Differentiation: Maclaurin Series & Small angle approximation

1. **AJC/II/1**

A curve with equation $y = \mathrm{f}(x)$ passes through the point $(0,1)$ and satisfies the relation

$$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + xy = \sqrt{1+x^2}\ .$$

By further differentiation of this result, find the Maclaurin's series for y in ascending powers of x, up to and including the term in x^2. [3]

Deduce the approximate value of $\int_0^{0.01} \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x$, giving your answer correct to 3 decimal places. [2]

2. **ACJC/I/11**

(i) Given that $y = \cot\left(2x + \frac{\pi}{4}\right)$, find an equation relating $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y. [2]

Hence show that

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = k\left[y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{dx}\right)^2\right],$$

where k is a constant to be found. [2]

(ii) Find the Maclaurin's series for y up to and including the term in x^3 if x is sufficiently small for powers of x higher than x^3 to be neglected. [2]

(iii) Using the Maclaurin's expansion for y, estimate the value of $\operatorname{cosec}^2\left(\frac{13\pi}{50}\right)$, giving your answer in the form $a + b\pi + c\pi^2$, where a, b and c are constants to be determined. [4]

3. **IJC/I/9**

Given that $y = \sin\left[\ln(1-3x)\right]$, prove that

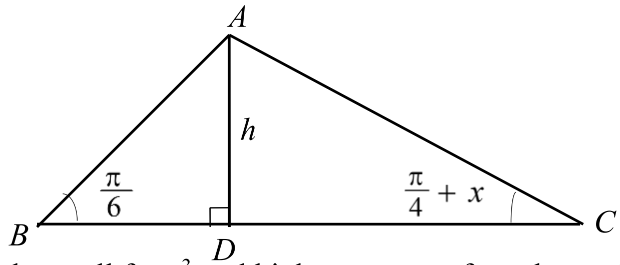
$$(1-3x)^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3(1-3x)\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0\ .$$

Hence find Maclaurin's series for y, up to and including the term in x^3. [6]

Deduce an expansion for $\cos\left[\ln(1-3x)\right]$ up to and including the term in x^2. [2]

4. **PJC/II/1**

The diagram shows triangle *ABC*. It is given that the height *AD* is h units, $\angle ABD = \dfrac{\pi}{6}$ and $\angle ACD = \dfrac{\pi}{4} + x$.



Show that if x is sufficiently small for x^3 and higher powers of x to be neglected, then

$$BC \approx h\,(1+\sqrt{3} - 2x + 2x^2)\,.$$ [5]

5. **TPJC/2010/II/5**

Given that $y = \ln(1+2x)$, show that $(1+2x)\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$. [1]

(i) By repeated differentiation of this result, find the Maclaurin's expansion for y up to and including the term in x^5. [5]

(ii) By writing down the Maclaurin's expansion of $\ln(1-2x)$, show that

$$\ln\left(\frac{1+2x}{1-2x}\right) = 2\left[2x + \frac{8}{3}x^3 + \frac{32}{5}x^5 + \ldots\right].$$ [2]

(iii) Use the series in part **(ii)** for $x = \dfrac{1}{4}$, find the exact value of $\displaystyle\sum_{r=0}^{\infty} \frac{1}{(2r+1)2^{2r+1}}$. [3]

6. **AJC/2010/I/3**

Given that $\ln y = \sin^{-1} 2x$, show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}\sqrt{1-4x^2} = 2y$.

By repeated differentiation of this result,

(i) Find the series expansion of y in ascending powers of x, up to and including the term in x^3. [4]

(ii) Hence deduce the approximate value of $e^{\frac{\pi}{3}}$, giving your answer in the form of $\dfrac{1}{8}\left(a + b\sqrt{3}\right)$, $a, b \in \mathbb{Z}$ where a and b are to be determined. [2]

7. **ACJC/2011/I/5**

Given that $y = \mathrm{f}(x)$ where $\tan^{-1} y = 2\tan^{-1} x + \dfrac{\pi}{4}$ for $-0.4 \le x \le 0.4$, show that

$$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(1+y^2\right).$$

By further differentiation of this result, find the Maclaurin's series for y up to and including the term in x^3.

Denote the above Maclaurin's series for y by $\mathrm{g}(x)$.

Find the range of values of x for which the value of $\mathrm{g}(x)$ differs from $\mathrm{f}(x)$ by less than 0.5.

8. **NYJC/2012/I/6**

(a) In a triangle with vertices A, B and C, angle BAC is a right-angle and angle $ABC = \dfrac{\pi}{3} - x$.

(i) Show that $\dfrac{AB}{AC} = \dfrac{1+\sqrt{3}\tan x}{\sqrt{3}-\tan x}$. [1]

(ii) Hence, show that when x is small enough for x^2 and higher powers of x to be neglected, then $\dfrac{AB}{AC} \approx a + bx$, where a and b are exact constants to be determined. [3]

(b) A curve is defined by the equation

$$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} + xy = \sqrt{1+x^2}$$

and (0, 1) is a point on the curve.

(i) Find the Maclaurin's expansion of y up to and including the term in x^2. [3]

(ii) Hence, find the series expansion of e^y, up to and including the term in x^2. [3]

9. **ACJC/2014/1/3**

The curve $y=\mathrm{f}(x)$ passes through the point $(0,1)$ and satisfies the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{6-2y}{\cos 2x}$. Find the Maclaurin's series of $\mathrm{f}(x)$, up to and including the term in x^3. [4]

Using standard results given in the List of Formulae (MF27), express $\dfrac{1-\sin x}{\cos x}$ as a power series of x, up to and including the term in x^3. [3]

Using the two power series you have found, show to this degree of approximation, that $\mathrm{f}(x)$ can be expressed as $a(\tan 2x - \sec 2x) + b$ where a and b are constants to be determined. [2]

10. **JJC/2014/II/1**

(a) Given that $e^{y+x} = \cos x$, show that $\frac{dy}{dx} + \tan x + 1 = 0$.

(i) By further differentiation of this result obtain the Maclaurin's series for y in terms of x, up to and including the term in x^2. [3]

(ii) Let the result in **(i)** be $h(x)$. Find the set of values of x for which $h(x)$ is within ± 0.2 of the value of y. [2]

(b) Expand, in ascending powers of x, $\left(a - \frac{x}{3}\right)^n$ where a and n are integers and $n < 0$, up to and including the term in x^2. [2]

It is given that the coefficient of x is four times the coefficient of x^2 and the constant in the expansion is $\frac{1}{4}$. Find a and n. [3]

11. **AJC/2010/II/1**

Find the values of a and b if the expansion of $\frac{\sqrt{9+ax}}{1+bx^2}$ in ascending powers of x up to and including the term in x^2 is $3 + x + \frac{35}{6}x^2$. With these values of a and b, state the range of values of x for which expansion is valid. [6]

12. **MJC/2010/I/4**

(i) Express $f(x) = \frac{x-4}{(x+1)(3x+2)}$ in partial fractions.

Hence, expand $f(x)$ in ascending powers of x, up to and including the term in x^3. [4]

(ii) State the range of values of x for which this expansion is valid. [1]

(iii) Find the coefficient of x^n in this expansion. [2]

13. **IJC/2013/I/8**

Find the expansion of $\left(\frac{1+2x^2}{4-x}\right)^{\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^2. [4]

(i) Find the set of values of x for which the expansion is valid. [2]

(ii) By putting $x = \frac{1}{4}$, show that $\sqrt{30} \approx \frac{a}{b}$, where a and b are integers to be determined. [2]

14. **PJC/2011/I/2**

(i) Find the first three terms in the expansion of $\dfrac{1}{\sqrt{4+x^2}}$ in ascending powers of x. [3]

(ii) Hence find the first four terms in the expansion of $\dfrac{x+1}{\sqrt{4+x^2}}$. [2]

(iii) State the set of values of x for which this expansion is valid. [1]

15. **ACJC/2013/I/4**

In the triangle *PQR*, *PQ* = 3, *QR*= $\sqrt{2}$ and angle *PQR*= $\theta+\dfrac{\pi}{4}$ radians. Given that θ is a sufficiently small angle, show that $PR \approx \left(5+6\theta+3\theta^2\right)^{\frac{1}{2}} \approx a+b\theta+c\theta^2$, for constants *a, b* and *c* to be determined. [5]

16. **AJC/2015/I/5**

In the triangle *ABC*, angle $BAC=\theta$ radians, angle $ACB=\dfrac{\pi}{6}$ radians and $AC=\sqrt{3}$.

Given that θ is sufficiently small, show that

$$AB \approx \frac{2\sqrt{3}}{2+2\sqrt{3}\theta-\theta^2} \approx a+b\theta+c\theta^2,$$

where *a*, *b* and *c* are constants to be determined in exact form. [7]

17. **JJC/2018/2/1**

Given that $\mathrm{f}(x)=\mathrm{e}^{\sin x}$, use the standard series to find the series expansion for $\mathrm{f}(x)$ in the form $a+bx+cx^2+dx^3$, where *a*, *b*, *c* and *d* are constants to be determined.

Hence show that the first three non-zero terms for the expansion of $\dfrac{1}{\left(\mathrm{e}^{\sin x}\right)^2}$ in ascending powers of x is $1-2x+2x^2$. [4]

The function $y=\mathrm{g}(x)$ satisfies $4\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(y+1\right)^2$ and $y=1$ at $x=0$.

(i) By further differentiation, find the series expansion for $\mathrm{g}(x)$, up to and including the term in x^3.

Hence show that when x is small, $\mathrm{g}\left(x\right)-\mathrm{f}\left(x\right)\approx\dfrac{1}{4}x^3$. [5]

(ii) By using the result in **(i)**, justify whether $\mathrm{f}(x)$ is a good approximation to $\mathrm{g}(x)$ for values of x close to zero. [1]

18. **SAJC/2019/I/Q4**

It is given that $y=\sqrt{\mathrm{e}^{\cos x}}$.

(i) Show that $2\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\sin x=0$. Hence find the Maclaurin's expansion of y up to and including the term in x^2. [4]

(ii) Deduce the series expansion for $\mathrm{e}^{\sin^2\left(\frac{x}{2}\right)}$ up to and including the term in x^2. [3]

(iii) Using the series expansion from **(ii)**, estimate the value of $\int_0^{\sqrt{2}} \mathrm{e}^{\sin^2\left(\frac{x}{2}\right)} \mathrm{d}x$ correct to 3 decimal places. [2]

19. **TMJC/2021/II/Q4**

(a) Expand $\left(b-\dfrac{x}{2}\right)^n$ in ascending powers of x, where b is a positive constant and n is a negative constant, up to and including the term in x^2. [2]

It is given that the coefficient of x is four times the coefficient of x^2 and the constant term in the expansion is $\dfrac{1}{2}$. Find the **integer** values of b and n. [3]

(b) **(i)** Explain why it is not possible to obtain a Maclaurin series for $\ln\left(2x^2\right)$. [1]

(ii) A Taylor series is an expansion of a real function $\mathrm{f}(x)$ about a point $x=a$ and it is defined by

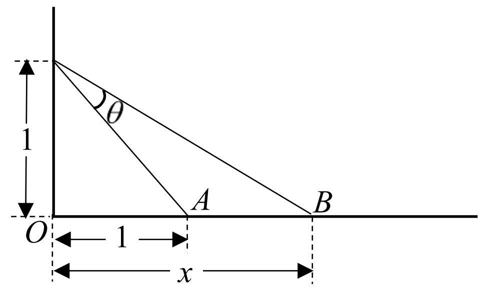
$$\mathrm{f}(x)=\mathrm{f}(a)+\mathrm{f}'(a)(x-a)+\frac{\mathrm{f}''(a)}{2!}(x-a)^2+\ldots+\frac{\mathrm{f}^{(n)}(a)}{n!}(x-a)^n+\ldots$$

where $\mathrm{f}^{(n)}(a)$ is the value of the nth derivative of $\mathrm{f}(x)$ when $x=a$.

Find the first three exact non-zero terms of the Taylor series for $\ln\left(2x^2\right)$ about the point $x=2$. [You need not simplify your answer.] [3]

20. **ASRJC/2022/I/Q4**

(i) Show that the first two non-zero terms of the Maclaurin series for $\tan\theta$ is given by $\theta+\frac{1}{3}\theta^3$.
You may use the standard results given in the List of Formulae (MF27). [2]



In the right-angle triangle OBC shown above, point A lies on OB such that $OA=1$, $OB=x$, where $x > 1$ and $OC=1$. It is given that angle COB is $\frac{\pi}{2}$ radians and that angle ACB is θ radians (see diagram).

(ii) Show that $AB=\dfrac{2\tan\theta}{1-\tan\theta}$. [2]

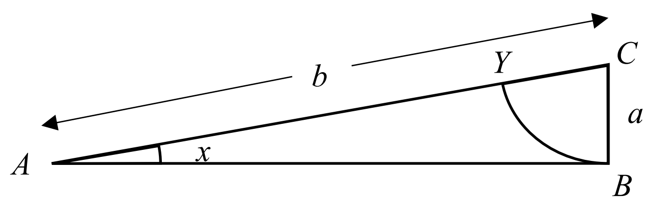
(iii) Given that θ is a sufficently small angle, show that

$$AB \approx a\theta + b\theta^2 + c\theta^3$$

for exact real constants a, b and c to be determined. [3]

21. **EJC Prelim 9758/2024/01/Q7**

For a right-angled triangle ABC with $BC=a$ and $AC=b$, $\measuredangle CAB = x$ and $\measuredangle ABC = \frac{\pi}{2}$ as shown below, a sector with radius a is extended from the vertex C. This sector meets the side AC at the point Y.



(a) Show that $\left(\dfrac{AC}{AY}\right)^2=\dfrac{1}{(1-\sin x)^2}$. [2]

(b) Using appropriate expansions from the List of Formulae (MF26), find the Maclaurin series for $f(x)=(1-\sin x)^{-2}$, up to and including the term in x^3. [3]

(c) Deduce the series expansion for $\dfrac{2\cos x}{(1-\sin x)^3}$, up to and including the term in x^2. [2]

(d) Using your answer in part **(b)** or otherwise, show that $\dfrac{AC}{AY}\approx 1+x+x^2$, when x is sufficiently small. [3]

22. **DHS Prelim 9758/2025/P1/Q3**

(a) Find the first three non-zero terms in the Maclaurin series for $\mathrm{e}^x \sin(x+\pi)$. [3]

(b) It is given that the three terms found in part **(a)** are equal to the first three terms in the series expansion of $ax(1+bx)^c$ for small x, where a, b and c are constants. Find the exact values of a, b and c. Use these values to find the coefficient of x^4 in the expansion of $ax(1+bx)^c$, giving your answer as a simplified rational number. [5]

23. **NYJC Prelim 9758/2025/P1/Q10**

In the question you may use expansions from the List of Formulae (MF27).

(a) Find the Maclaurin expansion of $\cos(t^3)$ in ascending powers of t, up to and including the term in t^{12}. Hence, find the Maclaurin series of $\int_0^x t^2 \cos(t^3)\mathrm{d}t$ up to and including the term in x^{15} given that x is small. [4]

(b) Use your expansion from part **(a)** to find an approximate value for $\int_0^{0.1} t^2 \cos(t^3)\,\mathrm{d}t$, correct to 5 decimal places. [1]

(c) Find $\int_0^x t^2 \cos(t^3)\mathrm{d}t$ in terms of x. Hence evaluate $\int_0^{0.1} t^2 \cos(t^3)\,\mathrm{d}t$, correct to 5 decimal places. [3]

(d) Comparing your answers to parts **(b)** and **(c)**, and with reference to the value of x, comment on the accuracy of your approximations. [1]

24. **VJC Prelim 9758/2025/P2/Q1**

It is given that $\mathrm{f}(x)=\ln(a+x)$, $x \in \mathbb{R}$, $x>-a$, where a is a constant.

(a) Using the standard series from the List of Formulae (MF27), find the series expansion for $\mathrm{f}(x)$, up to and including the term in x^3. [2]

It is given that $a=1$.

(b) Hence, or otherwise, show that the series expansion of $\sin[\mathrm{f}(x)]$, up to and including the term in x^3 is given by $x-\frac{1}{2}x^2+\frac{1}{6}x^3$. [2]

(c) Deduce the Maclaurin series for $\cos[\mathrm{f}(x)]$ up to and including the term in x^2. [2]

(d) Find $\int_1^3 \left(x-\frac{1}{2}x^2+\frac{1}{6}x^3\right)\mathrm{d}x$. Without the use of a calculator or any further calculation, explain, whether this value is a good approximation to the value of $\int_1^3 \sin[\mathrm{f}(x)]\mathrm{d}x$. [2]

Answers

1 $1+x-\frac{1}{2}x^2+\cdots;0.010$

2 (i) $\frac{\mathrm{d}y}{\mathrm{d}x}=-2(1+y^2);\ k=-4$ (ii) $\cot\left(2x+\frac{\pi}{4}\right)\approx 1-4x+8x^2-\frac{64}{3}x^3$
(iii) $a=2, b=-\frac{1}{25}, c=\frac{1}{1250}$

3 $-3x-\frac{9}{2}x^2-\frac{9}{2}x^3$; $1-\frac{9}{2}x^2$

5 (i) $y=2x-2x^2+\frac{8}{3}x^3-4x^4+\frac{32}{5}x^5+\ldots\ldots$; (iii) $\frac{1}{2}\ln 3$

6 (i) $y\approx 1+2x+2x^2+\frac{8x^3}{3}$ (ii) $a=11,\ b=5$

7 $1+4x+8x^2+20x^3$; $-0.376<x<0.252$

8 (a)(ii) $a=\frac{1}{\sqrt{3}},\ b=\frac{4}{3}$, (b)(i) $y=1+x-\frac{x^2}{2}+\ldots$, (ii) $\mathrm{e}(1+x)$

9 $\mathrm{f}(x)=1+4x-4x^2+\frac{16}{3}x^3+\ldots\ldots$; $\frac{1-\sin x}{\cos x}=1-x+\frac{x^2}{2}-\frac{x^3}{3}+\ldots$; $a=2$, $b=3$

10 (a)(i) $y=-x-\frac{x^2}{2}+\ldots$; (a)(ii) $-1.12<x<1.12$; (b) $a=2,\ n=-2$

11 $a=6$ and $b=-2$; $|x|<\frac{1}{\sqrt{2}}$

12(i) $\mathrm{f}(x)=\frac{5}{x+1}-\frac{14}{3x+2}$ (ii) $-2+\frac{11}{2}x-\frac{43}{4}x^2+\frac{149}{8}x^3+\ldots$ (iii) $(-1)^n\left[5-7\left(\frac{3}{2}\right)^n\right]$

13 $\frac{1}{2}\left(1+\frac{1}{8}x+\frac{131}{128}x^2\right)+\ldots$ (i) $\left\{x\in\mathbb{R}: -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}\right\}$ (ii) $\frac{11215}{2048}$

14(i) $\frac{1}{2}-\frac{1}{16}x^2+\frac{3}{256}x^4$ (ii) $\frac{1}{2}+\frac{1}{2}x-\frac{1}{16}x^2-\frac{1}{16}x^3$ (iii) $-2<x<2$

15 $a=\sqrt{5},\ b=\frac{3\sqrt{5}}{5},\ c=\frac{3\sqrt{5}}{25}$

16 $a=\sqrt{3},\ b=-3,\ c=\frac{7\sqrt{3}}{2}$

17 $a=1,\ b=1,\ c=\frac{1}{2}$ and $d=0$, (i) $\mathrm{g}(x)=1+x+\frac{x^2}{2}+\frac{x^3}{4}+\ldots\ldots$

18 (i) $y=\sqrt{e}-\frac{\sqrt{e}}{4}x^2+\cdots$ (ii) $1+\frac{1}{4}x^2+\cdots$ (iii) 1.650

19 (a) $b^n\left(1-\frac{n}{2b}x+\frac{n(n-1)}{8b^2}x^2+\ldots\right)$, $b=2$ and $n=-1$ (b)(ii) $\ln 8+(x-2)-\frac{1}{4}(x-2)^2+\ldots$

20 $a=2,\ b=2,\ c=\frac{8}{3}$

21 (b) $1+2x+3x^2+\frac{11}{3}x^3+...$ (c) $2+6x+11x^2+...$

22 (a) $-x-x^2-\frac{x^3}{3}+\cdots$ (b) $a=-1, b=\frac{1}{3}, c=3$. Coeff $x^4 = -\frac{1}{27}$

23(a) $\cos(t^3)=1-\frac{(t^3)^2}{2!}+\frac{(t^3)^4}{4!}+\cdots=1-\frac{t^6}{2}+\frac{t^{12}}{24}+\cdots$; $\int_0^x t^2\cos(t^3)\mathrm{d}t \approx \frac{x^3}{3}-\frac{x^9}{18}+\frac{x^{15}}{360}$

23(b) $\int_0^{0.1} t^2\cos(t^3)\,\mathrm{d}t \approx 0.00033$

23(c) $\int_0^x t^2\cos(t^3)\mathrm{d}t=\frac{1}{3}[\sin(t^3)]_0^x=\frac{1}{3}\sin(x^3)$ $\int_0^{0.1} t^2\cos(t^3)\,\mathrm{d}t=\frac{1}{3}\sin(0.1^3)=0.00033$ (5 d.p.)

23(d) The approximation was accurate up to 5 d.p. Approximation was good since $x=0.1$ is close to zero.

24(a) $\ln a+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}+...$; (c) $1-\frac{1}{2}x^2+...$; (d) 3