MOE Advanced Level Higher 3 (H3) Math

Mathematical Proofs and Reasoning

Lecture 3

Telegram Chat

Content

Definition of Sequences

Definition

A (real) sequence is a list of numbers in definite order:

$$a_1, a_2, a_3, \ldots, a_n, \ldots \ldots$$

The number a_n is called the n^{th} term of the sequence.

Example

1. https://en.wikipedia.org/wiki/List_of_integer_sequences
2. https://oeis.org/

Examples

1. $$a_1 = 2,\ a_2 = 3,\ a_3 = 5,\ a_4 = 7,\ a_5 = 11, \ldots$$

   List of positive prime numbers. There is no known formula to generate the n-th term.

2. $$a_1 = 2,\ a_2 = 4,\ a_3 = 8,\ a_4 = 16,\ a_5 = 32, \ldots$$

   $a_n = 2^n$ are the powers of 2.

3. $$a_1 = 1,\ a_2 = 3,\ a_3 = 7,\ a_4 = 15,\ a_5 = 31,\ a_6 = 63,\ a_7 = 127, \ldots$$

   $a_n = 2^n - 1$ are the Mersenne numbers.

4. $$a_1 = 1,\ a_2 = 2,\ a_3 = 6,\ a_4 = 24,\ a_5 = 120, \ldots$$

   $a_n = n!$.

Fibonacci Sequence

The Fibonacci sequence is defined as such. Let $a_0 = 0$, $a_1 = 1$, and

$$a_{n+1} = a_n + a_{n-1}, \qquad n \geq 1.$$

Let's write out a few more terms.

$$a_2 = 1, \quad a_3 = 2, \quad a_4 = 3, \quad a_5 = 5, \quad a_6 = 8, \quad a_7 = 13, \ldots\ldots$$

The general term is given by the formula

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad n \geq 1.$$

The derivations are omitted.

Golden Ratio

The number

$$\phi = \frac{1+\sqrt{5}}{2}$$

is known as the golden ratio. See https://en.wikipedia.org/wiki/Golden_ratio.

- It is one of the roots of the polynomial $x^2 - x - 1 = 0$.
- It is an irrational number.
- Frequently appear in geometry and architecture.
- Believed to define beauty.
- It is the limit of the ratio of the Fibonacci sequence,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \phi.$$

Observation

- Observe that

$$a_0 = 0 < 1 = 2^0, \quad a_1 = 1 < 2^1, \quad a_2 = 1 < 4 = 2^2, \quad a_3 = 2 < 8 = 2^3, \quad a_4 = 3 < 16 = 2^4.$$

- It seems like the difference between a_n and 2^n is larger as n gets larger.
- So, we might guess (correctly) that for all $n \geq 0$,

$$a_n < 2^n.$$

- But there are infinitely many n to check, it is not possible to check them all.
- Moreover, a_n is obtained from the lower terms. If the lower terms satisfy the inequality, we might be able to use them to show that a_n satisfies the inequality too.

Mathematical Induction

The Principle of Mathematical Induction can be used to prove statements of the form $\forall n \in \mathbb{Z}^+,\ P(n)$ is true.

1. First prove the base case: $P(1)$ is true.
2. Inductive step: Prove that $P(k) \Rightarrow P(k+1)$ is true.

We can then conclude that $P(n)$ is true for all $n \in \mathbb{Z}^+$.

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots\cdots \Rightarrow P(k) \Rightarrow P(k+1) \Rightarrow \cdots .$$

Example

Prove that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}, \qquad \forall n = 1, 2, 3, \ldots$$

- $P(1)$: $\displaystyle\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1+1} = \frac{1}{2}$, $\displaystyle\frac{1}{1+1} = \frac{1}{2}$. So, $P(1)$ is true.
- Suppose $P(k)$ is true. Show that $P(k+1)$ is true, that is $\displaystyle\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \text{by induction hypothesis} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}
\end{aligned}$$

This proves $P(k) \Rightarrow P(k+1)$, and hence $P(n)$ is true for all $n \geq 1$.

Example

For every positive integer n,

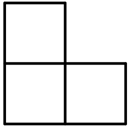
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

- Base case: $1^2 = 1 = \dfrac{1(1+1)(2(1)+1)}{6}$.
- Suppose $P(k)$ is true. Show that $P(k+1)$ is true, that is $\displaystyle\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$.

  First note that $\dfrac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \dfrac{(k+1)(k+2)(2k+3)}{6}$.

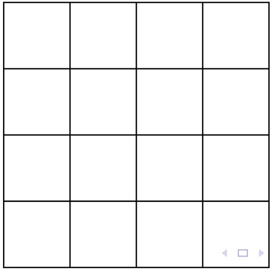
$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6} \\ &= \frac{(k+1)\left[2k^2+7k+6\right]}{6} = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.\end{aligned}$$

Discussion

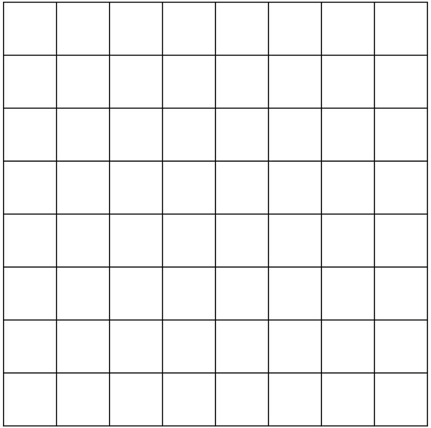
Tiling with triominoes  .

For any positive integer n, show that if one square is removed from a $2^n \times 2^n$ checkerboard, the remaining squares can be completely covered by triominoes.

Base case: 



Discussion



Mathematical Induction, Variation I

To prove statements of the form $\forall n \in \mathbb{Z},\ n \geq n_0,\ P(n)$ is true. That is, the base case is n_0 instead of 1.

1. Base case: $P(n_0)$ is true.
2. Inductive step: Prove that $P(k) \Rightarrow P(k+1)$ is true for $k \geq n_0$.

We can then conclude that $P(n)$ is true for all $n \geq n_0$.

Example

Theorem

For every positive integer $n \geq 4$,

$$n! > 2^n.$$

Proof.

- Base case: Prove that $P(4)$ is true. $24 = 4! > 16 = 2^4$.
- Suppose $P(k)$ is true. Show that $P(k+1)$ is true, that is $(k+1)! > 2^{k+1}$.

$$\begin{aligned}(k+1)! &= (k+1)k! > (k+1)2^k \quad \text{by induction hypothesis,} \\ &> 2 \times 2^k \quad \text{since } k \geq 4,\ k+1 \geq 5 > 2 \\ &= 2^{k+1}\end{aligned}$$

This proves $P(k) \Rightarrow P(k+1)$, and hence $P(n)$ is true for all $n \geq 4$. □

Mathematical Induction, Variation II

To prove $\forall n \in \mathbb{Z}^+,\ P(n)$ is true.

1. Base case: Prove that $P(1)$ is true.
2. Inductive step: Prove that if $P(1)$, $P(2)$, ..., $P(k)$ are true, then $P(k+1)$ is true.

This is often known as the Principle of Strong Induction since the inductive step is replaced with a "stronger hypothesis that requires that all terms before $P(k+1)$ be true.

Another variation:

1. Base case: Prove that $P(1)$, ..., $P(n_0)$ are true.
2. Inductive step: Prove that if $P(k)$, $P(k+1)$, ..., $P(k+n_0-1)$ are true, then $P(k+n_0+1)$ is true.

Fibonacci Sequence

Let $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for all $n \geq 1$. Show that for all $n \geq 0$,

$$a_n < 2^n.$$

- Shown previously that $P(0)$ and $P(1)$ are true.
- Suppose $P(k)$ and $P(k+1)$ are true. Show that $P(k+2)$ is true.

$$\begin{aligned} a_{k+2} &= a_{k+1} + a_k < 2^{k+1} + 2^k \text{ using the induction hypothesis that } P(k) \text{ and } P(k+1) \text{ are true} \\ &= 2^k(2+1) = 2^k(3) < 2^k(4) \text{ using that fact that } 3 < 4 \\ &= 2^k\left(2^2\right) = 2^{k+2} \end{aligned}$$

Here we need $P(k)$ and $P(k+1)$ to be true since the $k+2$ terms involves k and $k+1$ terms. Hence, we need to prove $P(0)$ and $P(1)$ for the base case.

Fact

We will state but not prove the following fact.

Theorem

All the variation of principal of Mathematical induction are equivalent.

Discussion

Claim:

All rabbits are of the same color.

- Base case: Since there is only 1 rabbit, clearly it has the same color.
- Suppose it is true that any k number of rabbits are of the same color. Show that any $k+1$ number of rabbits are of the same color.

  Consider a set of $k+1$ rabbits; order the rabbit. Removing the first rabbit from the set, we now have a set of k rabbits. So, by the induction hypothesis, they must have the same color. Now remove the last rabbit, and add back in the set the first rabbit, and again we have a set of k rabbits, and thus, they are of the same color. This shows that all the $k+1$ rabbits are of the same color, which concludes the proof of the inductive step.
- Hence, we conclude that all rabbits are of the same color.

What is wrong with the proof?

Introduction to Limits

A frog is 1 meters away from a wall.

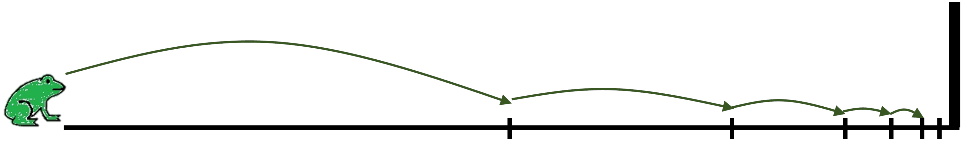
After $t = 1$hr, the frog leaps $\frac{1}{2}$ meter towards the wall.

After $t = 2$hrs, the frog leaps a further $\frac{1}{4}$ meters towards the wall.

After $t = 3$hrs, the frog leaps a further $\frac{1}{8}$ meters towards the wall.

And so on, that is, after $t = n$hrs, the frog leaps a further $\frac{1}{2^n}$ meters towards the wall.

Will the frog reach the wall? If it never reach the wall, how near from the wall can the frog be?



The wall is known as the limit, that is, the frog can be as near to the wall as possible, but it may not be able to reach it.

Introduction to Limits

Zeno's paradox

A tortoise and a hare have decided to settle once and for all who is the fastest, so they organize a race to be run over 100 meters. The hare, feeling confident, lets the tortoise have a 50-meter head start.

The starting pistol goes off, and they both start racing toward the finish.

The hare, traveling at twice the speed of the tortoise, reaches the 50-meter mark where the tortoise started.

The tortoise, meanwhile, has traveled only 25 meters, so is at the 75-meter mark.

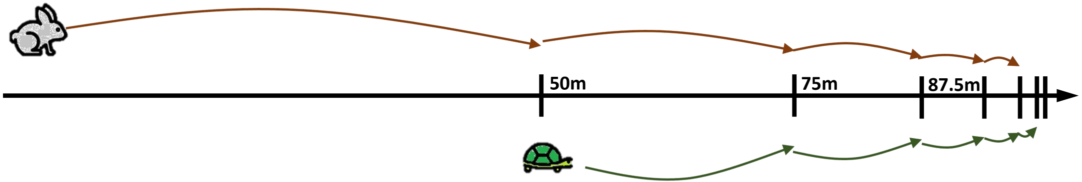
The hare then hops to the 75-meter mark, but the tortoise is still slightly ahead, at the 87.5-meter mark.

The rabbit again hops to where the tortoise was, but in that time, the tortoise has moved a little bit further still.

And so on.

Zeno's paradox

No matter how many times the hare tries to catch up, the tortoise remains slightly ahead. And so, Zeno said, the hare can never overtake the tortoise!



Can you explain this paradox?

Example of Limits

Let $a_n = \frac{n}{n+1}$. Then

- $a_1 = \frac{1}{2}$
- $a_{10} = \frac{10}{11} \approx 0.9090909$
- $a_{100} = \frac{100}{101} \approx 0.990099$
- $a_{1000} = \frac{1000}{1001} \approx 0.999001$
- $a_{100,000} = \frac{100000}{100001} \approx 0.99999$

As n increases, $\frac{n}{n+1}$ is closer and closer to 1. We say that the limit of the sequence $\{\frac{n}{n+1}\}$ is 1, denoted as

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

Convergence of Infinite Series

Definition

Let $\{a_n\}$ be a sequence and $\sum_{k=1}^{n} a_n$ be the sum of its first n terms. The (infinite) series associated to $\{a_n\}$ is

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_n.$$

A series $\sum_{k=1}^{n} a_n$ is said to be convergent if the limit $\lim_{n\to\infty} \sum_{k=1}^{n} a_n$ exists. Otherwise, it is said to be divergent.

Geometric Series

Theorem

Let $\{a_n = ar^n\}$ be a geometric sequence with first term $a_0 = a$ and common ratio r. Then the sum to n-th term is given by the formula

$$\sum_{k=0}^{n} ar^k = \frac{a(1-r^n)}{1-r}.$$

Proof.

Exercise. May proof using Mathematical induction. □

Geometric Series

Theorem

A geometric series $\sum_{k=0}^{\infty} ar^k$ *is convergent if* $-1 < r < 1$. *In this case,*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \quad \textit{for } -1 < r < 1.$$

Sketch of Proof.

Follows from the previous theorem, and the fact that if $|r| < 1$, then $r^n \to 0$ as $n \to \infty$. $\square$

Example

Is it true that $1 = 0.9999999\cdots$?

$$0.9999\cdots \quad = \quad \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots + \frac{9}{10^n} + \cdots$$

The sequence $a_n = \frac{9}{10^n}$ is a geometric progression with first term $a = \frac{9}{10}$ and common ratio $r = \frac{1}{10}$. So its sum to infinity is

$$\lim_{n\to\infty} S_n = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1.$$

Example

$$0.142857142857\cdots = \frac{1}{7}$$

$$0.142857142857\cdots \quad = \quad \frac{142857}{1,000,000} + \frac{142857}{1,000,000^2} + \cdots + \frac{142857}{1,000,000^n} + \cdots$$

This is a geometric series with first term $a = \dfrac{142857}{1,000,000}$ and common ratio $r = \dfrac{1}{1,000,000}$. Hence,

$$0.142857142857\cdots \frac{\frac{142857}{1{,}000{,}000}}{1 - \frac{1}{1{,}000{,}000}} = \frac{142857}{1,000,000} \times \frac{1,000,000}{999,999} = \frac{1}{7}.$$

Property of Rational Number

Theorem

Every decimal expansion that either terminates or eventually has repeating decimals is a rational number.

Proof.

The case where the decimal expansion terminates is clear. Suppose a real number r has repeating decimals. May write it as

$$r = k + \frac{a_0}{10^l} + \frac{a_0}{(10^l)^2} + \cdots$$

where a_0 are the digits in a repeating decimals, and l is the length of the repeating decimals. Note that k is a real number of terminating decimals, hence is a rational number. Then

$$r = k + \frac{\frac{a_0}{10^l}}{1 - \frac{1}{10^l}} = k + \frac{a_0}{10^l - 1} \text{ is a rational number.}$$

Remark

The converse of the previous theorem is also true. That is, every rational number has a decimal representation that either terminates or eventually repeats. The proof is beyond the scope of the syllabus.

Discussion

Prove that $S = \sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$.

- Consider the series $S_1 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = \sum_{k=0}^{\infty} (-1)^k$.
- Observe that $1 - S_1 = 1 - 1 + 1 - 1 + \cdots = S_1$. Hence, $S_1 = \frac{1}{2}$.
- Consider $S_2 = 1 - 2 + 3 - 4 + \cdots$.
- Then

$$\begin{aligned} 2S_2 = S_2 + S_2 \quad &= 1 \quad -2 + 3 - 4 + \cdots \\ &\qquad\;\; +1 - 2 + 3 - 4 + \cdots \\ &= 1 \quad -1 + 1 - 1 + \cdots = S_1 = \frac{1}{2} \end{aligned}$$

- Hence, $S_2 = \frac{1}{4}$.

Discussion

- Next,

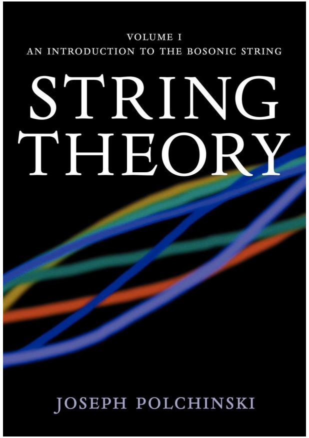
$$\begin{aligned} S - S_2 = & \; 1+2+3+4+5+6+\cdots \\ - & \; (1-2+3-4+5-6+\cdots) \\ = & \; 4+8+12\cdots = 4(1+2+3+\cdots) = 4S \end{aligned}$$

- Manipulating, we conclude

$$3S = -S_2 \quad \Rightarrow \quad S = -\frac{1}{3} \times \frac{1}{4} = -\frac{1}{12}.$$

Discussion

This result is derived and used in Physics, in particular, in String Theory.



22 *1 A first look at strings*

into why Lorentz invariance would be lost in the light-cone approach for the wrong A or D.

First, we assert that the operator ordering constant in the Hamiltonian for a free field comes from summing the zero-point energies of each oscillator mode, $\frac{1}{2}\omega$ for a bosonic field like X^μ. Equivalently, it always works out that the natural operator order is averaged, $\frac{1}{2}\omega(aa^\dagger + a^\dagger a)$, which is the same as $\omega(a^\dagger a + \frac{1}{2})$. In H this would give

$$A = \frac{D-2}{2}\sum_{n=1}^{\infty} n\,, \tag{1.3.31}$$

the factor of $D-2$ coming from the sum over transverse directions. The zero-point sum diverges. It can be evaluated by regulating the theory and then being careful to preserve Lorentz invariance in the renormalization. This leads to the odd result

$$\sum_{n=1}^{\infty} n \to -\frac{1}{12}\,. \tag{1.3.32}$$

Riemann Zeta Function

Definition

The Riemann zeta function, denoted by the Greek letter ζ (zeta), is a function of a complex variable defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s}, \quad \text{for } \Re(s) > 1,$$

and its analytic continuation elsewhere.

- For a complex number $s = x + iy$, define the real part to be $\Re(s) = x$.
- For s with $\Re(s) \geq 1$, $\zeta(s)$ is defined, but not given by the series on the right.
- Indeed, it is true that $\zeta(-1) = -\dfrac{1}{12}$.

Riemann Zeta Function

- It is related to the Riemann hypothesis, one of the Millennium prize problems, which offers US$1 million to anyone who can solve it.
- https://en.wikipedia.org/wiki/Riemann_zeta_function
- https://en.wikipedia.org/wiki/Riemann_hypothesis
- https://en.wikipedia.org/wiki/Millennium_Prize_Problems

Challenge

Prove the existence and uniqueness of the fundamental theorem of arithmetic using Principal of Mathematical Induction (and its variations).

Theorem (Fundamental Theorem of Arithmetic)

Every integer greater than 1 can be written as a product of primes in exactly one way, up to permutations of the prime.