H3 Mathematics Problem Solving
Lesson 1: What Is the Problem?

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How this deck connects to the book

Lesson design principle

Slides create the experience; the book consolidates the ideas.

- **Chapter 1:** What counts as a genuine mathematical problem?
- **Chapter 1:** First experience with the four stages through examples.
- **Chapter 2:** Make the stages explicit so that our thinking becomes visible and improvable.

Book link: Chapter 1; Chapter 2, Section 1

Lesson objectives

By the end of this lesson, you should be able to:

1. distinguish a routine exercise from a genuine problem;
2. describe Pólya's four problem-solving stages;
3. analyse a problem using the language of unknowns, data, assumptions and required conditions;
4. recognize that problem solving does not take place in a vacuum, but draws on mathematical resources.

Book link: Chapter 1, Section 1; Chapter 2, Sections 1–4

A routine task and a genuine problem

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Multiply 23451 by 2.

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Question 2

Which smallest whole number, when multiplied by 2, results in the last digit being shifted to the first digit?

- Question 1 is routine because a standard algorithm is readily available.
- Question 2 is a genuine problem if you have not seen it before.

Book link: Chapter 1, Section 1

What makes Question 2 a problem?

Working definition

A problem is a task for which there is no readily accessible algorithm that completely determines the method of solution.

This working definition is modified from Lester's original definition in [1].

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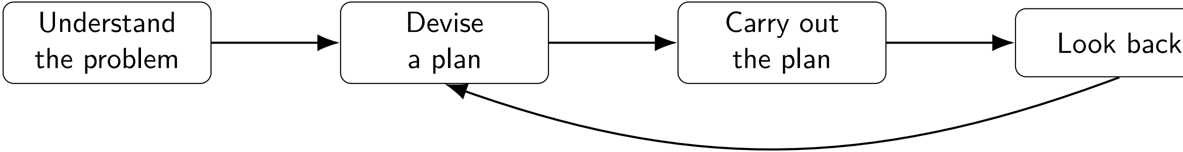
For problem solving to begin meaningfully, three things matter:

1. there is no immediate procedure;
2. the solver wants to know the answer;
3. some real effort is made.

Book link: Chapter 1, "What is a mathematical problem?"; Chapter 1, "Where are we heading to?"

The first problem-solving trajectory

For the digit-shift problem, the book walks us through four recurring phases:



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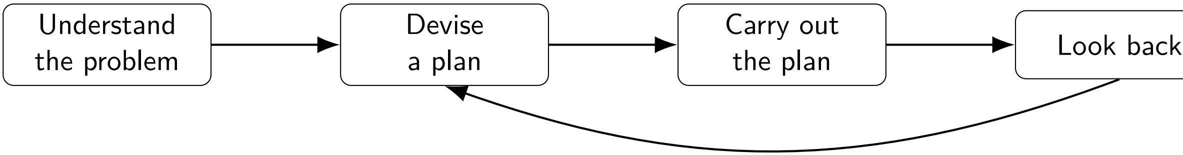

Today we first *experience* these stages, then we make them explicit.

Book link: Chapter 1, Sections 1.1–1.4; Chapter 2, Section 3

Stage 1: Understand the problem

Before solving, we must be able to answer:

- What form should the answer take?
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$$2 \times 12345 = 24690 \neq 51234.$$

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Key idea

Verification is part of understanding the problem, not something added only at the end.

Book link: Chapter 1, Section 1.1

Stage 2: Devise a plan

A first attempt may be guess and check:

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But random guessing is inefficient. So we impose structure.

Stage 2: Devise a plan

Assume the last digit is 2.

Try something!

Dirty your hands and work your way up!

Stage 2: Devise a plan

Assume the number has the form $10a + b$, where:

- b is the last digit, so $1 \le b \le 9$;
- a is the remaining non-zero block of digits.

Book link: Chapter 1, Section 1.2

Stage 3: Carry out the plan

If doubling shifts the last digit to the front, then:

$$2(10a + b) = 10^n b + a.$$

Hence

$$19a = (10^n - 2)b.$$

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So the problem reduces to finding the least positive integer n for which

$$10^n \equiv 2 \pmod{19}.$$

Book link: Chapter 1, Section 1.3

Stage 4: Look back

Using the remainder table gives $n = 17$. If we choose $b = 1$, we get a candidate

$$52, 631, 578, 947, 368, 421$$

that is close but wrong because an extra zero appears after doubling:

$$105, 263, 157, 894, 736, 842.$$

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Lesson

Looking back is not optional. It is where errors are exposed and understanding deepens.

Book link: Chapter 1, Section 1.4

Interim summary from Chapter 1

From the opening chapter, we keep four big messages:

1. A mathematical problem requires genuine thinking.
2. Problem solving begins only when effort begins.
3. The four stages already appear in the first worked example.
4. Mathematical resources must be accessed intentionally.

Book link: Chapter 1, "Where are we heading to?"

Why make problem solving explicit?

Chapter 2 begins here

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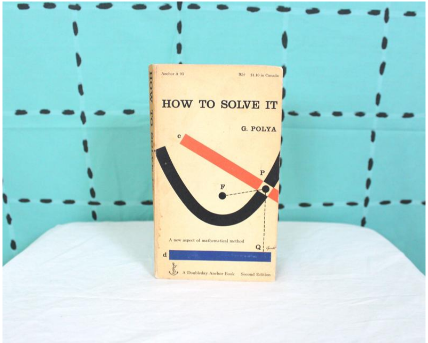
Why does this matter?

- When we are stuck, we need language to diagnose where we are stuck.
- There is no "right formula" that automatically solves unfamiliar problems.
- We improve our chances by taking intentional steps forward.

Book link: Chapter 2, Section 1

Polya's problem solving strategy



The problem-solving framework we adopt in this course (and also the book) comes from George Polyá's book *How to Solve It* ([2]).

Pólya's framework

Stage	What we are trying to do
Understand the problem	See clearly what is given, what is wanted, and what would count as a solution.
Devise a plan	Search for a workable route: a representation, a heuristic, a simplification, a related problem.
Carry out the plan	Execute carefully, while remaining alert to adjustments and new discoveries.
Look back	Check, reflect, extend, and extract reusable ideas.

Book link: Chapter 2, Sections 2–3

Stage 1 made explicit: understanding the problem

A student who understands a problem should be able to identify:

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- the **data**;
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- the **required conditions**.

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Useful prompts:

- Can you paraphrase the problem in your own words?
- What information is central, and what information is incidental?

Book link: Chapter 2, Section 3.1

Groundhog problem

Figure: A sketch of a groundhog

Groundhog problem: understanding the problem

Example

A groundhog has infinitely many holes arranged in a line. Each day it moves a fixed number of holes in one fixed direction. A farmer checks one hole at midnight each day. What strategy guarantees eventual capture?

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Clarifying questions from the book:

1. What exactly does "fixed number of holes in one fixed direction" mean?
2. May the farmer re-check a previously checked hole?
3. Do we know the groundhog's position on Day 0?

Book link: Chapter 2, Section 3.1

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So we simplify intentionally:

- assume the groundhog starts at Hole 0;
- assume it moves to the right with speed $k > 0$;
- ask: what if it were a 1-groundhog, then a 2-groundhog, then a 3-groundhog, and so on?

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Heuristic in action

Reduce the number of unknowns; solve a simpler version first.

Book link: Chapter 2, Section 3.2

Groundhog problem: carrying out the plan

If on Day n we test the possibility that the speed is n, then the hole to inspect is:

$$1, 4, 9, 16, 25, \ldots$$

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This works because systematic listing ensures that no positive integer speed is missed.

Book link: Chapter 2, Section 3.3

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Important principle

No problem is completely exhausted. Looking back often produces new mathematics.

Book link: Chapter 2, Section 3.4

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Content strand for this lesson: inequalities

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Consider the book's example:

$$((n-1)x+y)^n \geq n^n x^{n-1} y$$

for every positive integer n and positive real numbers x, y.

Book link: Chapter 1, Section 2; Chapter 2, Section 4

Understanding the inequality problem

What kind of task is this?

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This helps us focus on structure rather than on isolated symbols.

Book link: Chapter 1, Section 2.1

Use of technology

Let us exploit Desmos (`https://www.desmos.com/3d/tehbofdtct`) to give us some intuition.

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So we choose

$$x_1 = x_2 = \cdots = x_{n-1} = x, \qquad x_n = y.$$

Book link: Chapter 1, Section 2.2

Carrying out the inequality plan

Applying AM-GM to

$$\underbrace{x + x + \cdots + x}_{n-1 \text{ copies}} + y$$

gives

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Message

The proof works because we matched the structure of the problem to a known resource.

Book link: Chapter 1, Section 2.3

Summary of Lesson 1

1. A mathematical problem is not the same as a routine exercise.
2. Pólya's four stages give us a language for discussing thinking.
3. The stages are not a rigid script; we often move back and forth.
4. Effective problem solving depends on both heuristics and mathematical resources.

Book link: Chapter 1, "Where are we heading to?"; Chapter 2, Section 5

Homework and book follow-up

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- Chapter 2, Sections 1–3.

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- Exercise 1(c): the chain of reciprocal inequalities;
- Exercise 5(ii): the AM-GM inequality problem from H3 2020;
- Exercise 11: the airport walkway puzzle.

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- Exercise 11: the airport walkway puzzle.

Instruction: For each solution, include one short paragraph on what you tried first and what changed along the way.

Closing reflection

Exit question

When you were stuck today, were you stuck because you lacked a formula, or because you had not yet decided how to understand the problem and plan the next move?

Think mathematically. Reflect deliberately.

References

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