ANS

CEDAR GIRLS' SECONDARY SCHOOL
Preliminary Examination 2021
Secondary Four

CANDIDATE NAME: Worked Solutions

CENTRE NUMBER: | | | | |

INDEX NUMBER: | | | | |

ADDITIONAL MATHEMATICS
Paper 2

4049/02
14 September 2021

2 hours 15 minutes

Candidates answer on the Question Paper.

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your centre number, index number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.
DO **NOT** WRITE ON ANY BARCODES.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use
90

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{r} a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$cosec^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

Answer **all** the questions.

1 **(a)** Express $\dfrac{16x-14x^2-1}{(3x-1)(2x+1)^2}$ in partial fractions. [5]

$$\frac{16x-14x^2-1}{(3x-1)(2x+1)^2}=\frac{A}{3x-1}+\frac{B}{2x+1}+\frac{C}{(2x+1)^2}$$

$$16x-14x^2-1=A(2x+1)^2+B(3x-1)(2x+1)+C(3x-1)$$

Let $x=-\frac{1}{2}$, Let $x=\frac{1}{3}$, Let $x=0$,

$-\frac{5}{2}C=-\frac{25}{2}$ $\quad$ $\frac{25}{9}A=\frac{25}{9}$ $\quad$ $B=-3$

$C=5$ $\quad$ $A=1$

$$\frac{16x-14x^2-1}{(3x-1)(2x+1)^2}=\frac{1}{3x-1}-\frac{3}{2x+1}+\frac{5}{(2x+1)^2}$$

(b) A right pyramid has a square base of side $\left(\sqrt{6}+\sqrt{3}\right)$ m and a height of h m. The volume of the pyramid is $\left(6+\sqrt{8}\right)\text{m}^3$. Without using a calculator, show that h can be expressed as $a+b\sqrt{2}$, where a and b are integers. [3]

$$\frac{1}{3}\left(\sqrt{6}+\sqrt{3}\right)^2 \times h = 6+\sqrt{8}$$

$$h=\frac{18+6\sqrt{2}}{\left(\sqrt{6}+\sqrt{3}\right)^2}$$

$$=\frac{18+6\sqrt{2}}{9+6\sqrt{2}}$$

$$=\frac{18+6\sqrt{2}}{9+6\sqrt{2}}$$

$$=\frac{6+2\sqrt{2}}{3+2\sqrt{2}}$$

$$=\frac{6+2\sqrt{2}}{3+2\sqrt{2}}\times\frac{3-2\sqrt{2}}{3-2\sqrt{2}}$$

$$=10-6\sqrt{2}$$

2 **(a)** Given that $p > \frac{1}{3}$, explain why the equation $3^{2x+1} = 6(3)^{x-1} - p$ has no real solutions. [4]

$3^{2x+1} = 6(3)^{x-1} - p$

$3.3^{2x} = 6(3)^x 3^{-1} - p$

$3.3^{2x} - 2(3)^x + p = 0$

Let $3^x = a$,
$3a^2 - 2a + p = 0$

Discriminant $= (-2)^2 - 4(3)(p)$
$= 4 - 12p$

Given $p > \frac{1}{3}$,
$-12p < -4$,
$4 - 12p < 0$

Since discriminant < 0, the equation has no real solutions.

(b) Find the **exact** range of values of the constant a for which the line $y = 2x - \frac{a^2}{2}$ intersects the curve $y = x^2 - ax - 4$ at two distinct points. [5]

$$x^2 - ax - 4 = 2x - \frac{a^2}{2}$$
$$x^2 + (-a-2)x + \frac{a^2}{2} - 4 = 0$$

$$\text{Discriminant} = (-a-2)^2 - 4(1)\left(\frac{a^2}{2} - 4\right)$$
$$= -a^2 + 4a + 20$$

Since line intersects the curve at 2 distinct points,

Discriminant > 0,
$-a^2 + 4a + 20 > 0$,
$a^2 - 4a - 20 < 0$

Let $a^2 - 4a - 20 = 0$

$$a = \frac{4 \pm \sqrt{16 - 4(1)(-20)}}{2}$$
$$= \frac{4 \pm \sqrt{96}}{2}$$
$$= \frac{4 \pm 4\sqrt{6}}{2} = 2 \pm 2\sqrt{6}$$

$$2 - 2\sqrt{6} < a < 2 + 2\sqrt{6}$$

3 The equation of a curve is $y = \cos^3\left(\frac{x}{2}+\frac{\pi}{4}\right)$.

Find the value(s) of x for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ where the equation of the normal to the curve is $x = k$, where k is a constant. [4]

$$y = \cos^3\left(\frac{x}{2}+\frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = \frac{3}{2}\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)\left(-\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)$$

$$\frac{3}{2}\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)\left(-\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\right) = 0$$

$$\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right) = 0 \quad \text{or} \quad -\sin\left(\frac{x}{2}+\frac{\pi}{4}\right) = 0$$

$$\frac{x}{2}+\frac{\pi}{4} = \frac{\pi}{2}, \qquad \frac{x}{2}+\frac{\pi}{4} = 0$$

$$x = \frac{\pi}{2} \quad \text{or} \quad x = -\frac{\pi}{2}$$

4 A glass of hot liquid is placed on a table to cool. The following table shows the measured values of the temperature of the liquid, T°C after n minutes.

n (minutes)	10	20	30	50
T (°C)	76.5	57.4	45.9	34.6

(i) On the grid on page 9, plot $\ln(T-28)$ against n and draw a straight line graph. [2]

(ii) Use your graph to estimate the temperature of the liquid after 35 minutes. [2]

From the graph, $\ln(T-28)=2.64$

$T=42.0$°C

(iii) It is given that the formula connecting T and n is $T=28+ae^{-kn}$, where a and k are constants.

Use your graph to estimate

(a) the value of a and of k, [4]

$T=28+ae^{-kn}$

$\ln(T-28)=\ln(ae^{-kn})$

$\ln(T-28)=-kn+\ln a$

Gradient $=-k$

$y-$intercept $=\ln a$

$$\text{Gradient}=\frac{3.88-2.88}{10-30}$$

$$=-\frac{1}{20}$$

$k=\frac{1}{20}$ or 0.05

$y-$intercept $=4.38$

$\ln a=4.38$

$a=e^{4.38}=79.8$ (3 s.f)

(b) the time taken for the temperature to fall to half of its initial value. [2]

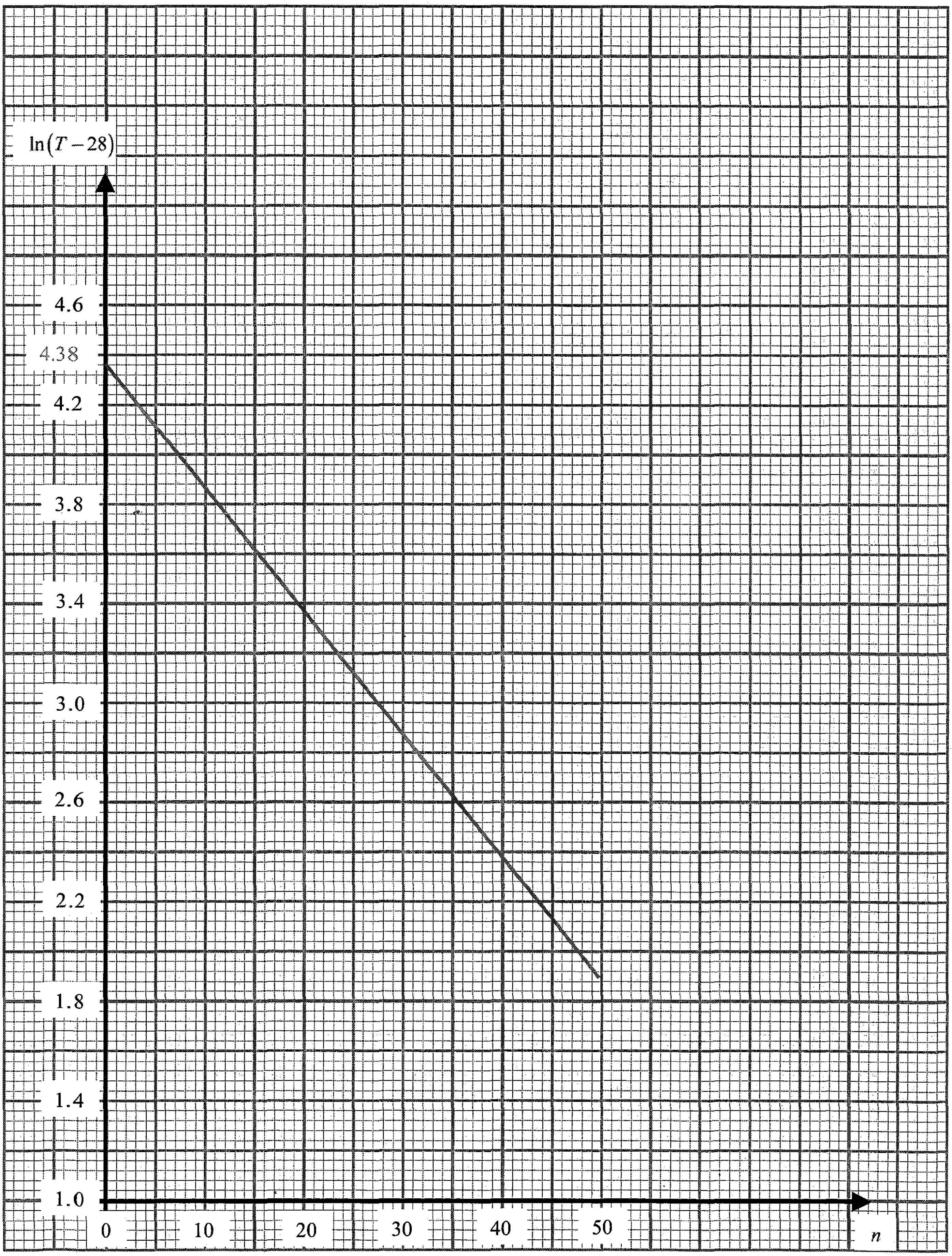
$\ln(T-28)=4.38$

Initial temperature $=107.8$°C

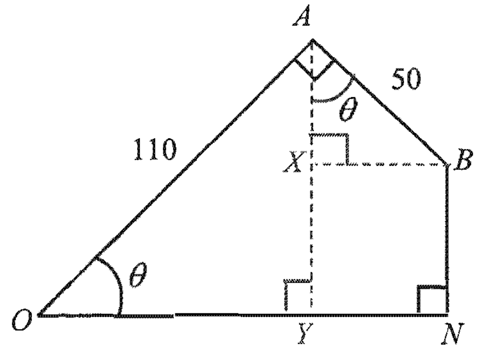
Half of its initial temperature $=53.9$°C

$\ln(53.9-28)=3.2542$

From graph, time taken $=23$ minutes


ln(T − 28)
4.6
4.38
4.2
3.8
3.4
3.0
2.6
2.2
1.8
1.4
1.0
0
10
20
30
40
50
n

5



A farmer puts fences around the perimeter of his land $OABN$.

Angle OAB = angle $ONB = \frac{\pi}{2}$ and acute angle $AON = \theta$ radians.

$OA = 110$ m and $AB = 50$ m.

The length of fencing needed is L m.

(i) Show clearly that $L = p + 60\cos\theta + 160\sin\theta$, where p is a constant. [4]

$$\cos\theta = \frac{OY}{110}$$
$$OY = 110\cos\theta$$

$$\sin\theta = \frac{AY}{110}$$
$$AY = 110\sin\theta$$

$$\angle BAX = \theta$$

$$\cos\theta = \frac{AX}{50}$$
$$AX = 50\cos\theta$$

$$\sin\theta = \frac{BX}{50}$$
$$BX = 50\sin\theta$$
$$= YN$$

$$BN = AY - AX$$
$$= 100\sin\theta - 50\cos\theta$$

$$L = 110 + 110\cos\theta + 50\sin\theta + 110\sin\theta - 50\cos\theta + 50$$
$$= 160 + 60\cos\theta + 160\sin\theta \text{ (shown)}$$

(ii) Express L in the form $p+R\cos(\theta-\alpha)$ where $R>0$ and α is acute. [3]

$$L=160+60\cos\theta+160\sin\theta$$
$$=160+R\cos(\theta-\alpha)$$

$$R=\sqrt{60^2+160^2}$$
$$=\sqrt{29200} \quad \text{or} \quad 170.88 \text{ (5 s.f)}$$

$$\tan\alpha=\frac{160}{60}$$
$$\alpha=1.2120 \text{ (5 s.f)}$$

$$L=160+\sqrt{29200}\cos(\theta-1.21) \quad \text{or} \quad 160+171\cos(\theta-1.21)$$

(iii) Given that the farmer uses exactly 315 m of fencing, find the value of θ. [3]

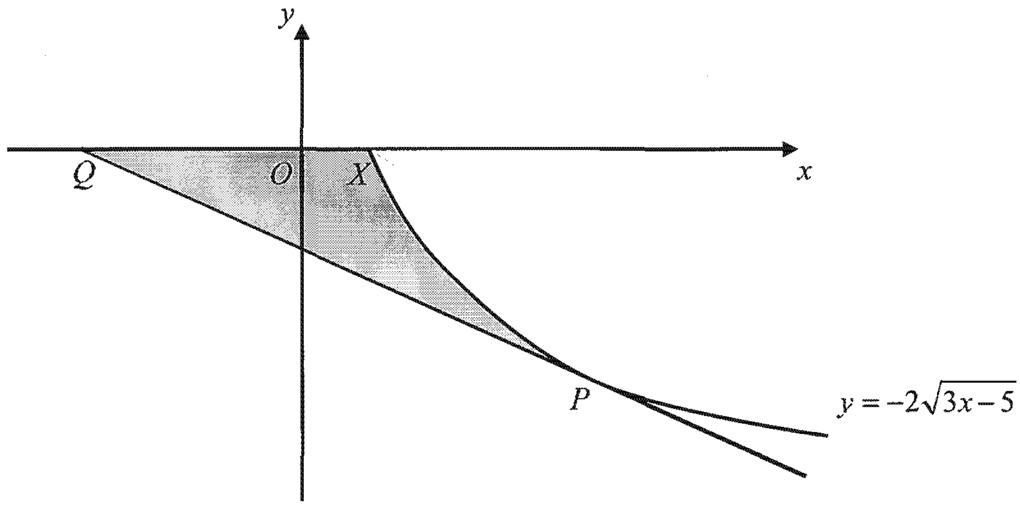
$$160+\sqrt{29200}\cos(\theta-1.2120)=315\,, \quad -1.2120<\theta-1.2120<0.35880$$

$$\cos(\theta-1.2120)=0.90707$$

$$\theta-1.2120=0.43453 \text{ (rej)}, \quad 2\pi-0.43453 \text{ (rej)}, \quad -0.43453$$

$$\theta=0.777\text{(3 s.f)}$$

6



The diagram shows part of the curve $y=-2\sqrt{3x-5}$, $x \geq \frac{5}{3}$ passing through the point P where $x=10$. The curve meets the x-axis at the point X.
The tangent to the curve at P meets the x-axis at the point Q.

(i) Show that the x-coordinate of Q is $-\frac{20}{3}$. [4]

$y=-2\sqrt{3x-5}$

$\frac{dy}{dx}=\frac{-3}{\sqrt{3x-5}}$

When $x=10$,

$\frac{dy}{dx}=-\frac{3}{\sqrt{30-5}}=-\frac{3}{5}$

and $y=-10$

Equation of tangent

$y+10=-\frac{3}{5}(x-10)$

$y=-\frac{3}{5}x-4$

When $y=0$, $\frac{3}{5}x=-4$

$x=-\frac{20}{3}$

(ii) Find the area of the shaded region *PQX*. [5]

When $y = 0$, $-2\sqrt{3x-5} = 0$

$$3x - 5 = 0$$

$$x = \frac{5}{3}$$

Area of shaded region *PQX*

$$= \frac{1}{2}\left(10 + \frac{20}{3}\right)(10) - \left|(-2)\int_{\frac{5}{3}}^{10} (3x-5)^{\frac{1}{2}}\ dx\right|$$

$$= \frac{250}{3} - \left|\left[\frac{(-2)(3x-5)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(3)}\right]_{\frac{5}{3}}^{10}\right|$$

$$\frac{250}{3} - \left|\left[-\frac{4}{9}(3x-5)^{\frac{3}{2}}\right]_{\frac{5}{3}}^{10}\right|$$

$$\frac{250}{3} - \left|-\frac{4}{9}\left(25^{\frac{3}{2}} - 0\right)\right|$$

$$\frac{250}{3} - \frac{500}{9}$$

$$= \frac{250}{9} \text{ or } 27.8 \text{ (3 s.f) sq units}$$

7 **(a)** Show that the solution of the equation $\frac{(\log_x y)^4}{2\log_y x}+\log_3 9=-14$ can be written in the form $y=x^k$, where k is a constant. [4]

$$\frac{(\log_x y)^4}{2\log_y x}+\log_3 9=-14$$

$$\frac{(\log_x y)^4}{\frac{2\log_x x}{\log_x y}}+2=-14$$

$$\frac{(\log_x y)^5}{2}=-16$$

$$(\log_x y)^5=-32$$

$$\log_x y=-2$$

$$y=x^{-2}$$

(b) Show that $\log_5(2x-7)-\log_5(x+4)=1$ has no real solutions. [3]

$\log_5\left(\frac{2x-7}{x+4}\right)=1$

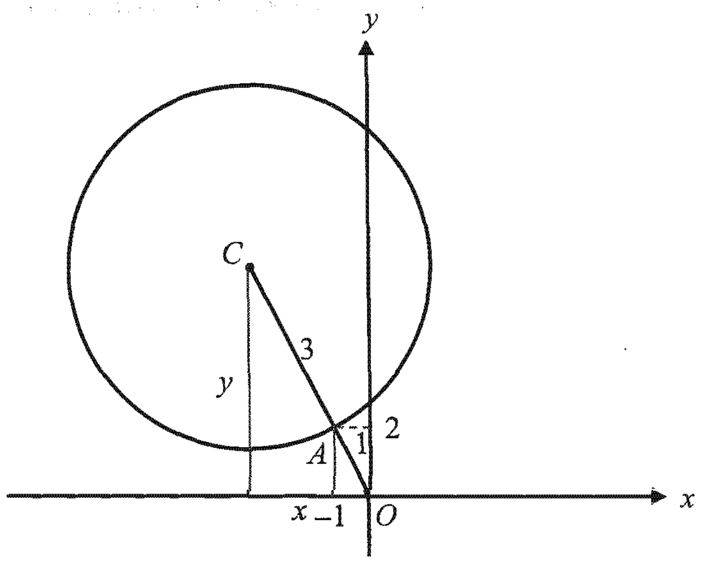
$\frac{2x-7}{x+4}=5$

$2x-7=5x+20$

$x=-9$

Sub $x=-9$ into $\log_5(2x-7)$ and $\log_5(x+4)$,
since $\log_5(-25)$ and $\log_5(-5)$ are undefined, the equation has no real solutions.

8



The diagram shows a circle, centre C passing through point A $(-1, 2)$.
A line from the origin O meets C such that $CA : AO$ is 3 : 1.

(i) Show that the centre of the circle is $(-4, 8)$ and hence find the equation of the circle. [3]

Let the vertical distance of the centre of circle from the x-axis be y and the horizontal distance of the centre of the circle from the y-axis be x.

Using similar triangles,

$\frac{x}{1} = \frac{4}{1}$, $\quad \frac{y}{2} = \frac{4}{1}$

$x = 4 \quad y = 8$

Thus, centre of circle is $(-4, 8)$

$$\text{Radius} = \sqrt{(-4+1)^2 + (8-2)^2}$$
$$= \sqrt{45}$$

Equation of circle

$(x+4)^2 + (y-8)^2 = 45$

(ii) The circle is reflected along the y-axis.
State the equation of the reflected circle. [2]

Centre of reflected circle = $(4, 8)$

Equation of circle

$(x-4)^2 + (y-8)^2 = 45$

(iii) Explain whether the point $(7, 6)$ lies on the tangent to the circle at A. [3]

Gradient of $AC = \dfrac{8-2}{-4+1} = -2$

Gradient of tangent to circle at $A = \dfrac{1}{2}$ (tan $\perp$ rad)

Equation of tangent at A

$$y-2=\frac{1}{2}(x+1)$$
$$y=\frac{1}{2}x+\frac{5}{2}$$

When $x=7, y=\frac{1}{2}(7)+\frac{5}{2}=6$

Yes, the point (7, 6) lies on the tangent to the circle at A.

(iv) Find the equation of another tangent to the circle that has the same gradient as the tangent to the circle at A. [2]

The point on the circle whose tangent has the same gradient as the tangent at A would be the other end point of the diameter passing through A.

Let this end point be (p, q)

$$\frac{p-1}{2}=-4$$
$$p=-7$$

$$\frac{q+2}{2}=8$$
$$q=14$$

The end point is $(-7, 14)$

Equation of another tangent to the circle:

$$y-14=\frac{1}{2}(x+7)$$
$$y=\frac{1}{2}x+\frac{35}{2}$$

9 (i) Find the value of the constant k for which $y = 2e^{-2x}x^2$ is a solution of the equation

$$4\frac{dy}{dx} + \frac{d^2y}{dx^2} + 4y = ke^{-2x}.$$ [6]

$y = 2e^{-2x}x^2$

$\frac{dy}{dx} = 4xe^{-2x} - 4x^2e^{-2x}$

$= e^{-2x}(4x - 4x^2)$

$\frac{d^2y}{dx^2} = e^{-2x}(4-8x) - 2e^{-2x}(4x-4x^2)$

$= e^{-2x}(8x^2 - 16x + 4)$

$4\frac{dy}{dx} + \frac{d^2y}{dx^2} + 4y$

$= 4e^{-2x}(4x-4x^2) + e^{-2x}(8x^2-16x+4) + 8e^{-2x}x^2$

$= e^{-2x}(16x - 16x^2 + 8x^2 - 16x + 4 + 8x^2)$

$= 4e^{-2x}$

Hence, $k = 4$

[3]

(ii) Find the exact coordinates of the stationary points of the curve $y=2e^{-2x}x^2$.

For stationary points,

$\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$e^{-2x}(4x-4x^2)=0$

$e^{-2x}=0$ (reject) or $4x-4x^2=0$

$x(1-x)=0$

$x=0$ or $x=1$

$y=0$ $y=\frac{2}{e^2}$

The stationary points are $(0,0)$ and $\left(1,\frac{2}{e^2}\right)$

(iii) Find the nature of the stationary points. [3]

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=e^{-2x}(8x^2-16x+4)$

When $x=0$,

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=e^0(4)$

$=4$ (>0)

(0, 0) is a minimum point

When $x=1$,

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=e^{-2}(8-16+4)$

$=\frac{-4}{e^2}$ (<0)

$\left(1,\frac{2}{e^2}\right)$ is a maximum point

10 **(i)** Express $\frac{9x^2}{3x-1}$ in the form $ax+b+\frac{c}{3x-1}$, where a, b and c are constants. [1]

$$\frac{9x^2}{3x-1}=3x+1+\frac{1}{3x-1}$$

(ii) Find $\int \frac{9x^2}{3x-1}\,\mathrm{d}x$. [3]

$$\int \frac{9x^2}{3x-1}\,\mathrm{d}x$$
$$=\int 3x+1+\frac{1}{3x-1}\,\mathrm{d}x$$
$$=\frac{3}{2}x^2+x+\frac{1}{3}\ln(3x-1)+C$$

(iii) Given that $y=3x^2\ln(3x-1)$, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2\left(\frac{3}{3x-1}\right)+6x\ln(3x-1)$$
$$=\frac{9x^2}{3x-1}+6x\ln(3x-1)$$

(iv) Hence, evaluate $\int_1^2 x\ln(3x-1)\ \mathrm{d}x$. [5]

$$\int_1^2 \frac{9x^2}{3x-1}+6x\ln(3x-1)\ \mathrm{d}x=\left[3x^2\ln(3x-1)\right]_1^2$$

$$\left[\frac{3}{2}x^2+x+\frac{1}{3}\ln(3x-1)\right]_1^2+\int_1^2 6x\ln(3x-1)\ \mathrm{d}x=12\ln 5-3\ln 2$$

$$\left(6+2+\frac{\ln 5}{3}\right)-\left(\frac{3}{2}+1+\frac{\ln 2}{3}\right)+6\int_1^2 x\ln(3x-1)\ \mathrm{d}x=12\ln 5-3\ln 2$$

$$6\int_1^2 x\ln(3x-1)\ \mathrm{d}x=17.234-5.8054$$

$$\int_1^2 x\ln(3x-1)\ \mathrm{d}x=\frac{11.4286}{6}=1.90\ (3\text{ s.f})$$

End of Paper