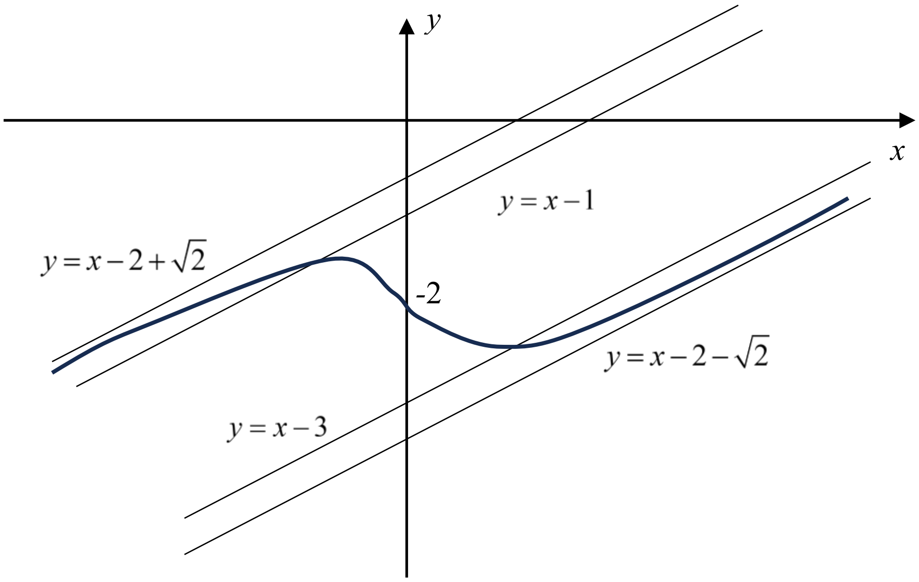
Q1	Suggested Answers	
(a)	$y = mx + c$ $\frac{\mathrm{d}y}{\mathrm{d}x} = m$ Substitute into the DE: $m = x^2 + (mx+c)^2 - 2x(mx+c) - 4x + 4(mx+c) + 3$ Comparing coefficients: x^2: $0 = 1 + m^2 - 2m \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$ Constant: $m = c^2 + 4c + 3$ $c^2 + 4c + 2 = 0$ $c = \frac{-4 \pm \sqrt{16-8}}{2} = -2 \pm \sqrt{2}$ Thus the solutions are $y = x - 2 + \sqrt{2}$ and $y = x - 2 - \sqrt{2}$.	
(b)	$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2 - 2xy - 4x + 4y + 3$ $= (y-x)^2 + 4(y-x) + 3$ $= 0$ Thus $(y-x+3)(y-x+1) = 0$ The equations of the two lines are $y = x - 3$ and $y = x - 1$.	
(c)	$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2 - 2xy - 4x + 4y + 3$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 4 + 4\frac{\mathrm{d}y}{\mathrm{d}x}$ $= 2x - 2y - 4$ since $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at stationary points If the stationary point lies on $y = x - 3$, then $x - y - 3 = 0$ Then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x - 2y - 4 = 2(x - y - 3 + 1) = 2 > 0$. Thus the stationary point is a minimum point. If the stationary point lies on $y = x - 1$, then $x - y - 1 = 0$ Then $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2x - 2y - 4 = 2(x - y - 1 - 1) = -2 < 0$. Thus the stationary point is a maximum point.	
(d)		



Q2	Suggested Answers	
(a)	$\mathrm{f}(n)=(n+3)^2+2$ $\mathrm{g}(n)=(n-2)^2+2$ $\mathrm{h}(n)=(n-1)^2+4$ Note that $\mathrm{f}(n)=\mathrm{g}(n+5)$ for every $n\in\mathbb{Z}$. i.e. $n \mapsto n+5$ is a bijection on $\mathbb{Z}$. Hence the functions have the same range.	
	To find common values in the range of f and h, consider $(n+3)^2+2=(s-1)^2+4$ $(n+3)^2-(s-1)^2=2$ Since there are no perfect squares that differ by 2, there are no integer solutions. There are no common values in the ranges of f and h.	
(b)(i)	$a^2+ab+b^2=\left(a+\frac{1}{2}b\right)^2+\frac{3}{4}b^2\geq 0$	
(b)(ii)	$(a-1-b)((a-1)^2+(a-1)b+b^2)$ $=(a-1)^3-b(a-1)^2+(a-1)^2b-(a-1)b^2+(a-1)b^2-b^3$ $=(a-1)^3-b^3$	
(c)	$\mathrm{p}(n)=n^3-3n^2+7n=(n-1)^3+4n+1$ To find common integers in the range of p and q, consider $(n-1)^3+4n+1=s^3+4s-6$ $(n-1)^3-s^3=4s-6-4n-1$ $(n-1-s)\left((n-1)^2+(n-1)s+s^2\right)=4(s-n+1)-11$ $(n-1-s)\left((n-1)^2+(n-1)s+s^2\right)+4(n-1-s)=-11$	

	$(n-1-s)\underbrace{\left((n-1)^2+(n-1)s+s^2+4\right)}_{\geq 4}=-11$ Thus $(n-1-s)=-1$ and $(n-1)^2+(n-1)s+s^2+4=11$ Thus $n=s$. $(s-1)^2+(s-1)s+s^2+4=11$ $3s^2-3s-6=0$ $(s+1)(s-2)=0$ $s=-1$ or 2 Thus the values common to both ranges are $\text{q}(-1)=-11$ and $\text{q}(2)=10$.	

Q3	Suggested Solutions	
a(i)	$\int_0^a \text{f}(x)\,\text{d}x=\int_a^0 \text{f}(a-u)(-1)\,\text{d}u \quad (\text{with } x=a-u)$ $=\int_0^a \text{f}(a-u)\,\text{d}u$ $=\int_0^a \text{f}(a-x)\,\text{d}x$	
a(ii)	Let $I=\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x+\cos^n x}\,\text{d}x$. $I=\int_0^{\frac{\pi}{2}} \frac{\cos^n\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)+\cos^n\left(\frac{\pi}{2}-x\right)}\,\text{d}x \quad (\text{by part (a)(i)})$ $=\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\cos^n x+\sin^n x}\,\text{d}x \quad \left(\begin{array}{l}\because \sin\left(\frac{\pi}{2}-x\right)=\cos x\\ \& \ \cos\left(\frac{\pi}{2}-x\right)=\sin x\end{array}\right)$ $2I=\int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x+\cos^n x}\,\text{d}x+\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\cos^n x+\sin^n x}\,\text{d}x$ $=\int_0^{\frac{\pi}{2}} \frac{\cos^n x+\sin^n x}{\sin^n x+\cos^n x}\,\text{d}x$ $=\frac{\pi}{2}$ $I=\frac{\pi}{4}$	

b	$\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x=\int_0^a (a-x)\,\mathrm{f}(a-x)\,\mathrm{d}x \quad (\text{by part (a)(i)})$ $=\int_0^a (a-x)\,\mathrm{f}(x)\,\mathrm{d}x \quad (\because \mathrm{f}(x)=\mathrm{f}(a-x))$ $=a\int_0^a \mathrm{f}(x)\,\mathrm{d}x-\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x$ $2\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x=a\int_0^a \mathrm{f}(x)\,\mathrm{d}x$ $\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x=\frac{a}{2}\int_0^a \mathrm{f}(x)\,\mathrm{d}x$	applying result from (a)(i) and applying given property of f
c(i)	$I_n=\int_0^{\frac{\pi}{2}}\cos^n(2x)\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{2}}\cos(2x)\cos^{n-1}(2x)\,\mathrm{d}x$ $=\left[\frac{1}{2}\sin(2x)\cos^{n-1}(2x)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}-(n-1)\sin^2(2x)\cos^{n-2}(2x)\,\mathrm{d}x$ $=(n-1)\int_0^{\frac{\pi}{2}}\left(1-\cos^2(2x)\right)\cos^{n-2}(2x)\,\mathrm{d}x$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}(2x)-\cos^n(2x)\,\mathrm{d}x$ $=(n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}(2x)\,\mathrm{d}x-(n-1)\int_0^{\frac{\pi}{2}}\cos^n(2x)\,\mathrm{d}x$ $(1+n-1)I_n=(n-1)I_{n-2}$ $nI_n=(n-1)I_{n-2}$	
c(ii)	$\int_0^{\pi}x\cos^8(2x)\,\mathrm{d}x$ $=\frac{\pi}{2}\int_0^{\pi}\cos^8(2x)\,\mathrm{d}x \quad \begin{pmatrix}\text{by part (b)}\\ \because \cos^8(2(\pi-x))=\cos^8(2x)\end{pmatrix}$ $=\frac{\pi}{2}\left(2\int_0^{\frac{\pi}{2}}\cos^8(2x)\,\mathrm{d}x\right) \quad \left(\because \cos^8(2x) \text{ symmetrical along } x=\frac{\pi}{2}\right)$ $=\pi\int_0^{\frac{\pi}{2}}\cos^8(2x)\,\mathrm{d}x$ $=\pi I_8$ $=\pi\left(\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times I_0\right)$ $=\frac{35\pi}{128}\int_0^{\frac{\pi}{2}}\cos^0 2x\,\mathrm{d}x$ $=\frac{35\pi}{128}\left(\frac{\pi}{2}\right)$ $=\frac{35}{256}\pi^2$	

Q4		
(a)	$\lvert z_1+z_2\rvert^2=\left(z_1+z_2\right)\left(z_1^*+z_2^*\right)$ $=\lvert z_1\rvert^2+\lvert z_2\rvert^2+z_1z_2^*+z_1^*z_2$ $=\lvert z_1\rvert^2+\lvert z_2\rvert^2+2\operatorname{Re}\left(z_1z_2^*\right)$ $\le\lvert z_1\rvert^2+\lvert z_2\rvert^2+2\lvert z_1z_2^*\rvert$ $=\lvert z_1\rvert^2+\lvert z_2\rvert^2+2\lvert z_1\rvert\lvert z_2\rvert$ $=\left(\lvert z_1\rvert+\lvert z_2\rvert\right)^2$ Hence $\lvert z_1+z_2\rvert\le\lvert z_1\rvert+\lvert z_2\rvert$.	
	Let P_n be the proposition that $\lvert z_1+z_2+\ldots+z_n\rvert\le\lvert z_1\rvert+\lvert z_2\rvert+\ldots+\lvert z_n\rvert$ for all $n\in\mathbb{Z}^+$. Clearly, P_1 and P_2 is true. Assume P_k is true for some *k*. Need to prove P_{k+1} is true. i.e. $\lvert z_1+z_2+\ldots+z_{k+1}\rvert\le\lvert z_1\rvert+\lvert z_2\rvert+\ldots+\lvert z_{k+1}\rvert$ $\lvert z_1+z_2+\ldots+z_{k+1}\rvert\le\lvert z_1+z_2+\ldots+z_k\rvert+\lvert z_{k+1}\rvert$ $\le\lvert z_1\rvert+\lvert z_2\rvert+\ldots+\lvert z_{k+1}\rvert$ P_k is true $\Rightarrow\mathrm{P}_{k+1}$ is true. Since P_1 and P_2 is true and P_k is true $\Rightarrow\mathrm{P}_{k+1}$ is true, hence by mathematical induction, P_n is true for all $n\in\mathbb{Z}^+$.	
(b)	Given $u_r-u_{r+1}^2>0$ and $u_{2023}-u_1^2>0$, $u_1-u_2^2,\ u_2-u_3^2,\ \ldots,\ u_{2022}-u_{2023}^2,\ u_{2023}-u_1^2$ is a sequence of positive numbers. Using AM-GM, $\sqrt[2023]{\left[\prod_{r=1}^{2022}(u_r-u_{r+1}^2)\right](u_{2023}-u_1^2)}\le\frac{1}{2023}\left[\sum_{r=1}^{2022}(u_r-u_{r+1}^2)+(u_{2023}-u_1^2)\right]$ Now, $\frac{1}{2023}\left[\sum_{r=1}^{2022}(u_r-u_{r+1}^2)+(u_{2023}-u_1^2)\right]$ $=\frac{1}{2023}\left[(u_1-u_2^2)+(u_2-u_3^2)+\ldots+(u_{2022}-u_{2023}^2)+(u_{2023}-u_1^2)\right]$ $=\frac{1}{2023}\left[(u_1-u_1^2)+(u_2-u_2^2)+\ldots+(u_{2022}-u_{2022}^2)+(u_{2023}-u_{2023}^2)\right]$ $=\frac{1}{2023}\sum_{r=1}^{2023}(u_r-u_r^2)$ $\le\frac{1}{2023}\left(\frac{1}{4}\right)(2023)$ since $(u_r-u_r^2)\le\frac{1}{4}$ $=\frac{1}{4}$	

	So $\sqrt[2023]{\left[\prod_{r=1}^{2022}(u_r - u_{r+1}^2)\right](u_{2023} - u_1^2)} \le \frac{1}{4}$ $\Rightarrow \quad W = \left[\prod_{r=1}^{2022}(u_r - u_{r+1}^2)\right](u_{2023} - u_1^2) \le \left(\frac{1}{4}\right)^{2023}$ $\Rightarrow \quad 4^{2025} W \le 4^{2025}\left(\frac{1}{4}\right)^{2023} = 16$	

<table>
<tr><th>Qn.</th><th>Suggested Solutions</th><th>Comments</th></tr>
<tr><td>5a</td><td>
Let $\mathrm{A}_1,\ldots,\mathrm{A}_m$ represent the different surnames and $\mathrm{B}_1,\ldots,\mathrm{B}_n$ represent the different birth months. For each boy (rows 1 to 18 in table below), put a tick under column A_i and B_j if his surname is A_i and birth month is B_j.

<table>
<tr><th></th><th colspan="3">Surnames</th><th colspan="3">Birth months</th></tr>
<tr><td>Boy</td><td>A_1</td><td>…</td><td>A_m</td><td>B_1</td><td>…</td><td>B_n</td></tr>
<tr><td>1</td><td>✓</td><td></td><td></td><td></td><td></td><td>✓</td></tr>
<tr><td>2</td><td></td><td></td><td>✓</td><td>✓</td><td></td><td></td></tr>
<tr><td>⋮</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>18</td><td></td><td></td><td>✓</td><td></td><td></td><td>✓</td></tr>
</table>
Let $a_1,\ldots,a_m,b_1,\ldots,b_n$ be the number of ticks in the columns $\mathrm{A}_1,\ldots,\mathrm{A}_m,\mathrm{B}_1,\ldots,\mathrm{B}_n$ respectively. $1 \le a_1,\ldots,a_m,b_1,\ldots,b_n \le 8$ since each column has 1 to 8 ticks.
Since each row has 2 ticks, the total number of ticks is $a_1+\ldots+a_m+b_1+\ldots+b_n = 18\times 2 = 36$.
Also, since all integers from 0 to 7 were seen in the boys' responses, and $a_1+\ldots+a_m+b_1+\ldots+b_n \ge 1+2+\ldots+8 = 36$.

Hence, $a_1,\ldots,a_m,b_1,\ldots,b_n$ is an arrangement of the integers 1, 2, …, 8, which implies that $m+n=8$ (there are 8 columns in total) and consequently $1 \le m,n \le 7$.

There is one column which has 8 ticks.
WLOG, suppose $a_1 = 8$, i.e. 8 boys have the same surname.
Since the number of birth months, n, is at most 7, by pigeonhole principle, there are at least 2 of these 8 boys with the same surname, that have the same birth month.
</td><td>
Note: the integers 0 to 7 appear at least once for surname <u>and/or</u> birth months.

Method to count and tabulate the data, or to present the data in a meaningful way

Count total number of ticks in 2 different ways and compare

Apply pigeonhole principle to show answer [AG]
</td></tr>
<tr><td>5b (i)</td><td>Note that the 4 groups are distinguishable because 4 different challenge questions.</td><td></td></tr>
</table>

	The 18 boys can split themselves up according to $6+4+4+4$ or $5+5+4+4$. Number of ways $= \underbrace{{}^4C_1}_{\text{Which group has 6 boys}} \times \binom{18}{6}\binom{12}{4}\binom{8}{4}\binom{4}{4} + \underbrace{{}^4C_2}_{\text{Which 2 groups have 5 boys each}} \times \binom{18}{5}\binom{13}{5}\binom{8}{4}\binom{4}{4}$ $=7204317120$	
5b (ii)	Each of the 6 players takes 5 dice, leaving 50 dice to be distributed. Number of ways $=\binom{50+6-1}{6-1}$ $=3478761$	
5c	Let A, B and C be the sets of questions that Alex, Ben and Charlie answered correctly respectively. Let a, b, c be the difficult questions that Alex, Ben and Charlie answered correctly respectively. $a+b+c$ $=\lvert A\rvert-\lvert A\cap B\rvert-\lvert A\cap C\rvert+\lvert A\cap B\cap C\rvert$ $\quad+\lvert B\rvert-\lvert A\cap B\rvert-\lvert B\cap C\rvert+\lvert A\cap B\cap C\rvert$ $\quad+\lvert C\rvert-\lvert A\cap C\rvert-\lvert B\cap C\rvert+\lvert A\cap B\cap C\rvert$ $=\lvert A\rvert+\lvert B\rvert+\lvert C\rvert-2\left(\lvert A\cap B\rvert+\lvert A\cap C\rvert+\lvert B\cap C\rvert\right)+3\lvert A\cap B\cap C\rvert$ $=2\left[\lvert A\rvert+\lvert B\rvert+\lvert C\rvert-\left(\lvert A\cap B\rvert+\lvert A\cap C\rvert+\lvert B\cap C\rvert\right)+\lvert A\cap B\cap C\rvert\right]$ $\quad-\left(\lvert A\rvert+\lvert B\rvert+\lvert C\rvert\right)+\lvert A\cap B\cap C\rvert$ $=2\lvert A\cup B\cup C\rvert-\left(\lvert A\rvert+\lvert B\rvert+\lvert C\rvert\right)+\lvert A\cap B\cap C\rvert$ $=2(50)-(40+30+20)+\lvert A\cap B\cap C\rvert$ $=10+\lvert A\cap B\cap C\rvert$ Hence, there were $a+b+c-\lvert A\cap B\cap C\rvert=10$ more difficult questions than easy questions.	
	Alternative Solution: Let A, B and C be the sets of questions that Alex, Ben and Charlie answered correctly respectively. Let $e=\lvert A\cap B\cap C\rvert$ where $A\cap B\cap C$ is the set of easy questions. By PIE, $\lvert A\cup B\cup C\rvert=\lvert A\rvert+\lvert B\rvert+\lvert C\rvert-\lvert A\cap B\rvert-\lvert A\cap C\rvert-\lvert B\cap C\rvert+\lvert A\cap B\cap C\rvert$ $50=40+30+20-\lvert A\cap B\rvert-\lvert A\cap C\rvert-\lvert B\cap C\rvert+e$ $\lvert A\cap B\rvert+\lvert A\cap C\rvert+\lvert B\cap C\rvert=40+e$	

	$\lvert Hard_A\rvert = \lvert A\rvert - \lvert A\cap B\rvert - \lvert A\cap C\rvert + e$ $\lvert Hard_B\rvert = \lvert B\rvert - \lvert A\cap B\rvert - \lvert B\cap C\rvert + e$ $\lvert Hard_C\rvert = \lvert C\rvert - \lvert A\cap C\rvert - \lvert B\cap C\rvert + e$ $\lvert Hard_{A\cup B\cup C}\rvert = \lvert A\rvert + \lvert B\rvert + \lvert C\rvert - 2\lvert A\cap B\rvert - 2\lvert A\cap C\rvert - 2\lvert B\cap C\rvert + 3e$ $= 40+30+20-2(40+e)+3e$ $\lvert Hard_{A\cup B\cup C}\rvert - e = 10$ Hence, there were 10 more difficult questions than easy questions.	
5d	Let F_i be the event where Alex met the i^{th} friend, $i=1,2,...,5$. Number of times Alex participated $= \left\lvert \bigcup_{i=1}^{5} F_i \right\rvert + 3$ $= \sum_i \lvert F_i\rvert - \sum_{i<j} \lvert F_i\cap F_j\rvert + \sum_{i<j<k} \lvert F_i\cap F_j\cap F_k\rvert$ $- \sum_{i<j<k<l} \lvert F_i\cap F_j\cap F_k\cap F_l\rvert + \left\lvert \bigcap_{i=1}^{5} F_i \right\rvert + 3$ $= \binom{5}{1}(11) - \binom{5}{2}(5) + \binom{5}{3}(3) - \binom{5}{4}(2) + \binom{5}{5}(1) + 3$ $= 29$	

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

Continued Fractions

A continued fraction for a real number x is an expression of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \ ,$$

or simply expressed as $x = [a_0; a_1, a_2, \ldots]$, where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{R}$ for each $i \geq 1$.

A continued fraction with $a_i \in \mathbb{Z}^+$ for each $i \geq 1$ is known as a simple continued fraction.

It is known that the simple continued fraction for x is finite if and only if x is rational.
For example, the number

$$\frac{9}{7} = 1 + \cfrac{1}{3 + \cfrac{1}{2}} = [1; 3, 2] \text{ is rational}$$

while the number

$$\ln 3 = 1 + \cfrac{1}{10 + \cfrac{1}{7 + \cfrac{1}{9 + \ddots}}} = [1; 10, 7, 9, \ldots] \text{ is irrational.}$$

Let $a_0, a_1, a_2, \ldots$ be a sequence of integers, with all the terms positive except perhaps a_0. We define two sequences of integers as follows:

$$p_{-1} = 1,\ p_0 = a_0,\ p_n = a_n p_{n-1} + p_{n-2},\ n \geq 1,$$
$$q_{-1} = 0,\ q_0 = 1,\ q_n = a_n q_{n-1} + q_{n-2},\ n \geq 1. \qquad (*)$$

For $x = [a_0; a_1, a_2, \ldots]$, and for each $n \in \mathbb{Z}^+$, we define the nth convergent of x as

$$r_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} = [a_0; a_1, a_2, \ldots, a_n] \ .$$

6 **(a)** **(i)** Show that $\sqrt{2}=1+\dfrac{1}{1+\sqrt{2}}$. Hence, write $\sqrt{2}$ as a simple continued fraction to show that it is irrational. [2]

(ii) The number $\pi = 3.14159265\ldots$ can be expressed as a simple continued fraction

$$\pi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \ddots}}}.$$

Find the values of b_0, b_1, b_2, and b_3. [2]

(iii) Let $x = [1; a_1, a_2, a_3, \ldots]$, where for each $n \in \mathbb{Z}^+$, $a_{2n-1} = 3$ and $a_{2n} = 4$. Find the value of x, giving your answer in the form $a + b\sqrt{c}$, where a and b are rational numbers and c is a positive integer. [3]

(b) Let $a_0, a_1, a_2, \ldots$ be a sequence of integers, with all the terms positive except perhaps a_0. Let p_n and q_n be defined by the recurrence relations in (*) as stated in the mathematical text.

(i) Prove that for each $n \in \mathbb{Z}^+$ and any positive real number y,

$$[a_0; a_1, a_2, \ldots, a_{n-1}, y] = \frac{yp_{n-1} + p_{n-2}}{yq_{n-1} + q_{n-2}}.$$ [5]

(ii) Hence, show that r_n, the nth convergent of $[a_0; a_1, a_2, \ldots]$, is given by $r_n = \dfrac{p_n}{q_n}$. [1]

(iii) Prove that for each $n \in \mathbb{Z}^+$,

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}, \text{ and}$$

$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n.$$ [4]

(iv) Explain why the nth convergent, $r_n = \dfrac{p_n}{q_n}$, is always in its lowest terms. [1]

(v) Prove that $r_2 < r_4 < r_6 < \ldots < r_5 < r_3 < r_1$. [2]

Qn	Suggested Solutions	Guidance
6a (i)	$\dfrac{1}{1+\sqrt{2}} = \dfrac{1}{1+\sqrt{2}} \times \dfrac{1-\sqrt{2}}{1-\sqrt{2}}$ $= \dfrac{1-\sqrt{2}}{1-2}$ $= \sqrt{2}-1$ Thus $\sqrt{2} = 1 + \dfrac{1}{1+\sqrt{2}}$ $\sqrt{2} = 1 + \dfrac{1}{1+\sqrt{2}}$ $= 1 + \dfrac{1}{1+\left(1+\dfrac{1}{1+\sqrt{2}}\right)}$	How find continue fraction: $\sqrt{2} = 1.4142...$ $= 1 + 0.4142...$ $= 1 + \dfrac{1}{2.4142...}$ $= 1 + \dfrac{1}{2 + 0.4142...}$ $= ...$ We can rewrite as proper fraction and then rewrite the remaining decimals in a reciprocal form. Repeatedly apply this process in the denominator

	$=1+\cfrac{1}{2+\cfrac{1}{1+\left(1+\cfrac{1}{1+\sqrt{2}}\right)}}$ $=1+\cfrac{1}{2+\cfrac{1}{2+\ddots}}$ $=[1;2,2,2,\ldots]$ Hence, $\sqrt{2}$ is an infinite continued fraction which shows that it is irrational.	
6a (ii)	$\pi=3+0.14159265\ldots$ $=1+\cfrac{1}{7.0625133...}$ $=1+\cfrac{1}{7+\cfrac{1}{15.99659...}}$ $=1+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1.0034633...}}}$ $=...$ Thus $b_0=3$ $b_1=7$ $b_2=15$ $b_4=1$. Alternatively, $b_0=\lfloor\pi\rfloor=3$ $b_1=\left\lfloor\cfrac{1}{\pi-3}\right\rfloor=7$ $b_2=\left\lfloor\cfrac{1}{\cfrac{1}{\pi-3}-7}\right\rfloor=15$ $b_4=\left\lfloor\cfrac{1}{\cfrac{1}{\cfrac{1}{\pi-3}-7}-15}\right\rfloor=1$	We can rewrite as proper fraction and then rewrite the remaining decimals in a reciprocal form. Repeatedly apply this process in the denominator Alternatively, by expressing $\pi=b_0+\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{b_3+\ddots}}}=b_0+\cfrac{1}{x_1}$, we take $b_1=\lfloor x_1\rfloor=\left\lfloor\cfrac{1}{\pi-b_0}\right\rfloor$, and so on.
6a (iii)	Given $x=[1;3,4,3,4,\ldots]$. Let $y=[0;3,4,3,4,\ldots]$. Thus $x=1+y$ and $y=\cfrac{1}{3+\ddots}=\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{3+\ddots}}}=\cfrac{1}{3+\cfrac{1}{4+y}}$ $=\cfrac{4+y}{13+3y}$	Observe how the recurring pattern is like to infinity and rewrite the expression in terms of y.

	$13y+3y^2=4+y$ $3y^2+12y-4=0$ $y=\frac{-12+\sqrt{192}}{6}\quad (\because y=[0;3,4,3,4,\ldots]>0)$ $y=-2+\frac{4}{3}\sqrt{3}$ $\therefore x=1+y=-1+\frac{4}{3}\sqrt{3}$	
6b (i)	Let T_n be the statement $[a_0;a_1,a_2,\ldots,a_{n-1},y]=\frac{yp_{n-1}+p_{n-2}}{yq_{n-1}+q_{n-2}}$ for each $n\in\mathbb{Z}^+$. When $n = 1$, LHS = $[a_0;y]=a_0+\frac{1}{y}$ RHS = $\frac{yp_0+p_{-1}}{yq_0+q_{-1}}=\frac{y(a_0)+1}{y(1)+0}$ (by definition (*) in qn) $=a_0+\frac{1}{y}$ Thus T_1 is true. Assume that T_k is true for some $k\in\mathbb{Z}^+$, i.e. $[a_0;a_1,a_2,\ldots,a_{k-1},y]=\frac{yp_{k-1}+p_{k-2}}{yq_{k-1}+q_{k-2}}$ for any $y\in\mathbb{R}^+$. For T_{k+1}: LHS $=[a_0;a_1,a_2,\ldots,a_{k-1},a_k,y]$ $=\left[a_0;a_1,a_2,\ldots,a_{k-1},a_k+\frac{1}{y}\right]$ $=\frac{\left(a_k+\frac{1}{y}\right)p_{k-1}+p_{k-2}}{\left(a_k+\frac{1}{y}\right)q_{k-1}+q_{k-2}}$ $\left(\text{by induction hypothesis, }\because a_k+\frac{1}{y}>0\right)$ $=\frac{(ya_k)p_{k-1}+p_{k-1}+yp_{k-2}}{(ya_k)q_{k-1}+q_{k-1}+yq_{k-2}}$ $=\frac{y(a_kp_{k-1}+p_{k-2})+p_{k-1}}{y(a_kq_{k-1}+q_{k-2})+q_{k-1}}$ $=\frac{yp_k+p_{k-1}}{yq_k+q_{k-1}}$ (by the given recurrence relations) This shows that T_k is true implies that T_{k+1} is true. By the principle of mathematical induction, T_n is true for each $n\in\mathbb{Z}^+$.	For questions where you need to show a statement is true for every integer, the induction approach may be applicable. Qn may not directly tell you that you need to use induction. This is the trick to link back to T_k by reducing "one term". For example, $[1;10,7,9]=1+\cfrac{1}{10+\cfrac{1}{7+\frac{1}{9}}}$ $=\left[1;10,7+\frac{1}{9}\right]$ You should clearly explain that $a_k+\frac{1}{y}>0$ so that you can apply the induction hypothesis. Marks not deducted here for this has been deducted in later part of qn.

6b (ii)	$r_n = [a_0; a_1, \ldots, a_{n-1}, a_n]$ $= \dfrac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$ (by part (b)(i) since $a_n > 0$) $= \dfrac{p_n}{q_n}$ (by given recurrence relations)	Show qn, working must be clear.
6b (iii)	Let S_n be the statement $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ for each $n \in \mathbb{Z}^+$. When $n = 1$, $p_1 q_0 - p_0 q_1 = (a_1 p_0 + p_{-1})(1) - a_0(a_1 q_0 + q_{-1})$ $= a_1 a_0 + 1 - a_0 a_1$ $= 1$ $= (-1)^{1-1}$ Hence, S_1 is true. Assume S_k is true for some $k \in \mathbb{Z}^+$, i.e. $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$. LHS of $S_{k+1} = p_{k+1} q_k - p_k q_{k+1}$ $= (a_{k+1} p_k + p_{k-1}) q_k - p_k (a_{k+1} q_k + q_{k-1})$ $= -(p_k q_{k-1} - p_{k-1} q_k)$ $= -(-1)^{k-1}$ (by induction hypothesis) $= (-1)^{(k+1)-1} =$ RHS of S_{k+1} This shows that S_k is true implies that S_{k+1} is true. By the principle of mathematical induction, S_n is true for each $n \in \mathbb{Z}^+$. Hence, $p_n q_{n-2} - p_{n-2} q_n$ $= (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2}(a_n q_{n-1} + q_{n-2})$ $= a_n (p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$ $= \begin{cases} a_n (p_0 \times 0 - 1 \times 1) & \text{if } n = 1 \\ a_n (-1)^{n-1-1} & \text{from previous result if } n \geq 2 \end{cases}$ $= (-1)^n a_n$	For questions where you need to show a statement is true for every integer, the induction approach will be applicable. Qn may not directly tell you that you need to use induction.
6b (iv)	From part **(b)(iii)**, $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ shows that if d divides both p_n and q_n, then d divides $(-1)^{n-1}$, and so d divides 1. This implies that $\gcd(p_n, q_n) = 1$.	$\dfrac{p_n}{q_n}$ in its lowest term means that the $\gcd(p_n, q_n) = 1$. i.e. p_n, q_n are coprime.

6b (v)	Consider $r_n - r_{n-1} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}$ $= \frac{p_n q_{n-1} - q_n p_{n-1}}{q_n q_{n-1}}$ $= \frac{(-1)^{n-1}}{q_n q_{n-1}}$ by 1st result in (b)(iii) and $r_n - r_{n-2} = \frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}}$ $= \frac{p_n q_{n-2} - p_{n-2} q_n}{q_n q_{n-2}}$ $= \frac{(-1)^n a_n}{q_n q_{n-2}}$ by 2nd result in (b)(iii) Since for all $n \in \mathbb{Z}_0^+$, $q_n > 0$, for each $m \in \mathbb{Z}^+$, we have 1. $r_{2m} - r_{2m-2} = \frac{(-1)^{2m} a_{2m}}{q_{2m} q_{2m-2}} > 0$, $\Rightarrow r_{2m-2} < r_{2m}$ $\Rightarrow r_2 < r_4 < r_6 < \ldots$ 2. $r_{2m+1} - r_{2m-1} = \frac{(-1)^{2m+1} a_{2m+1}}{q_{2m+1} q_{2m-1}} < 0$, $\Rightarrow r_{2m+1} < r_{2m-1}$. $\Rightarrow \ldots < r_5 < r_3 < r_1$ 3. $r_{2m} - r_{2m-1} = \frac{(-1)^{2m-1}}{q_{2m} q_{2m-1}} < 0$, $\Rightarrow r_{2m} < r_{2m-1}$. $\Rightarrow r_2 < r_1$ and $r_4 < r_3$ and … Combining, we have $r_2 < r_4 < r_6 < \ldots < r_5 < r_3 < r_1$.	The inequality to show is related to increasing/decreasing trend. Hence we consider the differences of r_n with even (or odd) indices. Recall by definition, $q_{-1} = 0$, $q_0 = 1$, $q_n = a_n q_{n-1} + q_{n-2}$, $n \geq 1$. Thus $q_n > 0$ for $n \geq 1$. Also, $a_1, a_2, \ldots > 0$. Note: The statement to be proven states that all even-indexed convergents are increasing and are all smaller than all the odd-indexed convergents which are decreasing.