2025 ASRJC H23 Math Preliminary Exam Solutions

<table>
<tr><td>1</td><td>Solution:</td><td></td></tr>
<tr><td></td><td>Consider the vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| \left|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right| \cos\theta$ where θ is the angle between vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$ax+by+cz = \sqrt{a^2+b^2+c^2}\sqrt{x^2+y^2+z^2}\cos\theta$

$ax+by+cz \;^2 = \; a^2+b^2+c^2 \quad x^2+y^2+z^2 \; \cos^2\theta$

Since $\cos^2\theta \le 1$,

Therefore $\; ax+by+cz \;^2 \le \; a^2+b^2+c^2 \quad x^2+y^2+z^2$</td><td>[M1]

[B1]</td></tr>
<tr><td></td><td>For equality to hold, $\cos^2\theta = 1 \Rightarrow \cos\theta = \pm 1 \Rightarrow \theta = 0$ or π.

Therefore a necessary and sufficient condition is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is parallel to $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.</td><td>[A1]</td></tr>
<tr><td></td><td>(i)
$\left(a^2(1)+b^2(1)+c^2(1)\right)^2 \le \left(a^4+b^4+c^4\right)\left(1^2+1^2+1^2\right)$

$\left(a^2+b^2+c^2\right)^2 \le 3\left[(a^2+b^2+c^2)^2 - 2\left((ab)^2+(bc)^2+(ca)^2\right)\right]$

$\Rightarrow \; 2\left(a^2+b^2+c^2\right)^2 \ge 6\left((ab)^2+(bc)^2+(ca)^2\right)$

$\Rightarrow \; \dfrac{\left(a^2+b^2+c^2\right)^2}{3} \ge (ab)^2+(bc)^2+(ca)^2$</td><td>[B1]

[M1]

[M1]</td></tr>
<tr><td></td><td>(ii) Let $x = p, y = 2q, z = r$ and $a = 8, b = 4, c = 1$.
LHS = $\; 8p+8q+r \;^2 = 243^2 = 59049$
RHS = $\; 8^2+4^2+1^2 \quad p^2+4q^2+r^2 \; = \; 81 \quad 729 \; = 59049$
Therefore these assigned values and variables satisfy
$ax+by+cz \;^2 = \; a^2+b^2+c^2 \quad x^2+y^2+z^2$

Therefore $\begin{pmatrix} p \\ 2q \\ r \end{pmatrix}$ is parallel to $\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$</td><td>[B1]

[M1]

[B1]</td></tr>
</table>

	ie $\begin{pmatrix} p \\ 2q \\ r \end{pmatrix} = k\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$ ie $p = 8k, 2q = 4k, r = k$ Substitute into $8p + 8q + r = 243$, $64k + 16k + k = 243$ $81k = 243$ $k = \frac{243}{81} = 3$ $\therefore \quad p = 24,\ q = 6,\ r = 3$	[M1] [A1]

2

Solution:

$f(r) > f(r+1), \quad r = 1, 2, 3, \ldots$

So for $r \in [k^n, k^{n+1}-1]$, the largest value $= f(k^n)$ and the smallest value $= f(k^{n+1})$ [M1]

No of terms in $[k^n, k^{n+1}-1]$ is $(k^{n+1}-1) - k^n + 1$

$$= k^{n+1} - k^n$$

$$= k^n(k-1) \quad \text{[B1]}$$

$$\therefore \; k^n(k-1) f(k^{n+1}) \le \sum_{r=k^n}^{k^{n+1}-1} f(r) \le k^n(k-1) f(k^n) \quad -(*) \quad \text{[AG1]}$$

(i) Simplify $(*)$

$$k^n(k-1)\frac{1}{k^{n+1}} \le \sum_{r=k^n}^{k^{n+1}-1} f(r) \le k^n(k-1)\cdot\frac{1}{k^n}$$

$$\frac{k-1}{k} \le \sum_{r=k^n}^{k^{n+1}-1} f(r) \le k-1 \quad \text{[M1]}$$

Let $k=2$, $$\frac{1}{2} \le \sum_{r=2^n}^{2^{n+1}-1} f(r) \le 1 \quad \text{[M1]}$$

$$\sum_{r=1}^{2^{N+1}-1} \frac{1}{r} = \sum_{n=0}^{N}\left(\sum_{r=2^n}^{2^{n+1}-1} f(r)\right)$$

$$\sum_{n=0}^{N}\left(\frac{1}{2}\right) \le \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \le \sum_{n=0}^{N}(1) \quad \text{[M1]}$$

$$\frac{1}{2}(N+1) \le \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \le N+1 \quad \text{[AG1]}$$

As $N \to \infty$, the lower bound $\left(\frac{N+1}{2}\right) \to \infty$.

Hence $\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge. [A1]

(ii) let $k=2$,

$$\sum_{r=2^n}^{2^{n+1}-1} \frac{1}{r^3} \leq 2^n \cdot \frac{1}{(2^n)^3} = \frac{1}{2^{2n}}$$ [M1]

$$\therefore \sum_{r=1}^{\infty} \frac{1}{r^3} \leq \sum_{n=0}^{\infty}\left(\sum_{r=2^n}^{2^{n+1}-1} \frac{1}{r^3}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{2^{2n}}\right)$$ [M1]

$$= \frac{1}{1-\frac{1}{2^2}}$$

$$= \frac{1}{\frac{3}{4}}$$

$$= \frac{4}{3}$$

$$= 1\tfrac{1}{3} \ \#$$ [AG1]

(iii) For 1 digit no, $\boxed{\ }$ 8

For 2 digit no, $\boxed{\ \ }$ 8 9

For 3 digit no, $\boxed{\ \ \ }$ 8 9 9 [M1]

$\therefore$ S(1000) has $8 + 8\cdot 9 + 8\cdot 9^2$

$= 8(1+9+9^2)$ [M1]

$= 8\left(\frac{1(1-9^3)}{1-9}\right)$

$= 9^3-1$ distinct numbers

For any n, there is a m digit no such that $n \leq m$

For m digit no, no of distinct numbers is $8\cdot 9^{m-1}$ [M1]

$$\sigma(n) \leq \sigma(m) = \sum_{r=1}^{10^{m+1}-1} \frac{1}{r} \leq 8(1) + 8\cdot 9\left(\frac{1}{10}\right) + 8\cdot 9^2\left(\frac{1}{10^2}\right) + \dots + 8\cdot 9^{m-1}\left(\frac{1}{10^{m-1}}\right)$$

(r contains no 2)

[M1]

$$= 8\left(1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \dots + \left(\frac{9}{10}\right)^{m-1}\right)$$

$$= 8\left(\frac{1-\left(\frac{9}{10}\right)^m}{1-\frac{9}{10}}\right)$$

$$= 80\left(1-\left(\frac{9}{10}\right)^m\right)$$ [M1]

≤ 80 for all m i.e. n as well.

3	Solution:	
	(i) For P_2, the train has 2 seats. $P_2 = P\ T_2$ sits in his assigned seat $= P\ T_1$ sits in his assigned seat $= \frac{1}{2}$ $P_3 = P\ T_3$ sits in his assigned seat $= P\ T_1$ sits in his assigned seat $+ P\ T_1$ sits in $S_2 \cap T_2$ sits in S_1 $= \frac{1}{3} + \left(\frac{1}{3} \times \frac{1}{2}\right)$ $= \frac{1}{2}$	[A1] [A1]
	(ii) If passenger T_1 sits in S_k, then $T_2, T_3, ..., T_{k-1}$ will sit in their assigned seats as they are all available for them. For T_k, since his assigned seat S_k has been taken, he would have to choose randomly from the remaining $n - k - 1 = n - k + 1$ seats. The situation is the same as a train having $n - k + 1$ seats and the first passenger choosing his seat randomly. So we have $P\ T_n$ sits in $S_n \mid T_1$ sits in $S_k = P_{n-k+1}$ $P_n = \sum_{k=1}^{n} P\ T_n$ sits in $S_n \cap T_1$ sits in S_k $= \sum_{k=1}^{n} P\ T_n$ sits in $S_n \mid T_1$ sits in $S_k \times P\ T_1$ sits in S_k $= \frac{1}{n} \sum_{k=1}^{n} P\ T_n$ sits in $S_n \mid T_1$ sits in S_k $= \frac{1}{n}\Big[P\ T_n$ sits in $S_n \mid T_1$ sits in $S_1 + \sum_{k=2}^{n-1} P\ T_n$ sits in $S_n \mid T_1$ sits in S_k $+ P\ (T_n$ sits in $S_n \mid T_1$ sits in $S_n) \Big]$ $= \frac{1}{n}\left[1 + \sum_{k=2}^{n-1} P_{n-k+1} + 0\right]$ Let $r = n - k + 1$, then when $k = 2, r = n - 1$ and when $k = n - 1, r = 2$. $P_n = \frac{1}{n}\left[1 + \sum_{k=2}^{n-1} P_{n-k+1}\right]$ $= \frac{1}{n}\left[1 + \sum_{r=2}^{n-1} P_r\right]$	[A2] [B1] [B1] [M1] [AG1]

	(iii) $P_4 = \frac{1}{4}\left[1+\sum_{r=2}^{3} P_r\right] = \frac{1}{4}\ 1+P_2+P_3\ = \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\right] = \frac{1}{2}$ Conjecture: $P_n = \frac{1}{2}$ We have already proven $P_1 = \frac{1}{2}$ ie the result is true for $n=1$. Assuming the result is true for $n = 2,.....,k$. ie $P_1 = P_2 = ... = P_k = \frac{1}{2}$.	[A1] [B1] [B1]
	$P_{k+1} = \frac{1}{k+1}\left[1+\sum_{r=2}^{k} P_r\right]$ $= \frac{1}{k+1}\left[1+\sum_{r=2}^{k}\left(\frac{1}{2}\right)\right]$ $= \frac{1}{k+1}\left[1+\frac{1}{2}\ k-1\ \right]$ $= \frac{1}{k+1}\left[\frac{k+1}{2}\right]$ $= \frac{1}{2}$ Hence by Mathematical Induction, $P_n = \frac{1}{2}$ is true for all positive integer n.	[M1] [A1]
4	Solution	
	(a)Since $n-1 = mp-1 = \ m-1\ p+\ p-1$ $x^{n-1} \equiv 1 \mod p$ $\Leftrightarrow x^{\ m-1\ p+(p-1)} \equiv 1(\mathrm{mod}\ p)$ $\Leftrightarrow x^{\ m-1\ p} x^{p-1} \equiv 1(\mathrm{mod}\ p)$ $\Leftrightarrow x^{\ m-1\ p} \equiv 1(\mathrm{mod}\ p)$ since $x^{p-1} \equiv 1(\mathrm{mod}\ p)$ by Fermat's Little Theorem $\Leftrightarrow\ x^{p\ \ m-1} \equiv 1(\mathrm{mod}\ p)$ $\Leftrightarrow\ x^{m-1} \equiv 1 \mod p$ since $x^{p-1} \equiv 1(\mathrm{mod}\ p) \Rightarrow x^p \equiv x(\mathrm{mod}\ p)$	[B1] [B1] [B1]
	(b) $n = p_1 p_2 .....p_k$ $\Rightarrow$ If $x^{n-1} \equiv 1 \mod n$ then $x^{n-1} = 1+sn$ for some integer n	

	$x^{n-1} = 1 + s\ \ p_1 p_2 \ldots p_n$	
	$x^{n-1} \equiv 1 \mod p_i$ for every p_i	[B1]
	$x^{\left(\frac{n}{p_i}\right)p_i - 1} \equiv 1 \mod p_i$	
	From (a), we have $x^{\left(\frac{n}{p_i}\right)-1} \equiv 1 \mod p_i$	[B1]
	$\Leftarrow\ x^{\left(\frac{n}{p_i}\right)-1} \equiv 1 \mod p_i$ for every p_i	
	From part (a), $x^{\left(\frac{n}{p_i}\right)p_i - 1} \equiv 1 \mod p_i$	[B1]
	$x^{n-1} \equiv 1 \mod p_i$ for every p_i	[B1]
	By Chinese Remainder Theorem,	
	$x^{n-1} \equiv 1 \mod n$	
	(c) Take any integer x with $\gcd(x,n)=1$. Then for each $p_i \mid n$, Fermat's	
	Little theorem gives $x^{p_i - 1} \equiv 1 \mod p_i$.	[B1]
	Since $p-1 \ \mid\ n-1$, we have $n-1 = k\ \ p-1$ for some integer *k*.	
	Then $x^{n-1} =\ x^{p_i-1}\ ^{k} \equiv 1^k \mod p_i \equiv 1 \mod p_i$.	[B1]
	This holds for every prime divisor p_i of n.	
	By Chinese Remainder Theorem, $x^{n-1} \equiv 1 \mod n$.	[B1]
5	Solution:	
	(i) There are n ways to pick a square from the 1st row. Suppose the square is in the jth column. "Block off" the squares in the 1st row and the squares in the jth column. The remaining $(n-1)$ squares can be picked from the $(n-1)$ unblocked rows and the $(n-1)$ unblocked columns in S_{n-1} ways. Hence $S_n = n\, S_{n-1}$.	[A1]
	(ii) If the square in row i and column j is selected, then map the integer i to the integer j. Conversely, if the integer i is mapped to the integer j, then select the square in row i and column j. $S_n = n!$	[B1] [A1]

	(iii) This is equivalent to a permutation of the integers $\{1, 2, 3, \ldots, n\}$ except that no integer is mapped to itself, e.g. the integer 1 cannot be mapped to itself (i.e. the 1^{st} slot) but to the other $(n-1)$ slots. There are $(n-1)$ slots to choose from. Suppose 1 is mapped to the rth slot. There are $(n - 1)$ position that 1 can occupy. Case 1 The integer r is mapped to 1^{st} slot (i.e. 1 and r swop places) We just need to do a derangement of all the integers other than 1 and r, which can be done in D_{n-2} ways. Case 2 The integer r is not mapped to 1^{st} slot (i.e. 1 and r don't swop places). The rth slot is now occupied by "1". For the time being, put "r" in the 1^{st} slot. Now do a derangement of all the slots other than the rth slot. Since the "r" in the 1^{st} slot cannot occupy the 1^{st} slot as every integer must be mapped to another position, this ensures that "r" is forced out of the 1^{st} slot. There are D_{n-1} ways to do this derangement. Combining cases 1 and 2, we have $D_n = (n-1)(D_{n-1} + D_{n-2})$. $D_1 = 0$ and $D_2 = 1$.	[B1] [B1] [B1] [A1]
	(iv) Let S be the set of all permutations. $\lvert S \rvert = n!$ Let B_k be the set containing all permutations of n objects where the k^{th} object is in its original position (k^{th} position). Since $\lvert B_j \rvert = (n-1)!$ for $1 \le j \le n$,we can choose any of the n integers to be fixed, $\sum_{j=1}^{n} \lvert B_j \rvert = \binom{n}{1}(n-1)!$. We can do the same for for $j \neq k$ and $1 \le j,k \le n$ and similarly when three, four etc. objects are fixed. $D_n = \lvert S \rvert - \lvert B_1 \cup B_2 \cup \ldots \cup B_n \rvert$ $= \lvert S \rvert - \left(\begin{array}{l} \sum_{j=1}^{n} \lvert B_j \rvert - \sum_{j \neq k} \lvert B_j \cap B_k \rvert + \sum_{j \neq k \neq l} \lvert B_j \cap B_k \cap B_l \rvert \\ + \ldots + (-1)^{n+1} \lvert B_1 \cap B_2 \cap \ldots \cap B_n \rvert \end{array} \right)$	[M1] [M1] [M1]

	$= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - ... + (-1)^n (n-n)!$ $= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... + (-1)^n \frac{1}{n!}\right)$	[B1]
	(v) $D_6 = 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right) = 360 - 120 + 30 - 6 + 1 = 265$	[A2]
	Answer the following questions In this question you will investigate these generalisations of the product rule further. **(a)** Find an expression for $\mathrm{D}^3(\mathrm{fg})$, the third derivative of the product fg, showing your workings clearly.	[2]
	The general form of $\mathrm{D}^n(\mathrm{fg})$, the n^{th} derivative of the product fg is as follows: For $n \in \mathbb{N}$, $\mathrm{D}^n(\mathrm{fg}) = \sum_{i=0}^{n} \binom{n}{i} (\mathrm{D}^i\mathrm{f})(\mathrm{D}^{n-i}\mathrm{g})$ where $\mathrm{D}^0\mathrm{f} = \mathrm{f}$. **(b) (i)** Show that $\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}$ for positive integers k and i satisfying $k \geq i$. **(ii)** Prove the result for general form of $\mathrm{D}^n(\mathrm{fg})$, the n^{th} derivative of the product fg.	[2] [9]
	The cubic polynomial $\mathrm{p}(x)$ has three distinct roots r_1, r_2 and r_3. **(c) (i)** Find an expression for $\mathrm{D}(\mathrm{p}(x))$. **(ii)** Prove that the tangent to the curve $y = \mathrm{p}(x)$ at the midpoint of any two of the roots intersects the curve at the third root.	[2] [5]

Question	Answer	Marks
(a)	$\mathrm{D}^3(\mathrm{fg}) = \mathrm{D}\left(\mathrm{D}^2(\mathrm{fg})\right)$ $= \mathrm{D}\left(\mathrm{f\,D^2g} + 2\mathrm{Df\,Dg} + \mathrm{g\,D^2f}\right)$ $= \mathrm{f\,D^3g} + \mathrm{Df\,D^2g} + 2\mathrm{D^2f\,Dg} + 2\mathrm{Df\,D^2g} + \mathrm{g\,D^3f} + \mathrm{DgD^2f}$ $= \mathrm{f\,D^3g} + 3\mathrm{Df\,D^2g} + 3\mathrm{D^2f\,Dg} + \mathrm{g\,D^3f}$	 **M1** **A1**
(b)(i)	$\binom{k}{i} + \binom{k}{i-1} = \frac{k!}{(k-i)!i!} + \frac{k!}{(k-i+1)!(i-1)!}$ $= \frac{k![k-i+1+i]}{(k-i+1)!i!}$ $= \frac{k!(k+1)}{(k+1-i)!i!}$ $= \frac{(k+1)!}{((k+1)-i)!i!}$ $= \binom{k+1}{i}$	 **M1** **B1**
(ii)	Base case $n=1$ $\sum_{i=0}^{1}\binom{1}{i}\left(\mathrm{D}^i\mathrm{f}\right)\left(\mathrm{D}^{1-i}\mathrm{g}\right) = \binom{1}{0}\left(\mathrm{D}^0\mathrm{f}\right)\left(\mathrm{D}^{1-0}\mathrm{g}\right) + \binom{1}{1}\left(\mathrm{D}^1\mathrm{f}\right)\left(\mathrm{D}^{1-1}\mathrm{g}\right)$ $= \mathrm{f\,Dg} + \mathrm{g\,Df}$ $= \mathrm{D}(\mathrm{fg})$ The result is true for $n = 1$	 **B1**
	Assume the result is true for $n = k, k \in \mathbb{Z}^+$, ie $\mathrm{D}^k(\mathrm{fg}) = \sum_{i=0}^{k}\binom{k}{i}\left(\mathrm{D}^i\mathrm{f}\right)\left(\mathrm{D}^{k-i}\mathrm{g}\right)$	 **M1**

	For $n = k+1$, $D^{k+1}(fg) = D\left(D^k(fg)\right)$ $= D\left(\sum_{i=0}^{k}\binom{k}{i}\left(D^i f\right)\left(D^{k-i} g\right)\right)$ $= \sum_{i=0}^{k}\binom{k}{i}\left(D^i f\right)\left(D^{k-i+1} g\right) + \sum_{i=0}^{k}\binom{k}{i}\left(D^{i+1} f\right)\left(D^{k-i} g\right)$ $= \sum_{i=0}^{k}\binom{k}{i}\left(D^i f\right)\left(D^{k-i+1} g\right) + \sum_{i=1}^{k+1}\binom{k}{i-1}\left(D^i f\right)\left(D^{k-i+1} g\right)$ $= f\ D^{k+1} g + \sum_{i=1}^{k}\binom{k}{i}\left(D^i f\right)\left(D^{k-i+1} g\right)$ $+ \sum_{i=1}^{k}\binom{k}{i-1}\left(D^i f\right)\left(D^{k-i+1} g\right) + \binom{k}{k}\left(D^{k+1} f\right)\left(D^0 g\right)$ $= f\ D^{k+1} g + \sum_{i=1}^{k}\left(\binom{k}{i} + \binom{k}{i-1}\right)\left(D^i f\right)\left(D^{k-i+1} g\right) + g\, D^{k+1} f$ $= \binom{k+1}{0} D^0 f\ D^{k+1} g + \sum_{i=1}^{k}\binom{k+1}{i}\left(D^i f\right)\left(D^{k-i+1} g\right) + \binom{k+1}{k+1} D^{k+1} f\ D^{k-(k+1)+1} g$ $= \sum_{i=0}^{k+1}\binom{k+1}{i}\left(D^i f\right)\left(D^{k+1-i} g\right)$	**M1** **M1** **M1** **M1** **M1** **A1**
	So the result is true for $n = 1$ and if true for $n = k$ is also true for $n = k + 1$. Hence by induction, the result is proven true for $n \in \mathbb{N}$.	**A1**
(c) (i)	$p(x) = a(x - r_1)(x - r_2)(x - r_3)$ $(Dp)(x) = a\left((x - r_2)(x - r_3) + (x - r_1)(x - r_3) + (x - r_1)(x - r_2)\right)$	**M1** **A1**
(ii)	$p\left(\frac{r_1 + r_2}{2}\right) = a\left(\frac{r_2}{2} - \frac{r_1}{2}\right)\left(\frac{r_1}{2} - \frac{r_2}{2}\right)\left(\frac{r_1 + r_2}{2} - r_3\right)$ $= -\frac{a}{8}(r_2 - r_1)^2(r_1 + r_2 - 2r_3)$	**M1**
	Gradient = $(Dp)\left(\frac{r_1 + r_2}{2}\right)$ $= a\left(\left(\frac{r_1 + r_2}{2} - r_2\right)\left(\frac{r_1 + r_2}{2} - r_3\right) + \left(\frac{r_1 + r_2}{2} - r_1\right)\left(\frac{r_1 + r_2}{2} - r_3\right) + \left(\frac{r_1 + r_2}{2} - r_1\right)\left(\frac{r_1 + r_2}{2} - r_2\right)\right)$ $= a\left(\left(\frac{r_1}{2} - \frac{r_2}{2}\right)\left(\frac{r_1 + r_2 - 2r_3}{2}\right) + \left(\frac{r_2}{2} - \frac{r_1}{2}\right)\left(\frac{r_1 + r_2 - 2r_3}{2}\right) + \left(\frac{r_2}{2} - \frac{r_1}{2}\right)\left(\frac{r_1}{2} - \frac{r_2}{2}\right)\right)$ $= a\left(\frac{r_2}{2} - \frac{r_1}{2}\right)\left(\frac{r_1}{2} - \frac{r_2}{2}\right)$ $= -\frac{a}{4}(r_2 - r_1)^2$	**M1**

	Eq of tangent at $x=\frac{r_1+r_2}{2}$ is $y=-\frac{a}{4}\left(r_2-r_1\right)^2 x+c$. $-\frac{a}{8}\left(r_2-r_1\right)^2\left(r_1+r_2-2r_3\right)=-\frac{a}{4}\left(r_2-r_1\right)^2\left(\frac{r_1+r_2}{2}\right)+c$ $\Rightarrow c=\frac{a}{4}\left(r_2-r_1\right)^2\left(\frac{r_1+r_2}{2}\right)-\frac{a}{8}\left(r_2-r_1\right)^2\left(r_1+r_2-2r_3\right)$ $=\frac{a}{8}\left(r_2-r_1\right)^2\left(\left(r_1+r_2\right)-\left(r_1+r_2-2r_3\right)\right)$ $=\frac{a}{4}\left(r_2-r_1\right)^2 r_3$	**M1** **M1**
	Eq of tangent at $x=\frac{r_1+r_2}{2}$ is $y=-\frac{a}{4}\left(r_2-r_1\right)^2 x+\frac{a}{4}\left(r_2-r_1\right)^2 r_3$. When $y=0$, $x=r_3$	**A1**