Topical Revision Worksheet 2

Topic: Equations and Inequalities

Question 1 [2008 EOY, Q3]	marks
It is given that $\left(p, \frac{1}{2}\right)$, where $p > 0$, is a solution to the pair of simultaneous equations $px + 2y = 10$ and $\frac{p}{x} + \frac{2}{y} = 5$. (i) Show that $p = 3$. **Solution:** Sub $x = p$ and $y = \frac{1}{2}$ into $px + 2y = 10$ to get $p^2 + 1 = 10 \Rightarrow p = 3$ since $p > 0$	[2]
(ii) Find the other solution (x, y) to the above pair of simultaneous equations. **Solution:** $3x + 2y = 10$ so $y = \frac{10-3x}{2}$ hence $\frac{3}{x} + \frac{4}{10-3x} = 5 \Rightarrow 3(10-3x) + 4x = 5x(10-3x)$ $\Rightarrow 15x^2 - 55x + 30 = 0 \Rightarrow 3x^2 - 11x + 6 = 0 \Rightarrow (3x-2)(x-3) = 0$ Hence $x = \frac{2}{3}$ or 3 so sub $x = \frac{2}{3}$ to get $y = 4$, i.e. $x = \frac{2}{3}$, $y = 4$	[4]
Question 2 [2009 EOY, Q7]	marks
(a) Given $f(x) = x^2 + (p+1)x + p + 1$, where p is real. Find the value(s) of p for which the graph of $y = f(x)$ is a tangent to the x-axis. **Solution:** $D = (p+1)^2 - 4(1)(p+1) = 0$ $p^2 + 2p + 1 - 4p - 4 = 0$ $p^2 - 2p - 3 = 0$ $(p-3)(p+1) = 0$ $p = 3$ or -1	[4]
(b) Find the range of values of x for which $(2x+1)(4x-3) < 18$. **Solution:** $8x^2 - 6x + 4x - 3 - 18 < 0$ $8x^2 - 2x - 21 < 0$ $(4x-7)(2x+3) < 0$ $\quad -\frac{3}{2} < x < \frac{7}{4}$	[3]

Question 3 [2010 EOY, Q17 EITHER i, ii]	marks
(a) A power generator works in such a way that it takes 10 minutes to warm up before it is able to generate electricity. It is expected to generate 500 units of electricity every 20 minutes after it warms up. (i) Write down an equation in E and t, where E units represent the amount of electricity generated and t minutes (where $t \geq 10$) represents the time after which the generator is turned on. **Solution:** $E = \dfrac{500(t-10)}{20}$ $= 25(t-10)$	[1]
(ii) Show that it is expected to take at least 5.5 hours to generate at least 8000 units of electricity. **Solution:** For $E > 8000$, $25(t-10) > 8000 \Rightarrow t - 10 > \dfrac{8000}{25}$ $\Rightarrow t > 330$ So it takes at least 330 minutes or 5.5 hours	

**Question 4 [2011 EOY, Q10b]	marks
The roots of the equation $x^2 - 4x + 1 = 0$ are α and β. Without finding the value of α and of β, construct a quadratic equation whose roots are $2\alpha + \dfrac{1}{\alpha}$ and $2\beta + \dfrac{1}{\beta}$. **Solution:** $\alpha + \beta = 4$ M1 & $\alpha\beta = 1$ *Sum* $= \left(2\alpha + \dfrac{1}{\alpha}\right) + \left(2\beta + \dfrac{1}{\beta}\right)$ $= 2(\alpha+\beta) + \dfrac{(\alpha+\beta)}{\alpha\beta}$ $= 2(4) + \dfrac{(4)}{1} = 12$ *Product* $= \left(2\alpha + \dfrac{1}{\alpha}\right) \times \left(2\beta + \dfrac{1}{\beta}\right)$ $= 4\alpha\beta + \dfrac{2(\alpha^2+\beta^2)}{\alpha\beta} + \dfrac{1}{\alpha\beta}$ $= 4(1) + \dfrac{2(14)}{(1)} + \dfrac{1}{(1)} = 33$ Quadratic equation : $x^2 - 12x + 33 = 0$ (Only quadratic expression is written, (not an equation, i.e. no equal sign) answer mark cannot be given.)	[5]

Question 5 [2012 EOY, Q9]	marks
(a) Given that $k\left(x^2-3x+4\right)+x-3$ is always positive for all real values of x. Find the range of values of k. **Solution:** $(a)\ k\left(x^2-3x+4\right)+x-3>0$ $\Rightarrow k>0$ and $\Delta<0$ $\Delta=9k^2-6k+1-16k^2+12k<0$ $7k^2-6k-1<0$ $(7k+1)(k-1)>0$ $k>1$ or $k<-\dfrac{1}{7}$ $\therefore k>1$	[4]
(b) The roots of the quadratic equation $x^2+2x+1-k^2=0$ are α and β, such that $\alpha^2-\beta^2=2$. Find the possible values of k. **Solution: $x^2+2x+1-k^2=0$ $\alpha+\beta=-2$ and $\alpha\beta=1-k^2$ $\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)=2$ $\therefore \alpha-\beta=-1$ $\therefore \alpha=-1.5, \beta=-0.5$ $k^2=0.25$ $k=\pm 0.5$	[5]
Question 6 [2013 EOY, Q4]	marks
(a) Find the range of values of h, $h\neq 0$, for which the line $y=x-h$ meets the curve $y = hx^2 + 7x$. **Solution:** $x-h=hx^2+7x \quad \Rightarrow \quad hx^2+6x+h=0$ As the line meets the curve i.e. $b^2-4ac\geq 0 \quad \Rightarrow \quad 6^2-4h^2\geq 0$ $h^2\leq 9$ $-3\leq h\leq 3, h\neq 0$	[3]
(b) Find the range of values of k for which $y = x^2 + (k-4)x + 1$ is always positive for all real values of x. **Solution:** As y is always positive and $a>0$, $\Rightarrow$ $b^2-4ac<0$ where $a=1,\ b=k-4$ and $c=1$ $(k-4)^2-4<0$ $k^2-8k+12<0$ $(k-2)(k-6)<0$ $\Rightarrow \quad 2<k<6$	[3]

Question 7 [2013 EOY, Q11]	marks
Solve the equation $\left\lvert x^2-3 \right\rvert = 2x$ **Solution:** Given $\left\lvert x^2-3 \right\rvert = 2x \quad \Rightarrow x^2-3=2x$ or $x^2-3=-2x$ Case 1 : $x^2-3=2x$ $x^2-2x-3=0$ $(x-3)(x+1)=0$ $x=3$ or -1 Case 2 : $x^2-3=-2x$ $x^2+2x-3=0$ $(x+3)(x-1)=0$ $x=-3$ or 1 But $\left\lvert x^2-3 \right\rvert = 2x \geq 0$ Hence after verifying, the answer is: $x=3$ or $x=1$ **Alternative Solution** $(x^2-3)^2=(2x)^2$ $x^4-10x^2+9=0$ $(x^2-9)(x^2-1)=0$ $x=\pm3,\pm1$ $\therefore x=3,1$	[4]
Question 8 [2014 EOY, Q5]	marks
Find the range of values of *m* for which $y=mx^2-(m+1)x+m$ is always positive for all real values of *x*. **Solution:** $y=mx^2-(m+1)x+m$ is always positive $\Rightarrow m>0$ and $\Delta<0$ $\Delta=(m+1)^2-4m^2<0$ $m^2+2m+1-4m^2<0$ $3m^2-2m-1>0$ $(3m+1)(m-1)>0$ $m<-\frac{1}{3}$ or $m>1$ $\because m$ is positive, $\therefore m>1$	[4]

Question 9 [2014 EOY, Q6]	marks
It is given that the equation $x^2+2k+1=(k+2)x$ has two real roots α and β. (i) Find the range of values of k. **Solution:** $x^2+2k+1=(k+2)x$ $x^2-(k+2)x+(2k+1)=0$ $\Delta=(k+2)^2-4(2k+1)\geq 0$ $k^2-4k\geq 0$ $k(k-4)\geq 0$ $k\leq 0$ or $k\geq 4$	[3]
(ii) If $\alpha^2+\beta^2=11$, determine the value(s) of k. **Solution: $\left.\begin{aligned}\alpha+\beta&=k+2\\ \alpha\beta&=2k+1\end{aligned}\right\}$ $\alpha^2+\beta^2=11$ $\Rightarrow(\alpha+\beta)^2-2\alpha\beta=11$ $(k+2)^2-2(2k+1)=11$ $k^2=9$ $k=\pm 3$ $\because \alpha,\beta$ are real, range of values for $k: k\leq 0$ or $k\geq 4$ $\therefore k=-3$	[4]
**Question 10 [2015 EOY, Q1]	marks
The roots of the quadratic equation $2x^2+1=5x$ are α and β. (i) Write down the values of $\alpha+\beta$ and $\alpha\beta$. **Solution:** $\alpha+\beta=\frac{5}{2}$ and $\alpha\beta=\frac{1}{2}$	[2]
(ii) Find an equation whose roots are given by $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$. **Solution:** $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ $=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=21$ $\qquad\frac{1}{\alpha^2}\times\frac{1}{\beta^2}=4$ Hence equation is $x^2-21x+4=0$	[3]

Question 11 [2015 EOY, Q2]	marks
(a) Explain why $2+4x-4x^2$ cannot be greater than 3 for all real values of *x*. **Solution:** $2-4x^2+4x=-4(x^2-x)+2$ $=-4\left(x-\frac{1}{2}\right)^2+3$ Since $\left(x-\frac{1}{2}\right)^2 \geq 0$, $-4\left(x-\frac{1}{2}\right)^2+3 \leq 3$, so $2-4x^2+4x$ cannot be greater than 3.	[3]
(b) Find the range of values of *p* such that $4px^2-4(p+2)x+6p+5$ is **always positive** for all real values of *x*. **Solution:** Need $4p>0$ and $[-4(p+2)]^2-4(4p)(6p+5)<0$ i.e. $p>0$ and $5p^2+p-4>0$ $\Rightarrow (5p-4)(p+1)>0$ $\Rightarrow p<-1$ or $p>\frac{4}{5}$ Hence $p>\frac{4}{5}$	[5]