

2022 HCI Prelim Paper 2_Modified [100 marks]

Section A: Pure Mathematics [50 marks]

- 1** Two curves C_1 and C_2 are given by the equations

$$C_1 : y = \cos\left(x^2 - \frac{\pi}{4}\right),$$

$$C_2 : y = 1 - \sin\left(x^2 - \frac{\pi}{4}\right),$$

for $0 \leq x \leq \frac{\pi}{2}$.

- (i) Find the exact x -coordinates x_1 and x_2 of the points of intersection of C_1 and C_2 . [3]

The region bounded by C_1 and C_2 between x_1 and x_2 is given by R .

- (ii) Find the exact volume when R is rotated by 2π radians about the y -axis. [3]

- 2** In this question, all matrices $\mathbf{A}$ are square matrices whose elements are real numbers. For each of the following statements, provide a proof if it is true or a counter example if it is false.

- (a) Every $\mathbf{A}$ that has zero as one of its eigenvalues is not invertible. [2]

- (b) Every $\mathbf{A}$ that is a diagonal matrix is diagonalisable. [2]

- (c) Every $\mathbf{A}$ that is diagonalisable is also a diagonal matrix. [2]

- (d) Every $\mathbf{A}$ that is diagonalisable has distinct eigenvalues. [2]

- 3** (i) Illustrate in an Argand diagram, the set of points z for which $\operatorname{Re}(z + z^2) \geq 0$. [4]

- (ii) Hence, or otherwise, determine the values of $\left|z + \frac{1+\sqrt{2}}{2}\right| - \left|z + \frac{1-\sqrt{2}}{2}\right|$ for z satisfying $\operatorname{Re}(z + z^2) = 0$. [3]

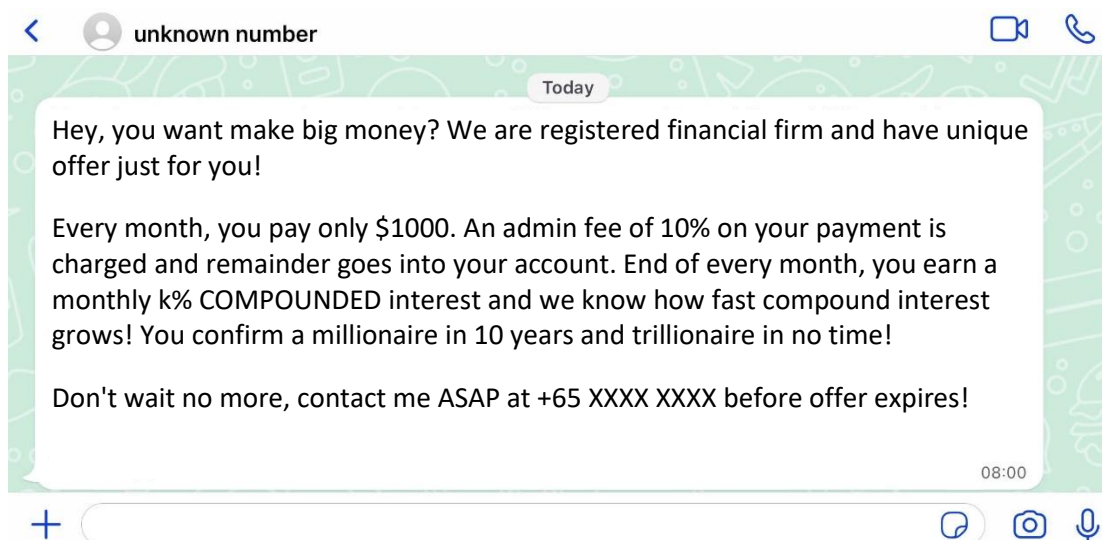
- 4** Let $\mathbf{A}$ and $\mathbf{B}$ be 2×2 matrices whose elements are real numbers.

- (i) Let T be the set of linear combinations of the columns of $\mathbf{AB}$. Show that T is a subset of the column space of $\mathbf{A}$. Hence show that T is a subspace. [4]

- (ii) Hence show that $\operatorname{rank}(\mathbf{AB}) \leq \operatorname{rank}(\mathbf{A})$. [2]

- (iii) Show that if $\mathbf{B}$ is invertible, then $\operatorname{rank}(\mathbf{AB}) = \operatorname{rank}(\mathbf{A})$. [2]

- 5 Adnan received the following message.



The viability of the offer is investigated. Assuming that the offer is taken up, let u_n be the amount of money in the account including interest at the end of the n^{th} month.

- (i) State the value of u_1 and write down a recurrence relation relating u_n and u_{n+1} for $n \geq 1$ in terms of k . [3]
- (ii) Find the general solution of the recurrence relation in part (i) in terms of k and n . [2]
- (iii) Suppose that the amount of money in the account is at least one million dollars after 10 years, find the minimum value of k . [2]
- (iv) Using the result from part (iii), advise Adnan if he should take up the offer. Give reason(s) to support your answer. [2]

- 6 By considering the substitution $\frac{dy}{dx} = p$,

- (a) solve the differential equation $\frac{d^2y}{dx^2} - \frac{1}{x} \frac{dy}{dx} = 1$, where $x > 0$. [5]

- (b) show that the differential equation $\frac{d^2y}{dx^2} = e^y \left(\frac{dy}{dx} \right)^3$ can be expressed as

$$\frac{1}{p^2} \frac{dp}{dy} = e^y.$$

It is given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = -1$.

Solve the differential equation, leaving your answer in the form $y = f(x)$, where $0 \leq x < 1$. [7]

Section B: Probability and Statistics [50 marks]

- 7 In a straw poll, 7 students were asked to rate the quality of their learning experiences on a scale of 1 to 10 (1 being the lowest and 10 being the highest) before and after the implementation of Home Based Learning (HBL). It is known that each student gave a different rating before and after the implementation of HBL. The results are shown in the table below, where a is a positive integer.

Student	A	B	C	D	E	F	G
Before HBL	4	8	7	7	1	9	9
After HBL	6	9	3	4	a	1	2

- (a) Given that there is no tied ranking among the students, carry out a Wilcoxon matched-pairs signed rank test, at the 5% significance level, to determine if the data supports the hypothesis that the quality of their learning experiences was lower after HBL was implemented. State an assumption for the Wilcoxon test to be valid in this context. [5]
- (b) Explain why Wilcoxon matched-pairs signed rank test is more appropriate than a paired-sample t -test in testing whether HBL lowers the quality of learning experiences. [1]
- 8 The continuous random variable X has cumulative distribution function F given by

$$F(x) = \begin{cases} 0, & x < 0; \\ \frac{1}{3}x^2 + kx, & 0 \leq x < 1; \\ \frac{1}{3}(2x-1) + k(2-x), & 1 \leq x \leq 2; \\ 1, & x > 2; \end{cases}$$

where k is a real constant.

- (i) Find the range of values that k can take. [3]
- (ii) Find in terms of k ,
- (a) the first quartile q where $P(X \leq q) = \frac{1}{4}$. [3]
- (b) the exact value of $P(Y \leq 4)$ given that $Y = 11X - 6X^2$. [3]

- 9 A beverage manufacturer produces instant coffee mix in sachets, and claims that the mass of sachets follows a normal distribution. A random sample of 100 sachets of the coffee mix is taken and the mass of each sachet measured. The data are summarised as follows.

Mass (g)	16-18	18-19	19-20	20-21	21-22	22-25
Frequency	9	25	27	16	19	4

- (i) Carry out a chi-squared goodness-of-fit test on the manufacturer's claim.
Discuss what the test indicates about the manufacturer's claim. You should refer to the level of significance and the p -value for your test. [9]
- (ii) It is now given that the population mean mass of sachets of coffee mix is known, and the mean mass obtained from the sample above is exactly the same as the population mean mass of sachets of coffee mix. With the help of a suitable diagram or otherwise, comment on the p -value thus obtained with reference to the p -value found in your test. [2]
- 10 The duration of a written examination for a particular subject is 3 hours. Candidates write their answers in a single 12-page answer booklet, and may request for additional 4-page answer booklets if they run out of writing space in the 12-page answer booklet.
In the first hour of one such examination, the number of additional 4-page answer booklets requested by candidates from classes A , B and C are assumed to follow independent Poisson distributions with means 0.2, 0.3 and λ respectively. The probability that no candidates from these 3 classes requested for additional 4-page answer booklets in the first hour is $\frac{1}{2}$.
- (i) Show that $\lambda = \ln k - \frac{1}{k}$, where k is a constant to be determined. [2]
- For the remaining parts of the question, use the value of λ found in part (i).
- (ii) Find the probability that at least one of the 3 classes has no candidates requesting for the additional 4-page answer booklets in the first hour of the examination. [2]
- (iii) Let M be the minimum number of additional 4-page answer booklets requested among the 3 classes in the first hour. Find $P(M > 1)$. [2]
- (iv) Find the probability that the mean number of additional 4-page answer booklets requested by the 3 classes in the first hour is 1, given that there are at least 2 classes with exactly 1 additional 4-page answer booklets requested by the 3 classes during the first hour. [3]
- (v) Let m denote the most likely total number of additional 4-page answer booklets requested by the 3 classes in the first hour of the examination. Find m .
Comment with justification if m is also the most likely total number of additional 4-page answer booklets requested by the 3 classes in the second hour of the examination. [3]

- 11** A medical student investigated the masses of new-born babies. A first baby is defined as the first child borne by a woman. A random sample of 20 new-born first babies was taken and their masses in kilograms (kg) measured. The 95% confidence interval of the average mass of new-born first babies constructed from these data is

(3.056, 3.384).

- (i) Explain the significance of a 95% confidence interval in this context. [1]
- (ii) By stating clearly an assumption for the method to be valid, find the value of an unbiased estimate of the population variance of masses of new-born first babies. [3]

The medical student wished to find out if the average mass of new-born first babies is lower than new-born babies who are not first babies (new-born non-first babies). She took another random sample consisting of new-born non-first babies and obtained the following data.

	Sample Size	Sum of masses (kg)	Sum of squares of masses (kg^2)
New-born non-first babies	20	66.865	224.161

- (iii) Carry out an appropriate test to determine if there is evidence, at the 10% level of significance, that the average mass of new-born first babies is indeed lower than new-born non-first babies. State two assumptions for the test to be valid. [6]
- (iv) A fellow medical student commented that, other than taking much larger sample sizes for both the samples mentioned above, a modification to the above test would have resulted in a better test. Explain briefly what the modified test procedure would have been, and why such a modification would have been better. [2]

End of Paper