

Topic: Polynomials

Question 1 [2008 EOY, Q11]	marks
(i) (a) Show that $2x+1$ is a factor of $2x^3 + x^2 - 8x - 4$. <u>Solution:</u> $\text{Sub } x = -\frac{1}{2}: 2\left(-\frac{1}{8}\right) + \frac{1}{4} - 8\left(-\frac{1}{2}\right) - 4 = -\frac{1}{4} + \frac{1}{4} + 4 - 4 = 0$ Hence $2x+1$ is a factor of $2x^3 + x^2 - 8x - 4$	[1]
(i) (b) Solve for x if $2x^3 + x^2 - 8x - 4 = 0$. <u>Solution:</u> By long division, $(2x+1)(x^2 - 4) = 0$ $\text{Hence } (2x+1)(x^2 - 4) = 0 \Rightarrow x = -\frac{1}{2} \text{ or } \pm 2$	[4]
(ii) The expression $4x^3 - 20x^2 + px + 12$ leaves a remainder of $-10x + q$ when divided by $2x^2 - 3x + 1$. Find the values of p and q. <u>Solution:</u> $4x^3 - 20x^2 + px + 12 \equiv (2x^2 - 3x + 1)Q(x) - 10x + q$ $4x^3 - 20x^2 + px + 12 \equiv (2x-1)(x-1)Q(x) - 10x + q$ $\text{Sub } x = \frac{1}{2}, 4\left(\frac{1}{8}\right) - 20\left(\frac{1}{4}\right) + \frac{1}{2}p + 12 = -10\left(\frac{1}{2}\right) + q \Rightarrow p = 2q - 25$ $\text{Sub } x = 1, 4 - 20 + p + 12 = -10 + q \Rightarrow p = -6 + q$ $\text{Hence } -6 + q = 2q - 25 \Rightarrow q = 19 \text{ and } p = 13$	[5]

Question 2 [2010 EOY, Q12]	marks
(a) Solve for x when $6x^3 + 7x^2 - 14x - 8 = 0$. <u>Solution:</u> By trial and error, when $x = -2$, $6x^3 + 7x^2 - 14x - 8 = 0$ hence $(x + 2)$ is a factor. By long division, $6x^3 + 7x^2 - 14x - 8 = (x + 2)(6x^2 - 5x - 4)$ Hence, $x = -2$ or $x = \frac{5 \pm \sqrt{25 - 4(-24)}}{12}$ [M1] $= -\frac{1}{2}$ or $\frac{4}{3}$	[3]
(b) It is given that $12x^3 + ax^2 + 15x + b \equiv (2x^2 + 5x - 3)Q(x) + 158x - 83$ where $Q(x)$ is a polynomial. (i) State the degree of $Q(x)$. <u>Solution:</u> Degree of $Q(x)$ is 1 as it is linear	[1]
(ii) Find the values of a and b. <u>Solution:</u> $12x^3 + ax^2 + 15x + b \equiv (2x^2 + 5x - 3)Q(x) + 158x - 83$ $12x^3 + ax^2 + 15x + b \equiv (2x - 1)(x + 3)Q(x) + 158x - 83$ Sub $x = \frac{1}{2}$ to get $a + 4b = -52$ Sub $x = -3$ to get $9a + b = -188$ Solve simultaneously to get $a = -20$ and $b = -8$	[3]

Question 3 [2011 EOY, Q11]	marks
(i) Given that $f(x) = x^3 - 2\sqrt{3}x^2 + 2(A - x)$ has a factor $(x + \sqrt{2})$ and the other two roots of the equation $f(x) = 0$ are $\sqrt{2}$ and k. Find the value of A and of k. <u>Solution:</u> $f(-\sqrt{2}) = 0$ $(-\sqrt{2})^3 - (2\sqrt{3})(-\sqrt{2})^2 + 2(A + \sqrt{2}) = 0 \quad f(x) = x^3 - 2\sqrt{3}x^2 + 2(2\sqrt{3} - x)$ $-2\sqrt{2} - 4\sqrt{3} + 2A + 2\sqrt{2} = 0 \quad = x^3 - 2\sqrt{3}x^2 - 2x + 4\sqrt{3}$ $A = 2\sqrt{3}$ $f(x) = (x - \sqrt{2})(x + \sqrt{2})(x - k)$ $2k = 4\sqrt{3} \Rightarrow k = 2\sqrt{3}$	[4]
(ii) Solve the equation $3x^3 = 2x^2 + 7x + 2$. <u>Solution:</u> $3x^3 - 2x^2 - 7x - 2 = 0$ Found 1 linear factor Correct factorization into linear and quadratic e.g. $(x - 2)(3x^2 + 4x + 1) = 0$ Correct factorization into 3 linear factors $(x - 2)(3x + 1)(x + 1) = 0$ $x = 2, -\frac{1}{3}, -1$	[4]

Question 4 [2012 EOY, Q8]	marks
The cubic polynomial $f(x)$ is such that the coefficient of x^3 is 2 and the roots of $f(x) = 0$ are $-1, 2k$ and k^2. It is given that $f(x)$ has a remainder of 168 when divided by $x - 2$. (i) Show that $k^3 - k^2 - 2k - 12 = 0$. <u>Solution:</u> $f(x) = 2(x+1)(x-2k)(x-k^2)$ <i>Ability to express f as a product of three linear factors and a constant 2.</i> $f(2) = 2(3)(2-2k)(2-k^2) = 168$ $4 - 2k^2 - 4k + 2k^3 = 28$ $k^3 - k^2 - 2k - 12 = 0$	[3]
(ii) Hence find a value for k and show that there are no other real values of k which satisfy this equation. <u>Solution:</u> $f(k) = k^3 - k^2 - 2k - 12$ $f(3) = 3^3 - 3^2 - 6 - 12 = 0$ <i>By trial and error, $k = 3$</i> $f(k) = (k-3)(ak^2 + bk + c) \text{ M1}$ $= (k-3)(k^2 + 2k + 4) \text{ M1}$ $k^2 + 2k + 4 = (k+1)^2 + 3 > 0$ <i>Hence, there is only one real value of k.</i>	[4]

(iii) Use your answer to part (ii) to solve the equation $12e^{-3x} + 2e^{-2x} + e^{-x} - 1 = 0$. <u>Solution:</u> $1 - e^{-x} - 2e^{-2x} - 12e^{-3x} = 0$ $e^{3x} - e^{2x} - 2e^x - 12 = 0$ $\text{Let } e^x = 3 \Rightarrow x = \ln 3$	[2]
Question 5 [2013 EOY, Q5] Given that $x^2 + x - 6$ is a factor of $2x^4 + x^3 - ax^2 + bx + a + b = 1$, find the value of a and of b. <u>Solution:</u> As $x^2 + x - 6 = (x - 2)(x + 3)$ Let $x = 2$ and subs into the eqn. $\Rightarrow a - b = 13$ ----- (1) Let $x = -3$ and subs into the eqn. $\Rightarrow 4a + b = 67$ ----- (2) Solve the above two equations, $a = 16$ and $b = 3$ $x = -3$	marks [4]
Question 6 [2014 EOY, Q7] (i) Solve the equation $2x^3 = 5x^2 - x - 2$. <u>Solution:</u> $2x^3 = 5x^2 - x - 2$ $2x^3 - 5x^2 + x + 2 = 0$ Let $f(x) = 2x^3 - 5x^2 + x + 2$ $\therefore f(1) = 0, \therefore (x - 1)$ is a factor of $f(x)$. Using long division, $(x - 1)(2x^2 - 3x - 2) = 0$ $(x - 1)(2x + 1)(x - 2) = 0$ $x = 1, -\frac{1}{2}$ or 2	marks [4]

(ii) Find the remainder when $(x-3)^{200} - (x-2)^{100}$ is divided by $x^2 - 5x + 6$. <u>Solution:</u> Let $f(x) = (x-3)^{200} - (x-2)^{100}$. When divided by $x^2 - 5x + 6$, $f(x) = (x^2 - 5x + 6)Q(x) + ax + b,$ where a and b are real constants. $f(x) = (x-2)(x-3)Q(x) + ax + b$ By Remainder Theorem, $f(2) = (2-3)^{200} - (2-2)^{100} = 1$ $\Rightarrow 2a + b = 1$ $f(3) = (3-3)^{200} - (3-2)^{100} = -1$ $\Rightarrow 3a + b = -1$ Solving the above two equations, $a = -2, b = 5$ $\therefore$ The remainder is $-2x + 5$.	[4]
Question 7 [2015 EOY, Q9] (i) Solve for x when $2(e^{3x} + 2) = e^x(7e^x + 5)$. <u>Solution:</u> $2(e^{3x} + 2) = e^x(7e^x + 5) \Rightarrow 2y^3 - 7y^2 - 5y + 4 = 0 \text{ where } y = e^x$ $y = -1$ is a root so $(y+1)(2y-1)(y-4) = 0 \Rightarrow e^x = -1 \text{ (rej.) or } \frac{1}{2} \text{ or } 4$ Hence $x = -\ln 2$ or $\ln 4$	marks [5]

(ii)

Given that $f(x) = 8x^4 + px^3 - 30x^2 + x + q$, where p and q are unknown constants, leaves a remainder of 2 when divided by $x - 2$ and a remainder of 9 when divided by $2x + 3$, find the values of p and q and hence find the remainder when $f(x)$ is divided by $2x^2 - x - 6$.

[6]

Solution:

$$f(2) = 2 \Rightarrow 8 + 8p + q = 0 \text{ and}$$

$$f\left(-\frac{3}{2}\right) = 9 \Rightarrow -27p + 8q = 300$$

Solving, $p = -4$ and $q = 24$

$$\text{Hence } f(x) = 8x^4 - 4x^3 - 30x^2 + x + 24$$

By long division, remainder is $6 - 2x$