Differentiation: Tangents and Normals

1. **AJC/2009/I/7**

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos^{-1}\left(\frac{1}{x^2}\right)\right]$ and simplify your answer. [2]

(b) The curve C has parametric equations $x = \ln(\sin t)$, $y = \cot t$, where $0 < t < \pi$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{\sin 2t}$. [2]

P is the point on the curve C where the normal is parallel to the line $2y = x - 2$. Find the equation of the normal at P. [2]

This normal intersects the x-axis at the point Q. R is a point on the curve C such $PQ = RQ$. Find the equation of the normal to the curve C at the point R. [2]

2. **AJC/2013/II/4**

A curve C has parametric equations

$$x = \frac{1}{t} + \tan^{-1}(t) \quad , \quad y = \frac{1}{t} - \tan^{-1}(t) \quad \text{where } t \in \mathbb{R}, t \neq 0.$$

The curve has an oblique asymptote $y = ax$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t and show that $a = 1$. [3]

(ii) Sketch C, showing clearly all the asymptote(s), axial intercepts and end points. [3]
Hence find the range of values of k such that the line $y = kx$ does not intersect C. [1]

(iii) Find the equation of the normal to C at the point where $t = 1$. Hence find the value of the parameter t at the point where the normal intersects C again. [3]

3. **RI/2010/I/10**

The curve C has equation $y = x\mathrm{e}^{-x}$ for $x > 0$ and $P(a, a\mathrm{e}^{-a})$ is a point on C.

(i) Sketch the curve C. [1]

(ii) Find, in terms of a, the equation of the tangent to the curve at P. [2]
This tangent cuts the $y-$axis at the point $Q(0, h)$.
Using differentiation, find, as a varies, the exact maximum value of h. [6]

4. **IJC/2011/II/3**

The curve C has parametric equations

$$x = a\left(1+\frac{1}{t}\right) \quad , \quad y = a\left(t - \frac{1}{t^2}\right),$$

where a is a positive constant and $t \neq 0$.

The point P on the curve has parameter $t = -\frac{1}{2}$. The tangent at P meets the curve again at the point Q.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [2]

(ii) Show that the tangent at P has equation $4y = 15x - 3a$. [3]

(iii) Find the coordinates of Q. [3]

(iv) By considering what happens to x and y for large values of t, find the equation of the asymptote of the curve C. Sketch the curve C. [2]

(v) Find, by a non-calculator method, the area of the region bounded by the curve C, the tangent at P and the x-axis. [4]

5. **NYJC/2011/I/4**

The parametric equations of a curve C are

$$x = a\sin 2t \qquad \text{and} \qquad y = a\cos t,$$

where a is a positive constant and $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

(i) Find the equation of the tangent to the curve at the point P where $t = \frac{\pi}{4}$. [3]

(ii) The normal to the curve at the point Q where $t = \frac{\pi}{3}$ intersects the x-axis at R. Find the coordinates of R and hence show that the area enclosed by the normal at Q, the tangent at P and the x-axis is $a^2\left(\frac{43\sqrt{3}}{48} - \frac{3}{2}\right)$. [5]

6. **ACJC/2011/I/12**

A curve with equation $y = \mathrm{f}(x)$ is also defined by the parametric equations

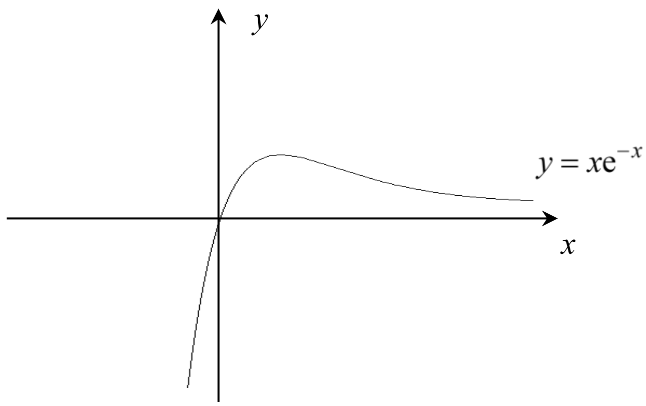
$$x = 1 - \mathrm{e}^{-t}, \qquad y = 1 + t^2, \qquad t \in \mathbb{R}.$$

(i) The point P on the curve has parameter p. Given that the tangent to the curve at P passes through the point (1, 0), find the value of p.

(ii) Given that the graph of $y = \mathrm{f}(x)$ has an inflexion point at $t = -1$, sketch, on separate diagrams, the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}'(x)$, showing clearly the exact coordinates of the turning point(s) and asymptote(s), if any.

7. **NYJC/2011/I/9**

The diagram shows a sketch of the curve $y = x\mathrm{e}^{-x}$.



(i) By differentiation, find the range of values of x for which the graph of $y = x\mathrm{e}^{-x}$ is decreasing. [3]

(ii) Determine the range of values of x for which the graph of $y = x\mathrm{e}^{-x}$ is decreasing and concave downwards. [3]

(iii) The tangent to the curve at $P(x, y)$ meets the y-axis at $R(0,\ h)$. Express h in terms of x, and find the greatest possible value of h. [6]

8. **PJC/2011/I/7**

A curve is defined by the parametric equations $x = \frac{a}{t^3}$, $y = \frac{a}{t}$ where a is a constant.

(i) Find the equations of the tangent and the normal at point P where $t = \frac{1}{2}$. [4]

(ii) Find the coordinates of the point where the tangent cuts the curve again. [2]

(iii) The tangent at P meets the x-axis at Q and the normal at P meets the x-axis at R. Show that the area of triangle PQR is $\frac{145}{6}a^2$. [2]

9. **IJC/2012/I/4**

The equation of a curve is given by

$$xy - 2y^2 + 4x^2 = 66.$$

(i) Find the exact coordinates of the points on the curve where the tangent is parallel to the y-axis. [4]

(ii) Show that every line parallel to the x-axis cuts the curve at two distinct points. [3]

10. **NYJC/2012/I/3**

The equation of a curve is given by $4x^3+3x^2y=y^3-2$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y, simplifying your answer. [2]

The curve meets the line $y=-x$ at point P. Find

(i) the coordinates of P and [2]

(ii) the equation of the tangent at P. [2]

The tangent to the curve at P cuts the x-axis at R and the normal to the curve at P cuts the y-axis at Q. If O denotes the origin, use the results above to give a geometrical description of the quadrilateral $OQPR$. [1]

11. **EJC Prelim 9758/2018/01/Q3[Modified]**

The parametric equations of curve C, is given as $x=at,\ y=at^3$, where a is a positive constant.

(i) The point P on the curve has parameter p and the tangent to the curve at point P cuts the y-axis at S and the x-axis at T. The point M is the midpoint of ST. Find a Cartesian equation of the curve traced by M as p varies. [5]

(ii) Find the exact area of triangle OPS, giving your answer in terms of a and p. [2]

12. **YJC/2013/I/7**

The equation of a curve is $y^2-xy=-1$.

(i) Find the equations of all tangents to the curve that are parallel to the y-axis. [4]

(ii) State and justify whether the curve has any stationary points. [2]

(iii) Find the area of the region bounded by the axes, and the normal to the curve at the point where the y-coordinate is 2. [5]

13. **SRJC/2013/I/10**[Modified]

(a) Find the coordinates of the point(s) to the curve

$$2x^2+xy-y^2=9$$

at which the tangent is parallel to the $y-$axis. [5]

(b) The curve C has parametric equations

$$x=t^2+t,\ y=4-t.$$

(i) The point P on the curve has parameter p. Show that the equation of the tangent at P is $(2p+1)(4-p-y)=x-p^2-p$. [2]

(ii) Hence, show that every tangent to the curve C does not meet the curve again. [3]

(iii) Find the acute angle the tangent at P makes with the x axis when p = 1. [1]

14. **DHS/2014/I/11**

A curve has parametric equations

$$x = \tan t, \quad y = 2\cos t, \quad \text{for } 0 \le t < \frac{\pi}{2}.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. What can be said about the tangent to the curve as $t \to 0$? Hence sketch the curve. [4]

(ii) The point P on the curve has a non-zero parameter p. Given that the normal to the curve at P passes through the origin, find the value of p.
Hence show that equation of the normal to the curve which passes through the origin is given by $y = x\sqrt{2}$. [5]

(iii) Find the exact area of the region bounded by the curve, the line $y = x\sqrt{2}$ and the y-axis. [5]

15. **SAJC/2019/I/5**

A curve C is determined by the parametric equations

$$x = at^2, \quad y = 2at, \quad \text{where } a > 0.$$

(i) Sketch C. [1]

(ii) Find the equation of the normal at a point P, with non-zero parameter p. [2]

(iii) Show that the normal at the point P meets C again at another point Q, with parameter q, where $q = -p - \frac{2}{p}$. Hence show that $|PQ|^2 = \frac{16a^2}{p^4}(p^2+1)^3$. [4]

(iv) Another point R on C with parameter r, is the point of intersection of C and the circle with diameter PQ. By considering the gradients of PR and QR, show that $p^2 - r^2 + 2\left(\frac{r}{p}\right) = 2$. [3]

16. **DHS/2018/2/5**

(a) A curve has the equation $(x+y)^2 = 4\mathrm{e}^{xy}$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [2]

(ii) Given that the curve cuts the positive y-axis at point A, find the equation of the tangent to the curve at A. [2]

(iii) The tangent to the curve at A meets the curve at another point B. Find the coordinates of B. [3]

(b) A closed cylinder is designed to contain a fixed volume of p cm^3 of liquid such that its external surface area is a minimum. Find the radius of the cylinder in terms of p in cm. [5]

17. **ACJC/2021/I/Q2**

A curve C has equation $\dfrac{x^3-2y^2}{x^2+3xy}=1$.

(i) The points P and Q on C each have x-coordinate 1. Find the exact gradients of the tangents to C at the points P and Q. [5]

(ii) Find the acute angle between the tangents to C at the points P and Q. [2]

18. **ASRJC/2022/I/Q7**

A curve C has parametric equations

$$x=\sin^3 t\ ,\quad y=\cos^2 t,\ -\frac{\pi}{2}<t<0\,.$$

The tangent at the point $P(\sin^3 p, \cos^2 p)$, $-\dfrac{\pi}{2}<p<0$, meets the x-axis and y-axis at Q and R respectively.

(i) By finding the equation of the tangent at the point P, show that the area of the triangle OQR is $-\dfrac{1}{12}\sin p\ (2+\cos^2 p)^2$. [6]

(ii) Find a cartesian equation of the locus of the mid-point of QR as p varies. You need not indicate its domain. [5]

19. **EJC Prelim 9758/2024/02/Q2**

A curve C has equation $y^{y+1}=x^x$, for $x>0$ and $y>0$.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [4]

(b) Hence find the equation of the tangent to the curve $y^{y+1}=x^x$ which is parallel to the x-axis. [3]

(c) Using the result $t>\ln t$ for all $t>0$, determine with reason whether there exists any tangent parallel to the y-axis. [2]

20. **NYJC Prelim 9758/2024/01/Q2**

The path traced by a moving particle is a curve C, given by

$$x=a\theta-a\sin\theta \text{ and } y=a-a\cos\theta,$$

where a is a positive constant and $0\le\theta<2\pi$.

(a) Show that the gradient of C at a point with parameter θ, is $\cot\left(\dfrac{1}{2}\theta\right)$. [2]

(b) Find the equation of the tangent to C at the point where $\theta=\dfrac{\pi}{3}$ and show that this tangent passes through the point $\left(\dfrac{1}{3}\pi a,\ 2a\right)$. [4]

21. **NYJC Prelim 9758/2025/P2/Q4**

The equation of a curve C is $x^3+xy+2y^3=k$, where k is a constant.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [2]

It is given that C has a tangent which is parallel to the y-axis.

(b) Show that the y-coordinate of the points of contact of the tangent with C must satisfy $216y^6+4y^3+k=0$. Hence show that $k \le \frac{1}{54}$. [6]

(c) Find the possible values of k in the case where the line $x=-6$ is a tangent to C. [2]

22. **ACJC Prelim 9758/2025/P1/Q11**

The curve C is defined by the parametric equations $x=at^2$, $y=\frac{a}{t}$, $t\neq 0$, where a is a positive constant.

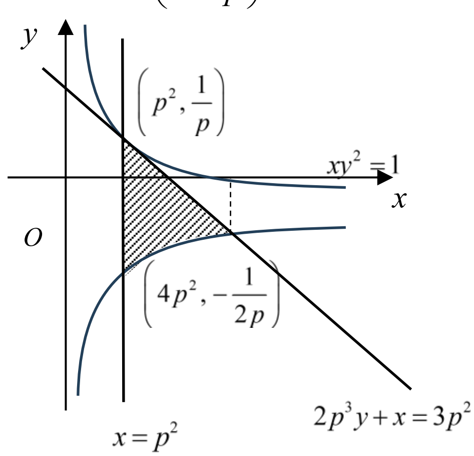
(a) Show that the equation of the tangent to the curve at the point $P\left(ap^2, \frac{a}{p}\right)$ is $2p^3y+x=3ap^2$. [2]

(b) Find the equations of the tangents to C at the points where $x=a$. Find also the acute angle between these tangents. [3]

(c) **(i)** Show that $(q-p)^2(q+2p)=q^3-3p^2q+2p^3$. [1]

(ii) The tangent to the curve at P cuts the curve again at $Q\left(aq^2, \frac{a}{q}\right)$. Find q in terms of p. [3]

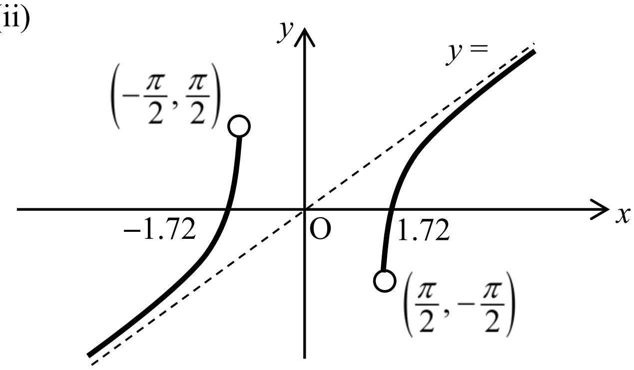
The following diagram shows the graph of C for $a=1$, and the Cartesian equation of the curve C is $xy^2=1$. The tangent to C at the point $\left(p^2, \frac{1}{p}\right)$ cuts the curve again at the point $\left(4p^2, -\frac{1}{2p}\right)$.



The region S is bounded by the tangent to C at the point $\left(p^2, \frac{1}{p}\right)$, the curve C and the line $x=p^2$.

(d) Find the area of S in terms of p. [3]

Answers

1	(a) $\dfrac{2}{x\sqrt{x^4-1}}$ (b) $y=\dfrac{1}{2}\left(x+\dfrac{1}{2}\ln(2)\right)+1$; $y=-\dfrac{1}{2}\left(x+\dfrac{1}{2}\ln(2)\right)-1$
2	(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2t^2+1$ (ii)  For $y=kx$ not to intersect the curve, $k\geq 1$ or $k\leq -1$ $3\left(y-1+\dfrac{\pi}{4}\right)=-\left(x-1-\dfrac{\pi}{4}\right)$, $t=$ - 8.41
3	(ii) $y=e^{-a}(1-a)(x-a)+ae^{-a}$, max $h=4\mathrm{e}^{-2}$
4	(i) $-t^2\left(1+\dfrac{2}{t^3}\right)$ (iii) $\left(\dfrac{5}{4}a,\dfrac{63}{16}a\right)$ (iv) $x=a$ (v) $a^2\left(\dfrac{1113}{640}+\ln 4\right)$
5	(i) $x=a$ (ii) R: $\left(\dfrac{a3\sqrt{3}}{4},0\right)$
6	(i)-1
7	(i) $x>1$; (ii) graph to be decreasing and concave downwards: $1<x<2$ (iii) $h=(x)^2\,\mathrm{e}^{-x}$, $h=4\mathrm{e}^{-2}$
8	(i) $y=\dfrac{1}{12}x+\dfrac{4}{3}a$, $y=-12x+98a$ (ii) $(-64a,-4a)$
9	(i) $(4,1)$, $(-4,1)$
10	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{4x^2+2xy}{y^2-x^2}$, (i) (–1, 1), (ii) $x=-1$, $OQPR$ is a square
11	(i) $y=-\dfrac{27x^3}{a^2}$, (ii) a^2p^4
12	(i) $x=2$ and $x=-2$; (ii) no stationary points; (iii) 10.0
13	(a) (2, 1) and (–2, –1) , (b)(iii) $\theta=0.322\,\text{rad}$

14	(i) $-2\sin t\cos^2 t$; the tangent becomes parallel to the x-axis/tangent is a horizontal line. (ii) $p=\frac{\pi}{4}$ (iii) $2\ln\left(\sqrt{2}+1\right)-\frac{\sqrt{2}}{2}$ unit^2
15	(ii) $y=2ap-p(x-ap^2)$, (iii) $\frac{16a^2}{p^4}(p^2+1)^3$
16	(a)(i) $\frac{2y\mathrm{e}^{xy}-y-x}{y+x-2x\mathrm{e}^{xy}}$ (a)(ii) $y=x+2$ (a)(iii) $(-2,0)$ (b) Minimum at $r=\left(\frac{p}{2\pi}\right)^{\frac{1}{3}}$
17	(i) $\frac{1}{3}$, $-\frac{11}{6}$ (ii) 79.8° or 1.39 rad
18	(ii) $54y^3=27y^2-4x^2$
19	(a) $\frac{y(1+\ln x)}{(y+1+y\ln y)}$ (b) $y=0.817$
20	(b) $y=\sqrt{3}x-\frac{\sqrt{3}}{3}\pi a+2a$
21	(a) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-y-3x^2}{x+6y^2}$ (b) $k\le\frac{1}{54}$ (c) $k=-220$ or -212
22	(b) $y=-\frac{1}{2}x+\frac{3a}{2}$, $y=\frac{1}{2}x-\frac{3a}{2}$ (c) $q=-2p$ (d) $\frac{11}{4}p$