1 Jane bought an electric car in January 2025. After purchase, the value of the car, $\$V$, diminishes over time and can be modelled by $V = 38000e^{-kt}$, where k is a positive constant and t is time measured in years. The value of the car is expected to be \$29000 after 3 years of driving. Jane intends to sell her car when its value drops to half of its original value.

Showing clear mathematical calculations, find the year that Jane is likely to sell her electric car. [5]

$$29000 = 38000e^{-3k}$$
$$e^{-3k} = \frac{29}{38}$$
$$-3k = \ln\frac{29}{38}$$
$$k = -\frac{1}{3}\ln\frac{29}{38} \approx 0.090096$$

$$19000 = 38000e^{-0.090096t}$$
$$e^{-0.090096t} = \frac{1}{2}$$
$$-0.090096t = \ln\frac{1}{2}$$
$$t \approx 7.69 \text{ years}$$

The year is likely to be 2032.

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2 **(i)** Given that the curve $y = ax^2 + 2bx - 1$ lies entirely below the x-axis, determine the conditions that must be applied to the constants a and b. [2]

$a < 0$ for maximum curve (not required)

$(2b)^2 - 4a(-1) < 0$ for no roots

$4b^2 + 4a < 0$

$b^2 + a < 0$ $\left(\text{or } a < -b^2\right)$

(ii) If a and b are both integers, state an example of the values of a and b which satisfy the conditions found in **(i)**. [2]

$a = -5$

$b = 1$

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3 **(a)** The variables x and y are defined such that $\log_2 x - 3\log_4 y = \log_9 3$.

(i) Give a reason why x and y must be positive numbers. [1]

x and y must be positive for
$\log_2 x$ and $\log_2 y$ to be defined / exist / calculable / computed.

(ii) Express x in terms of y. [5]

$$\log_2 x - 3\left(\frac{\log_2 y}{\log_2 4}\right) = \frac{1}{2}$$

$$\log_2 x - \frac{3\log_2 y}{2} = \frac{1}{2}$$
$$2\log_2 x - 3\log_2 y = 1$$
$$\log_2 x^2 - \log_2 y^3 = 1$$
$$\log_2 \frac{x^2}{y^3} = 1$$
$$\frac{x^2}{y^3} = 2$$
$$x^2 = 2y^3$$
$$x = \sqrt{2y^3}$$

(b) Solve the equation $5^{x+1} + 2\left(5^{-x}\right) = 11$, leaving non-exact value(s) of x in the form $\log_a b$. [4]

$$5\left(5^x\right) + 2\left(\frac{1}{5^x}\right) = 11$$

Let $w = 5^x$

$$5w + \frac{2}{w} = 11$$
$$5w^2 - 11w + 2 = 0$$
$$(5w-1)(w-2) = 0$$
$$w = \frac{1}{5} \quad \text{or} \quad 2$$
$$5^x = 5^{-1} \quad \text{or} \quad 2$$
$$x = -1 \quad \text{or} \quad \log_5 2$$

A1

4 **(a)** **(i)** The polynomials $P(x) = ax^3 + x^2 + bx + 10$ and $Q(x) = x^3 + ax^2 - 5x + b$ leave the same remainder when divided by $x + 2$.

Show that $4a + b = 4$. [2]

$$a(-2)^3 + (-2)^2 + b(-2) + 10 = (-2)^3 + a(-2)^2 - 5(-2) + b$$
$$-8a - 2b + 14 = -8 + 4a + 10 + b$$
$$-12a - 3b = -12$$
$$4a + b = 4$$

(ii) If $P'(x)$ has a factor of $3x - 2$, find the values of a and b. [4]

$$P'(x) = 3ax^2 + 2x + b$$

$$P'\left(\frac{2}{3}\right) = 0$$
$$3a\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right) + b = 0$$
$$\frac{4}{3}a + b = -\frac{4}{3}$$

$$4a + 3b = -4$$
$4a + b = 4$ from (i)

$$2b = -8$$
$$\left.\begin{aligned} b &= -4 \\ a &= 2 \end{aligned}\right\}$$

(b) Solve the equation $2x^3 + 4x^2 - 5x - 7 = 0$, expressing non-exact solutions in the form $a \pm \frac{1}{2}\sqrt{b}$ where a and b are constants to be determined. [5]

$$P(x) = 2x^3 + 4x^2 - 5x - 7$$
$$P(-1) = 2(-1)^3 + 4(-1)^2 - 5(-1) - 7 = 0$$
$x + 1$ is a factor of $P(x)$

$$P(x) = (x+1)(2x^2 + Bx - 7)$$

Term in x^2 :

$$4x^2 = Bx^2 + 2x^2$$
$$B = 2$$

$$P(x) = (x+1)(2x^2 + 2x - 7)$$

For $2x^2+2x-7=0$,

$$x=\frac{-2\pm\sqrt{2^2-4(2)(-7)}}{2(2)}$$
$$=\frac{-2\pm\sqrt{60}}{4}$$
$$=-\frac{1}{2}\pm\frac{1}{2}\sqrt{15}$$

$$\therefore x=-1,\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{15}$$

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5 The depth of water, h metres, in a shallow port on a particular day is modelled by the formula $h=7+2\sin\left(\frac{\pi}{6}t\right)$, where t is the number of hours after midnight.

(a) State the period of h. [1]

$$h=\frac{2\pi}{\pi/6}=12\text{ hours}$$

(b) Use the model to predict the time when the depth of water is at its lowest. [2]

h is lowest when $\sin\left(\frac{\pi}{6}t\right)=-1$

$$\frac{\pi}{6}t=\frac{3\pi}{2}$$
$$t=9$$

Time is 9 am (or 0900 hrs)

A supply boat docks at the port at 5.30 am and it takes the workers 3 hours to unload its cargo immediately after it docks. To leave the port safely, the supply boat requires the depth of water to be at least 5.6 metres.

(c) Determine the earliest time, to the nearest minute, that the supply boat can leave the port. [4]

$$7+2\sin\left(\frac{\pi}{6}t\right)=5.6$$
$$\sin\left(\frac{\pi}{6}t\right)=-\frac{7}{10}$$

basic angle $=\sin^{-1}\left(\frac{7}{10}\right)\approx 0.77539$

$$\frac{\pi}{6}t=\pi+0.77539\ (\approx 3.9169),2\pi-0.77539\ (\approx 5.5077)$$
$$t=7.4808\ ,10.519$$

Two critical timings are 7.29 am (rejected since boat is still unloading) and 10.31am. Hence, earliest time boat can leave the port is 10.31am or 10.32am (or 24hr format).

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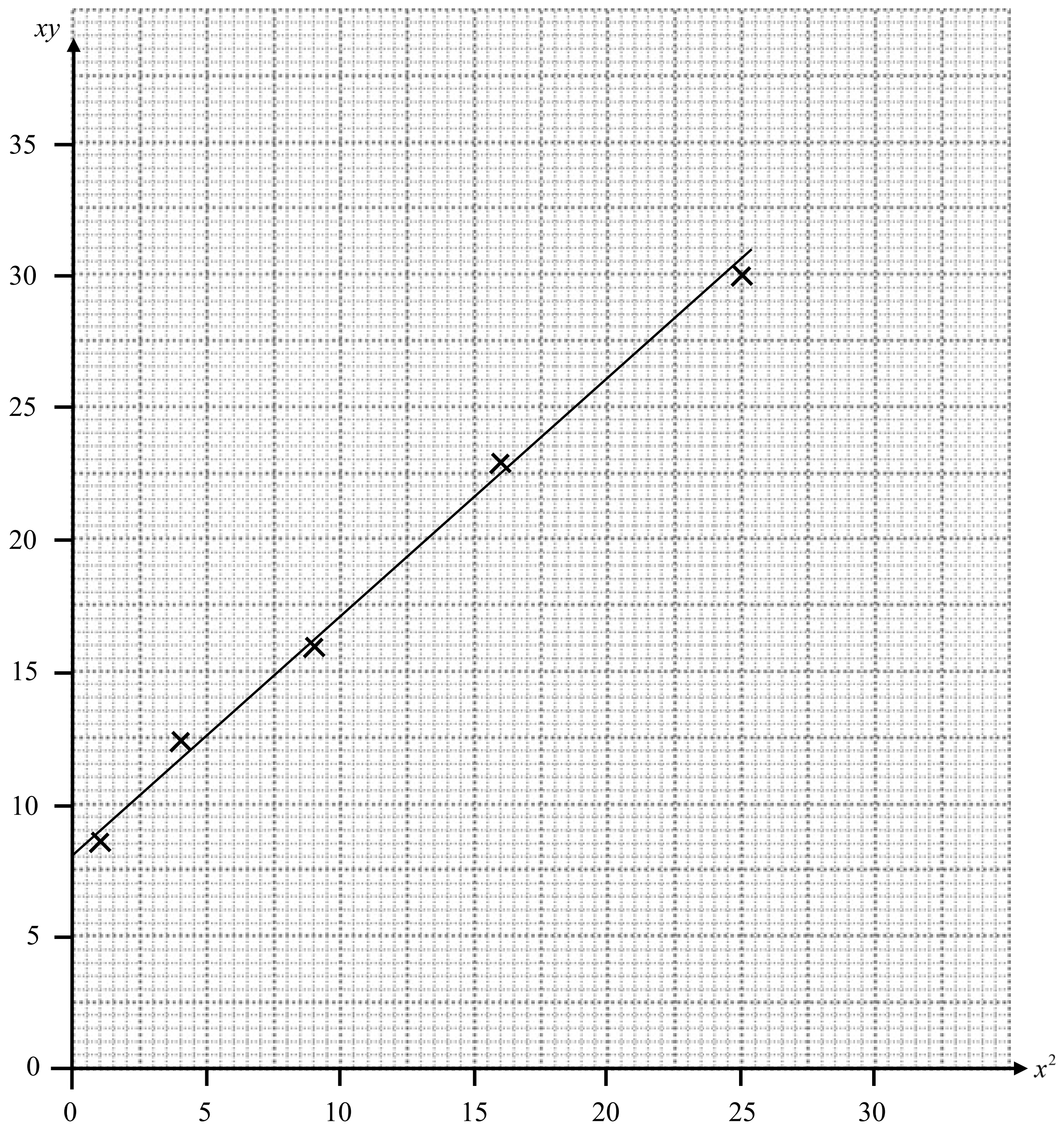
6 The variables x and y are known to be related by the formula $y = px + \frac{q}{x}$ where p and q are constants.

(a) The table below shows some experimental values of x and y.

x	1	2	3	4	5
y	8.55	6.20	5.32	5.72	6.00

x^2	1	4	9	16	25
xy	8.55	12.40	15.96	22.88	30.00

(i) On the grid below, plot xy against x^2 and draw a straight line graph. [2]



(ii) Use your graph to estimate

(a) the value of p and of q. [3]

$$xy = px^2 + q$$

$$p = \frac{26 - 17.5}{20 - 10.5} \qquad\qquad q = 8 \;\left(\text{acceptable } 7.5 \le q \le 9\right)$$

$$\approx 0.895 \;\left(\text{acceptable } 0.8 \le p \le 1.1\right)$$

(b) the positive value of x that satisfies the equation $px + \frac{q}{x} = \frac{20}{x}$. [2]

$$y = \frac{20}{x}$$
$$xy = 20$$

from graph, $x^2 = 13.25$

$$x = 3.64 \;\left(\text{acceptable } 3.60 \le x \le 3.75\right)$$

(b) Explain how another straight line graph can be obtained from $y = px + \frac{q}{x}$ by plotting $\frac{x}{y}$ against $\frac{1}{xy}$, expressing clearly the gradient and vertical intercept in terms of p and/or q.
You need not draw this straight line graph. [3]

$$xy = px^2 + q$$
$$1 = \frac{px}{y} + \frac{q}{xy}$$
$$\frac{px}{y} = -\frac{q}{xy} + 1$$
$$\frac{x}{y} = \left(-\frac{q}{p}\right)\frac{1}{xy} + \frac{1}{p}$$

gradient $= -\frac{q}{p}$

vertical intercept $= \frac{1}{p}$

7 The equation of a curve is $y = \dfrac{\ln 2x}{x}$.

(i) Find an expression for $\dfrac{d^2y}{dx^2}$ and show that $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + \dfrac{1}{x^2} = 0$. [5]

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{2x}\times 2\right) - \ln 2x}{x^2}$$
$$= \frac{1 - \ln 2x}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{2x}\times 2\right) - (1 - \ln 2x)(2x)}{x^4}$$
$$= \frac{-x - 2x + 2x\ln 2x}{x^4}$$
$$= \frac{2\ln 2x - 3}{x^3}$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + \frac{1}{x^2} = x\left(\frac{2\ln 2x - 3}{x^3}\right) + 2\left(\frac{1 - \ln 2x}{x^2}\right) + \frac{1}{x^2}$$
$$= \frac{2\ln 2x - 3 + 2 - 2\ln 2x + 1}{x^2}$$
$$= 0$$

(ii) Find the **exact** coordinates of the stationary point of the curve and use the second derivative test to determine if the stationary point is a maximum or minimum. [5]

$$\frac{dy}{dx} = \frac{1 - \ln 2x}{x^2} = 0$$

$$\ln 2x = 1$$
$$2x = e$$
$$x = \frac{e}{2}$$
$$y = \frac{\ln e}{e/2} = \frac{2}{e}$$

Stationary point is $\left(\dfrac{e}{2}, \dfrac{2}{e}\right)$ or $\left(\dfrac{e}{2}, 2e^{-1}\right)$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{e}{2}} = \frac{2\ln e - 3}{\left(\frac{e}{2}\right)^3}$$ **M1 -** substitute *their x* value into *their* $\frac{d^2y}{dx^2}$

$$= \frac{-8}{e^3} \left(\text{or} \quad \frac{-1}{\left(\frac{e}{2}\right)^3} \quad \text{or} \quad -0.398 \right)$$

$$< 0$$

Hence, $\left(\frac{e}{2}, \frac{2}{e}\right)$ is a maximum point.

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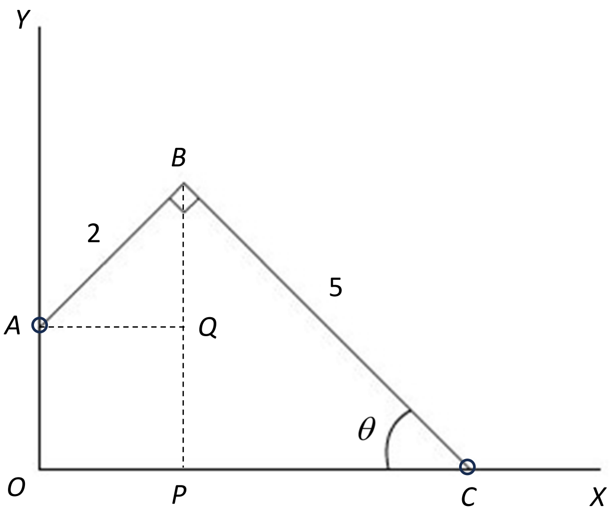
8 The diagram shows two rods AB and BC of length 2 m and 5 m respectively, joined together at B such that angle $ABC = 90°$. Small rings are attached to A and C so that A can move along the vertical pole OY and C can move along the horizontal pole OX. The rod BC makes an angle θ with OX.



(i) Show that $OA = 5\sin\theta - 2\cos\theta$. [2]

$$\frac{BP}{5} = \sin\theta \qquad\qquad \frac{BQ}{2} = \cos\theta$$

$$BP = 5\sin\theta \qquad\qquad BQ = 2\cos\theta$$

$$OA = BP - BQ = 5\sin\theta - 2\cos\theta$$

(ii) Express OA in the form $R\sin(\theta - \alpha)$ where $R > 0$ and $0° < \alpha < 90°$. [3]

$$OA = 5\sin\theta - 2\cos\theta = R(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$$

$$\left.\begin{aligned} R\cos\alpha &= 5 \\ R\sin\alpha &= 2 \end{aligned}\right\} \quad \begin{aligned} \tan\alpha &= \frac{2}{5} \\ \alpha &\approx 21.801° \end{aligned} \qquad \begin{aligned} R^2 &= 2^2 + 5^2 = 29 \\ R &= \sqrt{29} \end{aligned}$$

$$OA = \sqrt{29}\sin(\theta - 21.8°)$$

(iii) Given that A can be no higher than 3.5 m above O, calculate the greatest possible value of θ. [2]

$$\sqrt{29}\sin\left(\theta-21.801°\right)=3.5$$
$$\sin\left(\theta-21.801°\right)=\frac{3.5}{\sqrt{29}}$$
$$\theta-21.801°=\sin^{-1}\frac{3.5}{\sqrt{29}}\approx 40.536°$$

greatest $\theta = 62.3°$

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9 A particle moves in a straight line with a fixed point O. The velocity of the particle, v m/s, is given by $v=\dfrac{25}{2t+1}+10$ where t is time in seconds after passing O.

(a) Explain why the particle never change its direction of motion and will eventually travel at a speed close to 10 m/s. [2]

Since $t>0$, $\dfrac{25}{2t+1}>0$ and hence $v>0$. Therefore particle will never change its direction.

As t increases, $\dfrac{25}{2t+1}$ **decreases and approaches 0 / becomes very small**.

Therefore particle will eventually travel at a speed close to 10 m/s.

(b) Find the acceleration of the particle at $t=2$. [2]

$$v=25\left(2t+1\right)^{-1}+10$$

$$a=-25\left(2t+1\right)^{-2}\left(2\right)$$
$$=-\frac{50}{\left(2t+1\right)^{2}}$$

$$a_2=-\frac{50}{25}=-2\text{ m/s}^2$$

(c) Calculate the distance travelled by the particle in the fifth second of its motion. [4]

$$s = \frac{25\ln(2t+1)}{2} + 10t + c$$

When $t = 0$, $s = 0$, $c = 0$

$$s = \frac{25\ln(2t+1)}{2} + 10t$$

$$s_4 = \frac{25\ln 9}{2} + 40 \approx 67.465$$

$$s_5 = \frac{25\ln 11}{2} + 50 \approx 79.973$$

Distance travelled $= 79.973 - 67.465$

$= 12.5\text{m}$

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10 $A(1\ ,\ 7)$ and $B(9\ ,\ 7)$ lie on a circle C_1 with centre P. The line $4y + 3x = 31$ is a normal to the circle and passes through A.

(i) Use a geometrical property of circle to explain why the x-coordinate of P is 5. [2]

Perpendicular bisector of chord passes through the centre of circle.

Hence, since AB is horizontal chord,
P is vertically above/below the midpoint of AB.

$$\therefore x_p = \frac{1+9}{2} = 5$$

(ii) Find the equation of the circle C_1. [3]

$$4y + 3(5) = 31$$
$$4y = 16$$
$$y = 4$$

P is $(5\ ,\ 4)$

$$r = \sqrt{(9-5)^2 + (7-4)^2}$$
$$= \sqrt{25}$$
$$= 5$$

Equation is circle is $(x-5)^2 + (y-4)^2 = 25$

(iii) The tangent to the circle at A intersects the y-axis at D.

Find the coordinates of D. [3]

Gradient of normal to A is $-\frac{3}{4}$, hence gradient of tangent at A is $\frac{4}{3}$.

Tangent : $y = \frac{4}{3}x + c$

At $A(1\ ,\ 7)$,

$$7 = \frac{4}{3} + c$$

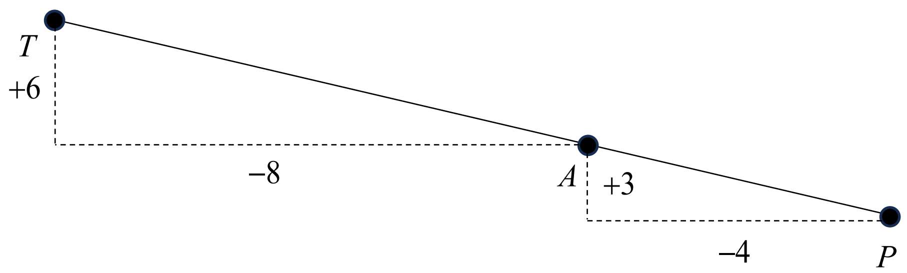
$$c = \frac{17}{3}$$

$$y = \frac{4}{3}x + \frac{17}{3}$$

D is $\left(0\ ,\ \frac{17}{3}\right)$

A larger circle C_2 has the same centre P and radius three times that of C_1. T is a point on C_2 such that P, A and T are collinear and the x-coordinate of T is negative.

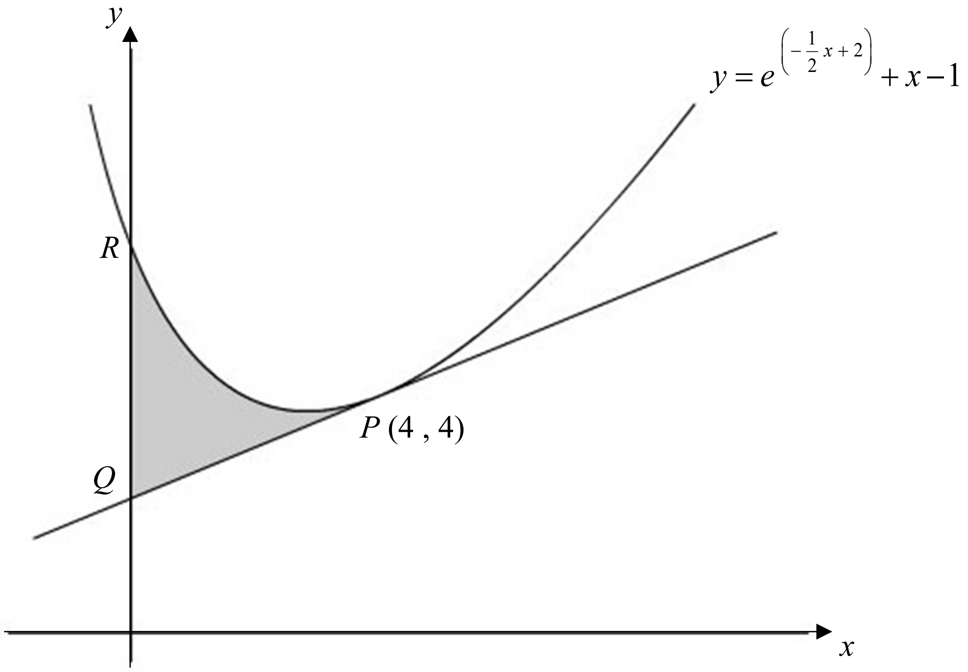
(iv) Determine the coordinates of T. [2]



$T(1-8\ ,\ 7+6)$

$T(-7\ ,\ 13)$

11 The diagram shows part of the curve $y = e^{\left(-\frac{1}{2}x+2\right)} + x - 1$ that intersects the y-axis at R. The tangent to the curve at $P(4,4)$ intersects the y-axis at Q.



Show that the area of the shaded region can be expressed as $ae^2 + b$ where a and b are constants to be determined. [8]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}e^{\left(-\frac{1}{2}x+2\right)} + 1$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=4} = -\frac{1}{2}e^0 + 1 = \frac{1}{2}$$

Tangent is $y = \frac{1}{2}x + c$

At P,

$$4 = \frac{1}{2}(4) + c$$
$$c = 2$$

$\therefore y = \frac{1}{2}x + 2$ and $y_Q = 2$

$$\text{Area of shaded} = \int_0^4 e^{\left(-\frac{1}{2}x+2\right)} + x - 1 \, \mathrm{d}x - \frac{1}{2}(4)(2+4)$$

$$= \left[\frac{e^{\left(-\frac{1}{2}x+2\right)}}{-\frac{1}{2}} + \frac{x^2}{2} - x \right]_0^4 - 12$$

$$= \left(-2e^0 + \frac{4^2}{2} - 4 \right) - \left(-2e^2 \right) - 12$$

$$= 2 + 2e^2 - 12$$

$$= 2e^2 - 10$$
