \le 1)]$ $= (0.0175230963)(0.0369363131)(0.0164164322)$ $= 0.0000106$ (3 s.f.)
(iv)	$P(\text{mean of 3 classes} = 1 \mid \text{mode of 3 classes} = 1)$ $= \dfrac{P(\text{mean of 3 classes} = 1 \cap \text{mode of 3 classes} = 1)}{P(\text{mode of 3 classes} = 1)}$ $= \dfrac{P(X_A=1, X_B=1, X_C=1)}{\begin{bmatrix} P(X_A=1, X_B=1, X_C=1) + P(X_A=1, X_B=1, X_C \ne 1) \\ +P(X_A=1, X_B \ne 1, X_C=1) + P(X_A \ne 1, X_B=1, X_C=1) \end{bmatrix}}$ $= \dfrac{P(X_A=1, X_B=1, X_C=1)}{\begin{bmatrix} P(X_A=1, X_B=1) + P(X_A=1, X_B \ne 1, X_C=1) \\ +P(X_A \ne 1, X_B=1, X_C=1) \end{bmatrix}}$ $= \dfrac{(0.1637461506)(0.2222454662)(0.1592229325)}{\begin{bmatrix} (0.1637461506)(0.2222454662) \\ +(0.1637461506)(0.7777545338)(0.1592229325) \\ +(0.8362538494)(0.2222454662)(0.1592229325) \end{bmatrix}}$ $= \dfrac{0.0057944154}{0.0862617259}$ $= 0.0671724956$ $= 0.0672$ (3 s.f.)
(v)	From (i), $X_A + X_B + X_C \sim \text{Po}(\ln 2)$ Using GC, <table><tr><td>x</td><td>$P(X_A+X_B+X_C=x)$</td></tr><tr><td>0</td><td>0.5</td></tr><tr><td>1</td><td>0.34657</td></tr><tr><td>2</td><td>0.12011</td></tr></table>$\therefore\ m=0$ Possible comments: • The number of 4-page additional booklets requested by the 3 classes in the 1st and 2nd hour may not be independent of each other. Hence the use of Poisson distribution with mean value $\ln 2$ for the 2nd hour may be inappropriate. • $m=0$ in the 2nd hour may not be realistic, and would likely be more than 0. By the 2nd hour, candidates would have attempted more questions and more likely to run out of writing space.
11(i)	A 95% confidence interval means that in the long run, if samples of new-born first babies are taken repeatedly and the interval (a,b) for each sample calculated, then for 95% of these samples taken, the calculated intervals (a,b) will contain the population mean mass of new-born first babies.

(ii)	Let X kg be mass of a new-born first baby. Let S_X^2 be unbiased estimator of population variance of new-born first babies. Assumption: The masses of new-born first babies follow a normal distribution. Since 95% confidence interval is $(3.056, 3.384)$, $\therefore\ \bar{x} = \frac{3.056+3.384}{2} = 3.220$ Since $n = 20$ is small, using t-distribution, confidence interval is $\bar{x} \pm t_{0.025}(19)\frac{s_x}{\sqrt{20}} = 3.220 \pm 2.093024022\frac{s_x}{\sqrt{20}} = (3.056, 3.384)$ Using $3.220 + 2.093024022\frac{s_x}{\sqrt{20}} = 3.384$, $2.093024022\frac{s_x}{\sqrt{20}} = 0.164$ $\therefore\ s_x = \sqrt{20}\,\frac{0.164}{2.093024022} = 0.350416569$ Hence required unbiased estimate of population variance, $s_x^2 = 0.1227917718 = 0.123$ (3 s.f.)
(iii)	Let Y kg be mass of a new-born non-first baby. Let S_Y^2 be unbiased estimator of population variance of new-born non-first babies. Let μ_X and μ_Y be population mean masses of new-born first and non-first babies respectively. Assumptions: 1. The masses of new-born non-first babies also follow a normal distribution. 2. The masses of new-born first and non-first babies are independent of each other. $H_0: \mu_X = \mu_Y$ $H_1: \mu_X < \mu_Y$ $\bar{y} = \frac{66.865}{20} = 3.34325 \quad s_y^2 = \frac{1}{20-1}\left[224.161 - \frac{66.865^2}{20}\right] = 0.0323467763$ Under H_0, using two-sample z-test, test statistic $Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_x^2}{20} + \frac{S_y^2}{20}}} \sim N(0,1)$ Level of Significance: 10% Reject H_0 if p-value ≤ 0.10 Assuming H_0 is true, using GC, z-value $= -1.399400628$, p-value $= 0.0808464914$ Since $p\text{-value} = 0.0808 < 0.10$, we reject H_0 and conclude that at the 10% level of significance, there is sufficient evidence that new-born first babies have population mean mass lower than new-born non-first babies.
(iv)	By taking a random sample of paired data consisting of new-born first and non-first babies from the same mother, a paired-sample t-test could be carried out. Such a paired-sample t-test is better as pairing eliminates the variability between different mothers.