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DIFFERENTIAL EQUATIONS

SYLLABUS

- Solving for the general solutions and particular solutions of differential equations of the form $\frac{dy}{dx} = f(x)g(y)$
 - Use direct integration to solve differential equations of the form $\frac{dy}{dx} = f(x)$;
 - Use the method of separable variables to solve differential equations of the form $\frac{dy}{dx} = g(y)$ and $\frac{dy}{dx} = f(x)g(y)$;
 - Use a given substitution to reduce a first-order differential equation to the form $\frac{dy}{dx} = f(x)g(y)$.
- Formulating a differential equation from a problem situation
 - Formulate a simple statement involving a rate of change as a differential equation, including the introduction of a constant of proportionality from a problem situation if necessary.
- Interpreting a differential equation and its solution in terms of a problem situation
 - Comment on the appropriateness of the models used and the assumptions made.

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LECTURE 1

Lesson Outline

- Introduction to ordinary differential equations.
 - Solving for the general solutions and particular solutions of first order differential equations of the form $\frac{dy}{dx} = f(x)g(y)$.
-

1 INTRODUCTION

A differential equation (DE) is an equation that contains one or more functions together with their derivatives.

DEs allow modelling of laws governing natural phenomena which involve quantities and their rates of change. Researchers solve the DEs to analyse the relationship between the variables and predict future behaviour. For this reason, DEs are used in physics, chemistry, biology and economics.

Example

A researcher observes that the rate of increase of the population of a particular species of bird is proportional to the number of birds, x , present at time t weeks.

This situation can be modelled by a differential equation in x ,

$$\frac{dx}{dt} = kx, \text{ where } k \text{ is a positive constant.}$$

Suppose $k = 0.01$. This means an increase (i.e. growth rate) of 0.01 birds per week for every existing bird at that instant.

At the instant when $x = 1000$, i.e. on the week that the number of birds is 1000, the rate of increase of the population is $0.01 \times 1000 = 10$ new birds.

At the instant when $x = 2000$, the rate of increase of population is $0.01 \times 2000 = 20$ new birds.

So, the rate of increase of population (at any instant) is the growth rate multiplied by the population at that instant.

Notice that the population is constantly increasing: the more birds we have, the more new baby birds we get.

The researcher can solve this differential equation, i.e. find x as a function of t , to predict the future population using this model.

DEFINITIONS

1. An **ordinary differential equation** is an equation relating x , y and the derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and so on.

For example, $y^2 \frac{dy}{dx} + 2xy = x$ and $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 4y = \sin x$.

2. The **order** of a differential equation is the order of the highest derivative that occurs in it.
3. The **degree** of a differential equation is the power of the highest derivative that occurs in it.

Differential Equation	Order	Degree
$x^2 \frac{dy}{dx} + y - 2 = 0$	1	1
$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - y = \sin x$	1	2
$\left(\frac{d^2y}{dx^2}\right)^4 + x^2 \left(\frac{dy}{dx}\right)^5 = e^{2x}$	2	4

4. The **general solution** of a differential equation of order n is a function that satisfies the differential equation and it **contains n arbitrary constants**.

Example

Solve the differential equation $\frac{dy}{dx} = 2x$. ---(1)

$$y = \int 2x \, dx = 2 \left(\frac{x^2}{2} \right) + C = x^2 + C$$

The general solution of the differential equation (of order 1) is $y = x^2 + C$. (It has one arbitrary constant, C .)

5. A **particular solution** of a differential equation is a solution of the differential equation which **contains no arbitrary constants**.

Example

$y = x^2 + 1$, $y = x^2 - 1$ and $y = x^2 + 2$ are some of the particular solutions of equation (1).

A particular solution of a differential equation can be found if **initial conditions** are given.

Example

Find the particular solution of $\frac{dy}{dx} = 2x$ given that $y = 7$ when $x = 2$.

This is an initial condition to find the specific value of C .

From (1), general solution is $y = x^2 + C$.

When $x = 2, y = 7, 7 = 2^2 + C \Rightarrow C = 3$

$\therefore$ The particular solution is $y = x^2 + 3$.

2 FIRST ORDER DIFFERENTIAL EQUATIONS**2.1 Differential Equations of the Form $\frac{dy}{dx} = f(x)$**

A differential equation of the form $\frac{dy}{dx} = f(x)$ can be solved by directly integrating the equation with respect to x .

For example, given the differential equation $\frac{dy}{dx} = x^2$, we integrate both sides w.r.t. x : $y = \int x^2 dx = \frac{1}{3}x^3 + C$.

This is the general solution for the given DE.

Example 1

Solve the following differential equations:

(a) $x \frac{dy}{dx} - 1 = 2x^2$

(b) $\frac{dw}{ds} = s^2 \ln s$

Solution

(a) Rearrange $x \frac{dy}{dx} - 1 = 2x^2$ into the form $\frac{dy}{dx} = f(x)$:

Integrate both sides w.r.t. x ,

Must use modulus for $\ln$.

(b) $\frac{dw}{ds} = s^2 \ln s$

Integrate both sides w.r.t. s ,

$$\int \frac{dw}{ds} ds = \int s^2 \ln s ds$$

RHS can be solved using integration by parts.

$$w = \frac{s^3}{3} \ln s - \int \frac{s^3}{3} \cdot \frac{1}{s} ds$$

$$w = \frac{s^3}{3} \ln s - \frac{1}{3} \int s^2 ds = \frac{s^3}{3} \ln s - \frac{s^3}{9} + C \quad \blacksquare$$

2.2 Differential Equations of the Form $\frac{dy}{dx} = g(y)$

We cannot integrate both sides with respect to x directly. We instead rearrange the differential equation $\frac{dy}{dx} = g(y)$ into the form $\frac{1}{g(y)} \frac{dy}{dx} = 1$.

Integrate both sides with respect to x ,

$$\int \frac{1}{g(y)} \frac{dy}{dx} dx = \int 1 dx \Rightarrow \int \frac{1}{g(y)} dy = \int 1 dx$$

We effectively **separated the variables** of the differential equation, putting the term containing y on the side with $\frac{dy}{dx}$.

For example, given the differential equation $\frac{dy}{dx} = 2y$, we rearrange the equation such that the variables are separated:

The variables are separated with y on LHS and 1 on RHS.	$\int \frac{1}{y} dy = 2 \int 1 dx$ $\ln y = 2x + C$ $ y = e^{2x+C}$ $y = \pm e^{2x+C}$
A is an arbitrary constant (independent of x).	$y = \pm e^{2x} e^C$ $y = A e^{2x}, A = \pm e^C$

Example 2

- (a) Find the general solution of the differential equation $y \frac{dy}{dx} - 1 = 2y^2$.
- (b) The gradient of a curve at the point (x, y) is given by $\frac{1+y^2}{y^2}$. Find the equation of the curve with this gradient function which passes through the origin.

Solution

(a) $y \frac{dy}{dx} - 1 = 2y^2$

$$\frac{dy}{dx} = \frac{2y^2 + 1}{y}$$

Do we need to include the modulus here?

We can stop here and leave the answer as above since the question did not require us to “express y in terms of x ”.

However, if we are required to express y in terms of x for **(a)**, the next few steps are required:

$$\ln(2y^2 + 1) = 4x + 4C$$

$$2y^2 + 1 = e^{4x+4C}$$

$$y^2 = \frac{e^{4x+4C} - 1}{2}$$

$$y^2 = \frac{e^{4x} \cdot e^{4C} - 1}{2}$$

$$y^2 = \frac{Ae^{4x} - 1}{2}, \text{ where } A = e^{4C}$$

$$y = \pm \sqrt{\frac{Ae^{4x} - 1}{2}}, \text{ where } A \text{ is an arbitrary constant.}$$

$$(b) \quad \frac{dy}{dx} = \frac{1+y^2}{y^2}$$

$$\frac{y^2}{1+y^2} \frac{dy}{dx} = 1$$

$$\left(1 - \frac{1}{1+y^2}\right) \frac{dy}{dx} = 1$$

$$\int \left(1 - \frac{1}{1+y^2}\right) dy = \int 1 dx$$

$$y - \tan^{-1} y = x + C$$

$$\text{At } (0,0), 0 = 0 + C \Rightarrow C = 0$$

$$\therefore \text{Equation of the curve is } y - \tan^{-1} y = x.$$

Use the given **initial condition** to obtain the **particular solution**.

2.3 Differential Equations of the Form $\frac{dy}{dx} = f(x)g(y)$

Can we generalise the method of solving $\frac{dy}{dx} = f(x)$ and $\frac{dy}{dx} = g(y)$ to solve differential equations of the form $\frac{dy}{dx} = f(x)g(y)$, now that the right-hand side is a product of a function of x and a function of y ?

A differential equation of the form $\frac{dy}{dx} = f(x)g(y)$ is said to be **separable** or to have **separable variables**. Rearranging the differential equation,

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x).$$

Integrate both sides with respect to x ,

$$\int \frac{1}{g(y)} \frac{dy}{dx} dx = \int f(x) dx \Rightarrow \int \frac{1}{g(y)} dy = \int f(x) dx.$$

We **separated the variables** of the differential equation, putting the term containing y on the side with $\frac{dy}{dx}$, and the term containing x on the other.

Example 3

Solve $(1+x)\frac{dy}{dx} = y$, expressing y in terms of x .

Solution

$$(1+x)\frac{dy}{dx} = y \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{1}{1+x}$$

Integrate both sides w.r.t. x ,

$$\therefore y = A(1+x), \text{ where } A = \pm e^C$$

■

Example 4

The gradient of a curve at any point (x, y) is given by the expression

$\frac{dy}{dx} = y + \frac{y}{x}$, $x \neq 0$. Find the equation of the curve if it passes through the point $(2, e^2)$.

Solution

The equation of the curve is $y = \frac{1}{2}xe^x$. ■

Extra Practice

Annex: Practice Questions 1 – 4

LECTURE 2

Lesson Outline

- Solve a first-order differential equation by using a given substitution to reduce it to the form $\frac{dy}{dx} = f(x)$, $\frac{dy}{dx} = g(y)$ or $\frac{dy}{dx} = f(x)g(y)$.
- Formulate and solve a differential equation from a problem situation.

3 REDUCTION OF A DIFFERENTIAL EQUATION BY MEANS OF A SUITABLE SUBSTITUTION

Not all differential equations can be solved by separating the variables. However, a suitable substitution may reduce the differential equation into the form $\frac{dy}{dx} = f(x)$, $\frac{dy}{dx} = g(y)$ or $\frac{dy}{dx} = f(x)g(y)$ which we can solve.

For example, given the differential equation $\frac{dy}{dx} = \frac{y-x+1}{y-x}$, we use the substitution $u = y - x$ **to reduce the equation into one with variables u and x** , which may then be solved by one of the methods discussed earlier.

Original DE in
variables y and x .

$$\frac{dy}{dx} = \frac{y-x+1}{y-x}$$

Step 1: Convert $\frac{dy}{dx}$ to $\frac{du}{dx}$ by differentiating the substitution equation: $u = y - x \Rightarrow \frac{du}{dx} = \frac{dy}{dx} - 1$

Step 2: Substitute $\frac{dy}{dx} = \frac{du}{dx} + 1$ and $u = y - x$ into DE.

$$\frac{du}{dx} + 1 = \frac{u+1}{u} \Rightarrow \frac{du}{dx} = \frac{1}{u}$$

Step 3: Solve the reduced DE in variables u and x .

Reduced DE in
variables u and x .

$$\int u \, du = \int 1 \, dx \Rightarrow \frac{u^2}{2} = x + C$$

Step 4: Final solution must be expressed in terms of the **original variables y and x** .

$$\frac{(y-x)^2}{2} = x + C$$

Example 5

Use the substitution $y = u + 3x$ to find the general solution of the differential equation $\frac{dy}{dx} = \frac{9x - 3y + 1}{3x - y + 1}$.

Solution

We want to reduce the differential equation in original variables **y and x** to a differential equation in variables **u and x**.

Step 1 Differentiate the given substitution to express $\frac{dy}{dx}$ in terms of $\frac{du}{dx}$.	$\frac{dy}{dx} = \frac{9x - 3y + 1}{3x - y + 1}$ Note that u is not a constant.
Step 2 Substitute $\frac{dy}{dx} = \frac{du}{dx} + 3$ and $y = u + 3x$ into the DE. Simplify to obtain a DE in terms of the variables u and x .	$\frac{du}{dx} = \frac{9x - 3y + 1}{3x - y + 1}$ Original DE in variables y and x . $\Rightarrow \frac{du}{dx} = \frac{2}{u - 1}$ Reduced DE in variables u and x .
Step 3 Solve the reduced DE.	$\frac{du}{dx} = \frac{2}{u - 1}$ $\Rightarrow \int (u - 1) du = \int 2 dx$ $\Rightarrow \frac{(u - 1)^2}{2} = 2x + C$ where C is an arbitrary constant
Step 4 Substitute the original variables back into the final answer.	Since $y = u + 3x$, $\therefore (y - 3x - 1)^2 = 4x + K$ where $K = 2C$ is an arbitrary constant

Example 6

Using the substitution $y = vx$, or otherwise, solve the differential equation

$$x \frac{dy}{dx} = y + x^2 \text{ where } y = 0 \text{ when } x = -1, \text{ expressing } y \text{ in terms of } x.$$

[J88/II/14b]

Solution

We want to reduce the differential equation in original variables **y and x** to a differential equation in variables **v and x** .

Step 1 Differentiate the given substitution to express $\frac{dy}{dx}$ in terms of $\frac{dv}{dx}$.	Note that v is not a constant. Therefore, we need to apply product rule in this case.
Step 2 Substitute $\frac{dy}{dx} = v + x \frac{dv}{dx}$ and $y = vx$ into the DE. Simplify to obtain DE in the two variables v and x .	$x \frac{dy}{dx} = y + x^2$ Original DE in variables y and x. $\Rightarrow \frac{dv}{dx} = 1$ Reduced DE in variables v and x.
Step 3 Solve the reduced DE.	
Step 4 Substitute the original variables back into the final answer.	Since $y = vx \Rightarrow v = \frac{y}{x}$, When $x = -1, y = 0$, $0 = 1 - C \Rightarrow C = 1$ $\therefore y = x^2 + x$

■

Extra Practice

Annex: Practice Question 5

4 FORMULATING AND SOLVING A DIFFERENTIAL EQUATION FROM A PROBLEM SITUATION

In some problems, the differential equation is not given directly, and we need to formulate the differential equation. Here are some examples:

Problem Situation	DE Formulation
When a cake is removed from the oven, its temperature decreases at a rate proportional to the positive difference between its temperature and the temperature of the room . The temperature of the room is constant at 25°C and T is the temperature of the cake t hours after removing from the oven. Form the differential equation.	
It is given that the population of an insect colony is x (in thousands) at time t (in months). The population increases at a rate that is proportional to its population x at time t. In any month, 3000 insects will die of natural causes. Form the differential equation.	
A contagious disease was found to infect a village with a population of 10000 people. Let x , in thousands, be the number of infected people t days after the start of the outbreak. The disease spread at a rate that was proportional to the product of the number of infected people and the number of non-infected people . Form the differential equation.	

Example 7

A radioactive substance decays so that the ¹rate of decrease of its mass at any time is proportional to the mass present at that time. For an experiment, an ²initial mass of 625 milligrams of a substance is prepared. Denoting by x the mass in milligrams remaining at time t days, write down a differential equation satisfied by x and show that $x = 625e^{-kt}$ where k is a positive constant. The ³mass of the substance is reduced to $\frac{4}{5}$ of its initial value in 30 days.

- (i) Determine the mass which is present 90 days after the preparation.
 (ii) Find the least number of days required for the substance to decay to less than one-fifth of its initial mass.

[Modified from JMB, Perkins & Perkins, Chapter 24]

Solution

$k > 0$ since the mass is decreasing with respect to time.	¹ Formulate the DE	Given: $\frac{dx}{dt} \propto x$
Must use modulus for ln. Do not solve for 'C' yet. Use $A = \pm e^C$ to simplify.	Solve for x in terms of t .	$ x = e^{-kt+C} = e^{-kt} \cdot e^C$ $x = \pm e^C e^{-kt}$ $= Ae^{-kt}, \text{ where } A = \pm e^C \text{ is an arbitrary constant}$
	² Use the initial condition to solve for A .	When $t = 0$, $x = 625$, $\therefore x = 625e^{-kt}$ (shown)
	³ Use the other initial condition to find k .	(i) When $t = 30$, $x = \frac{4}{5}(625)$,
	Put $t = 90$ to find x .	

Put $x < \frac{1}{5}(625)$ to find t .	(ii) For $x < \frac{1}{5}(625)$, $\therefore$ Least number of days = 217.
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LECTURE 3

Lesson Outline

- Interpret a differential equation and its solution in the context of a problem situation modelled by the equation.
- Comment on the appropriateness of the models used and the assumptions made.

Example 8 (Population Growth)

In a certain farm, the growth of the population of prawns is studied. The population of prawns at time t is denoted by x (in thousands). It was found that the rate of birth per day is twice of x and the rate of death per day is proportional to x^2 .

- (i) Given that there is no change in the population of prawns when its population hits 10000, show that $\frac{dx}{dt} = 2x - \frac{x^2}{5}$.
- (ii) Its owner decides to sell away 1800 prawns daily. Modify the differential equation above and show that the resulting differential equation can be written as

$$\frac{dx}{dt} = -\frac{1}{5}(x-9)(x-1).$$

Given that the initial population of prawns is 13000, solve this modified differential equation, expressing x in terms of t . Sketch a graph of x against t .

Deduce the long term implications on the population of prawns in the farm. [YJC/Prelims/2007/I/10]

Solution

"the rate of birth per day is twice of x and the rate of death per day is proportional to x^2 ."

- (i) Rate of change of population = Birth rate – Death rate

When $x = 10$, there is no change in the population.

When $x = 10$,

$$\Rightarrow 0 = 2(10) - a(10)^2$$

$$\Rightarrow a = \frac{1}{5}$$

$$\therefore \frac{dx}{dt} = 2x - \frac{x^2}{5} \text{ (shown)}$$

"no change in population of prawns when its population hits 10000"

(ii) Selling away 1800 prawns daily, the differential equation becomes

"Its owner decides to sell away 1800 prawns daily."

$$\Rightarrow \frac{dx}{dt} = -\frac{1}{5}(x-9)(x-1) \text{ (shown)}$$

Separating the variables, $\frac{1}{(x-9)(x-1)} \frac{dx}{dt} = -\frac{1}{5}$

Integrating both sides w.r.t. t :

$$\int \frac{1}{(x-9)(x-1)} dx = \int -\frac{1}{5} dt$$

$$\int \frac{1}{8(x-9)} - \frac{1}{8(x-1)} dx = \int -\frac{1}{5} dt$$

$$\ln|x-9| - \ln|x-1| = -\frac{8t}{5} + C, \text{ where } C \text{ is an arbitrary constant}$$

$$\ln \left| \frac{x-9}{x-1} \right| = -\frac{8t}{5} + C$$

$$\frac{x-9}{x-1} = Ae^{-\frac{8t}{5}}, \text{ where } A = \pm e^C$$

Must use modulus for $\ln$.

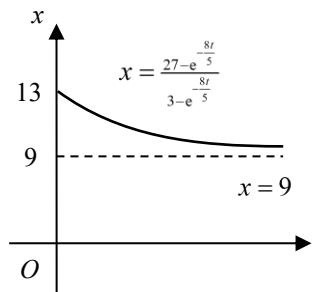
Use $A = \pm e^C$ to simplify.

"Given that the initial population of prawns is 13000"

$$\Rightarrow 3x - 27 = e^{-\frac{8t}{5}}(x-1)$$

$$\Rightarrow 3x - e^{-\frac{8t}{5}}x = 27 - e^{-\frac{8t}{5}}$$

$$\Rightarrow x = \frac{27 - e^{-\frac{8t}{5}}}{3 - e^{-\frac{8t}{5}}}$$



In the long term, as $t \rightarrow \infty$,

Hence, the population of prawns will stabilise at 9000 in the long term. ■

Example 9 (Newton's Law of Cooling)

Detectives arrive at a crime scene at 12 p.m. and found an unfinished cup of coffee at 45°C . In order to estimate what time the coffee was brewed, a fresh cup of coffee was made from the coffee machine in the same room, and its temperature was found to be 110°C . After leaving the new cup of coffee in the room for 5 minutes, the temperature dropped to 80°C .

Based on Newton's Law of Cooling, it is known that the rate at which the temperature of the coffee falls is proportional to the difference between the temperature of the coffee and its surroundings. Given that the crime scene is an air-conditioned room with temperature controlled at 25°C , at what time was the unfinished cup of coffee brewed? Leave your answer to the nearest minute.

[ACJC/Prelims/2011/I/11]

Solution

Let the temperature of the coffee be $\theta^{\circ}\text{C}$.

$$\frac{d\theta}{dt} \propto \theta - 25 \Rightarrow$$

$$\Rightarrow \int \frac{1}{(\theta - 25)} d\theta = -k \int 1 dt$$

$$\Rightarrow \ln|\theta - 25| = -kt + C$$

$$\Rightarrow$$

$$\text{When } t = 0, \theta = 110 \Rightarrow 110 - 25 = A \Rightarrow A = 85$$

$$\text{When } t = 5, \theta = 80 \Rightarrow 80 - 25 = 85e^{-5k} \Rightarrow -5k = \ln \frac{55}{85} \Rightarrow k = -\frac{1}{5} \ln \frac{55}{85}$$

$$\therefore \theta - 25 = 85e^{\left(\frac{1}{5} \ln \frac{55}{85}\right)t}$$

$$\text{When } \theta = 45, 45 - 25 = 85e^{\left(\frac{1}{5} \ln \frac{55}{85}\right)t}$$

$$\Rightarrow \left(\frac{1}{5} \ln \frac{55}{85}\right)t = \ln \frac{20}{85}$$

$$\Rightarrow t = 16.619 \text{ min} = 17 \text{ min (nearest minute)}$$

The estimated time when the coffee was brewed is 11.43am. ■

5 MORE WORKED EXAMPLES

Example 10

The current I in an electric circuit at time t satisfies the differential equation

$$4 \frac{dI}{dt} = 2 - 3I.$$

Find I in terms of t , given that $I = 2$ when $t = 0$. [6]

State what happens to the current in this circuit for large values of t . [1]

[N07/I/4]

Solution

Method I

$$4 \frac{dI}{dt} = 2 - 3I$$

$$\int \frac{1}{2-3I} dI = \frac{1}{4} \int 1 dt$$

$$-\frac{1}{3} \ln|2-3I| = \frac{1}{4}t + C$$

$$\ln|2-3I| = -\frac{3}{4}t - 3C$$

$$|2-3I| = e^{-\frac{3}{4}t - 3C}$$

$$2-3I = \pm e^{-\frac{3}{4}t - 3C}$$

$$2-3I = Ae^{-\frac{3}{4}t}, \text{ where } A = \pm e^{-3C}$$

$$\therefore I = \frac{1}{3} \left(2 - Ae^{-\frac{3}{4}t} \right).$$

As $I = 2$ when $t = 0$,

$$2 = \frac{1}{3}(2 - A) \Rightarrow A = -4.$$

$$\therefore I = \frac{1}{3} \left(2 + 4e^{-\frac{3}{4}t} \right)$$

$$= \frac{2}{3} \left(1 + 2e^{-\frac{3}{4}t} \right).$$

Method II (use the initial condition to 'remove' the modulus)

$$4 \frac{dI}{dt} = 2 - 3I$$

$$\int \frac{1}{2-3I} dI = \frac{1}{4} \int 1 dt$$

$$-\frac{1}{3} \ln|2-3I| = \frac{1}{4}t + C$$

As $I = 2$ when $t = 0$:

In 'removing' this modulus, consider the initial condition:

Note that $2-3I < 0$

when $t=0, I=2$.

Thus, for $\ln$ to be defined, we write $\ln(3I-2)$ and not $\ln(2-3I)$.

Hence,

$$-\frac{1}{3} \ln(3I-2) = \frac{1}{4}t - \frac{1}{3} \ln 4$$

$$\ln(3I-2) = -\frac{3}{4}t + \ln 4$$

$$3I-2 = e^{-\frac{3}{4}t + \ln 4} = 4e^{-\frac{3}{4}t}$$

$$3I = 2 + 4e^{-\frac{3}{4}t}$$

$$\therefore I = \frac{2}{3} \left(1 + 2e^{-\frac{3}{4}t} \right)$$

For large values of t , $e^{-\frac{3}{4}t} \rightarrow 0$, $\therefore I \rightarrow \frac{2}{3}$. ■

Example 11

A tank, with a rectangular base and having vertical sides of height h , is initially full of water. The water leaks out of a small hole in the horizontal base of the tank, such that the depth changes at a rate which, at any instant, is proportional to the square root of the depth of the water at that instant. If x is the depth of the water at time t (hours) after the leak started, write down a differential equation connecting x and t . If the tank is exactly half empty after one hour, find the further time that elapses before the tank becomes completely empty. [N79/I/7]

Solution

$$\text{Rate of change of depth of water} = \frac{dx}{dt} \propto \sqrt{x}$$

$$\Rightarrow \frac{dx}{dt} = kx^{\frac{1}{2}}, \text{ where } k \text{ is a constant, } k < 0.$$

k is negative since depth of water is decreasing with respect to time

$$\int x^{-\frac{1}{2}} dx = k \int 1 dt$$

$$2\sqrt{x} = kt + C, \text{ where } C \text{ is an arbitrary constant}$$

Initially, $t = 0$ and $x = h$,

$$2\sqrt{h} = k(0) + C$$

$$\Rightarrow C = 2\sqrt{h}$$

$$\therefore 2\sqrt{x} = kt + 2\sqrt{h}$$

$$\text{Then, } t = 1 \text{ and } x = \frac{h}{2},$$

$$2\sqrt{\frac{h}{2}} = k(1) + 2\sqrt{h}$$

$$\Rightarrow \sqrt{2h} = k + 2\sqrt{h}$$

$$\Rightarrow k = \sqrt{2h} - 2\sqrt{h} = (\sqrt{2} - 2)\sqrt{h}$$

$$\therefore 2\sqrt{x} = (\sqrt{2} - 2)\sqrt{h} t + 2\sqrt{h}$$

When the tank is completely empty, $x = 0$

$$\Rightarrow 0 = (\sqrt{2} - 2)\sqrt{h} t + 2\sqrt{h}$$

$$\Rightarrow t = \frac{-2\sqrt{h}}{(\sqrt{2} - 2)\sqrt{h}} = 3.41 \text{ (to 3 s.f.)}$$

It takes 3.41 hours for the tank to become completely empty.

From the time when the tank was half empty, it takes a further 2.41 hours for the tank to become completely empty. ■

For A-Level exam contextual questions that involve physical quantity such as time, you should give your answer as a decimal rather than exact form.

Example 12

As part of making water safe for consumption, vitamins are added into pure water. A vat maintains a constant 2000 litres of water, which is assumed to be an evenly mixed solution of pure water and vitamins.

At time t minutes after a tap is turned on, $(50 - 50e^{-t})$ litres of water containing 0.2g per litre of vitamins is added into the vat, while an equal volume of water is pumped out from the vat for consumption. At time t minutes, the vat contains y grams of vitamins.

- (i) Show that a differential equation relating y and t is given by

$$\frac{dy}{dt} = \left(10 - \frac{y}{40}\right)(1 - e^{-t}).$$

- (ii) If the vat contained 80g of vitamins initially, find y in terms of t .
 (iii) Water that is fit for consumption should have a vitamin concentration of at least 0.15g per litre. In the long run, is the water produced by this process fit for consumption? Justify your answer.

Solution

- (i) Concentration of vitamins in the vat is $\frac{y}{2000}$ grams per litre.

For each litre, we gain 0.2 grams and lose $\frac{y}{2000}$ grams of vitamins.

Total rate of change of vitamins =

$$\frac{dy}{dt} = \left(10 - \frac{y}{40}\right)(1 - e^{-t}) \quad (\text{shown})$$

(ii)
$$\int \frac{1}{\left(10 - \frac{y}{40}\right)} dy = \int (1 - e^{-t}) dt$$

$$10 - \frac{y}{40} = \pm e^{-\frac{C}{40}} e^{-\frac{t+e^{-t}}{40}} = Ae^{-\frac{t+e^{-t}}{40}}, \text{ where } A = \pm e^{-\frac{C}{40}}$$

When $t = 0, y = 80$:

$$10 - \frac{y}{40} = 8.2025e^{-\frac{t+e^{-t}}{40}}$$

$$\frac{y}{40} = 10 - 8.2025e^{-\frac{t+e^{-t}}{40}}$$

$$\therefore y = 400 - 328e^{-\frac{t+e^{-t}}{40}} \quad (\text{to 3 sf})$$

(ii) From the GC, as $t \rightarrow \infty$,

In the long run, the concentration of vitamins in the vat is 400g per 2000 litres, or equivalently, 0.2g per litre. This is more than 0.15g per litre.

Hence, the water produced would be fit for consumption. ■

ANNEX

Practice Questions on Differential Equations

- 1 Find the general solution of the following differential equations:

(a) $\frac{dy}{dx} = 2x + \cos x + e^x$

(b) $\frac{dw}{ds} = 9 + w^2$

(c) $\sin^2 \theta \frac{dy}{d\theta} = \frac{1}{y}$

(d) $\frac{dy}{dx} = y \ln x$, for $x > 0$

- 2 Solve the following differential equation, expressing y in terms of t ,

$$y = \frac{dy}{dt} + 5 \text{ given that } y = 6 \text{ when } t = 2.$$

- 3 The gradient of a curve at the point (x, y) is given by $\frac{1}{1-x}$. Find the equation of the curve with this gradient function which passes through the point $(-1, 1)$.

- 4 Show that the general solution of the differential equation $\frac{dy}{dx} = y + 1$ may be expressed as $y = Ae^x - 1$, where A is an arbitrary constant.

- 5 Using the substitution $v = x + y + 1$, solve the differential equation $\frac{dy}{dx} = (x + y + 1)^2$ where $y = 0$ when $x = -1$, expressing y in terms of x .

Answers

1 (a) $y = x^2 + \sin x + e^x + c$ (b) $s = \frac{1}{3} \tan^{-1} \left(\frac{w}{3} \right) + c$

(c) $y^2 = -2 \cot \theta + c$ (d) $y = ce^{x \ln x - x}$

2 $y = 5 + e^{t-2}$

3 $y = -\ln|1-x| + 1 + \ln 2$

5 $y = \tan(x+1) - x - 1$