

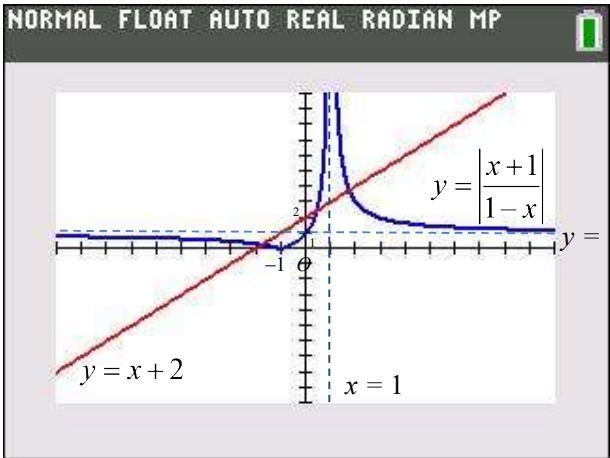


Raffles Institution
H2 Mathematics
Solution for 2015 A-Level Paper 1

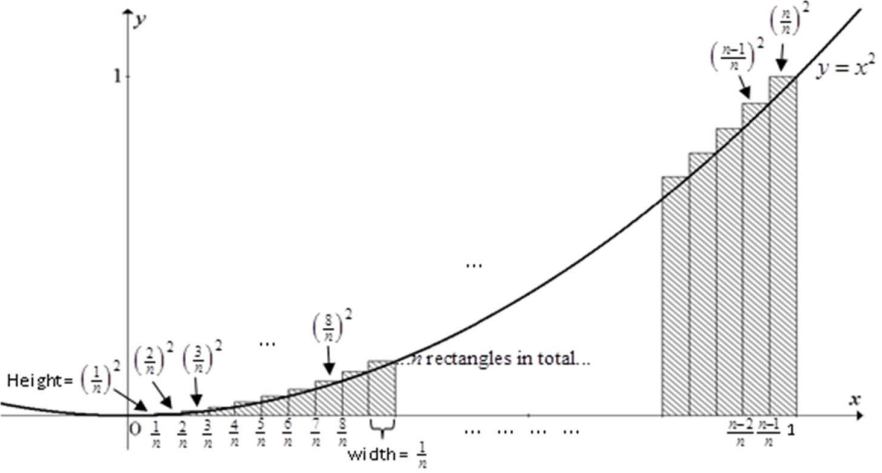
Question 1

No.	Suggested Solution	Remarks for Student
(i)	$y = \frac{a}{x^2} + bx + c$ $\frac{dy}{dx} = -\frac{2a}{x^3} + b$ At $x = 1$, $\frac{dy}{dx} = 2$, we have $-2a + b = 2 \quad \dots(1)$ At $(1.6, -2.4)$, we have $\frac{a}{(1.6)^2} + 1.6b + c = -2.4 \quad \dots(2)$ At $(-0.7, 3.6)$, we have $\frac{a}{(-0.7)^2} - 0.7b + c = 3.6 \quad \dots(3)$ Using GC to solve (1), (2) and (3): $a = -3.59345 \approx -3.593$, $b = -5.18691 \approx -5.187$, $c = 7.30274 \approx 7.303$	Can key in $\frac{1}{(1.6)^2}$
(ii)	Using GC, $x = -0.589$ (3 d.p.)	
(iii)	$y = -5.187x + 7.303$ (3 d.p.)	

Question 2

No.	Suggested Solution	Remarks for Student
(i)	 Label asymptotes: $y = 1$ and $x = 1$	
(ii)	From graphs and GC, the graphs intersect at $x = -1.73, 0.414, 1.73$ For the inequality to hold, we have $-1.73 < x < 0.414$ or $x > 1.73$	

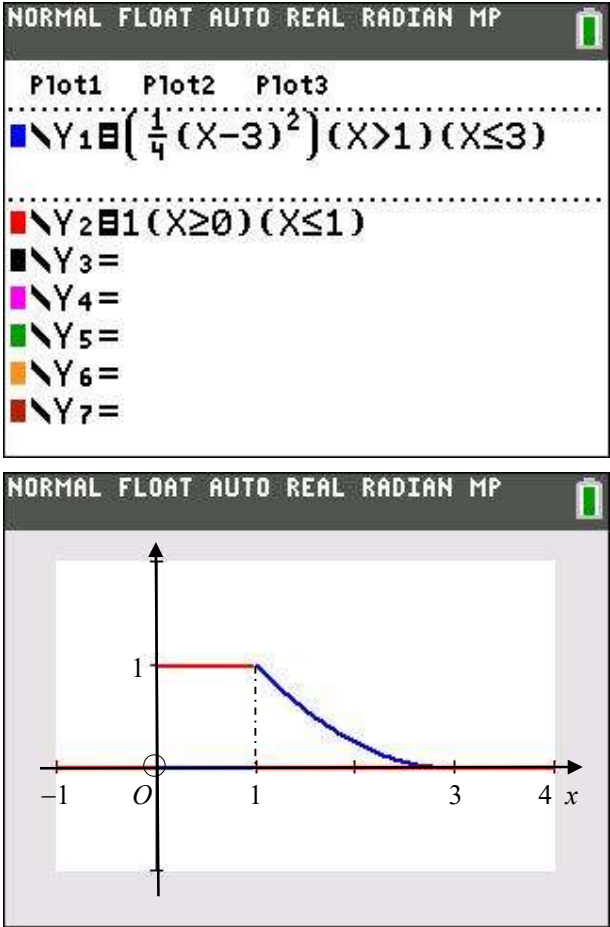
Question 3

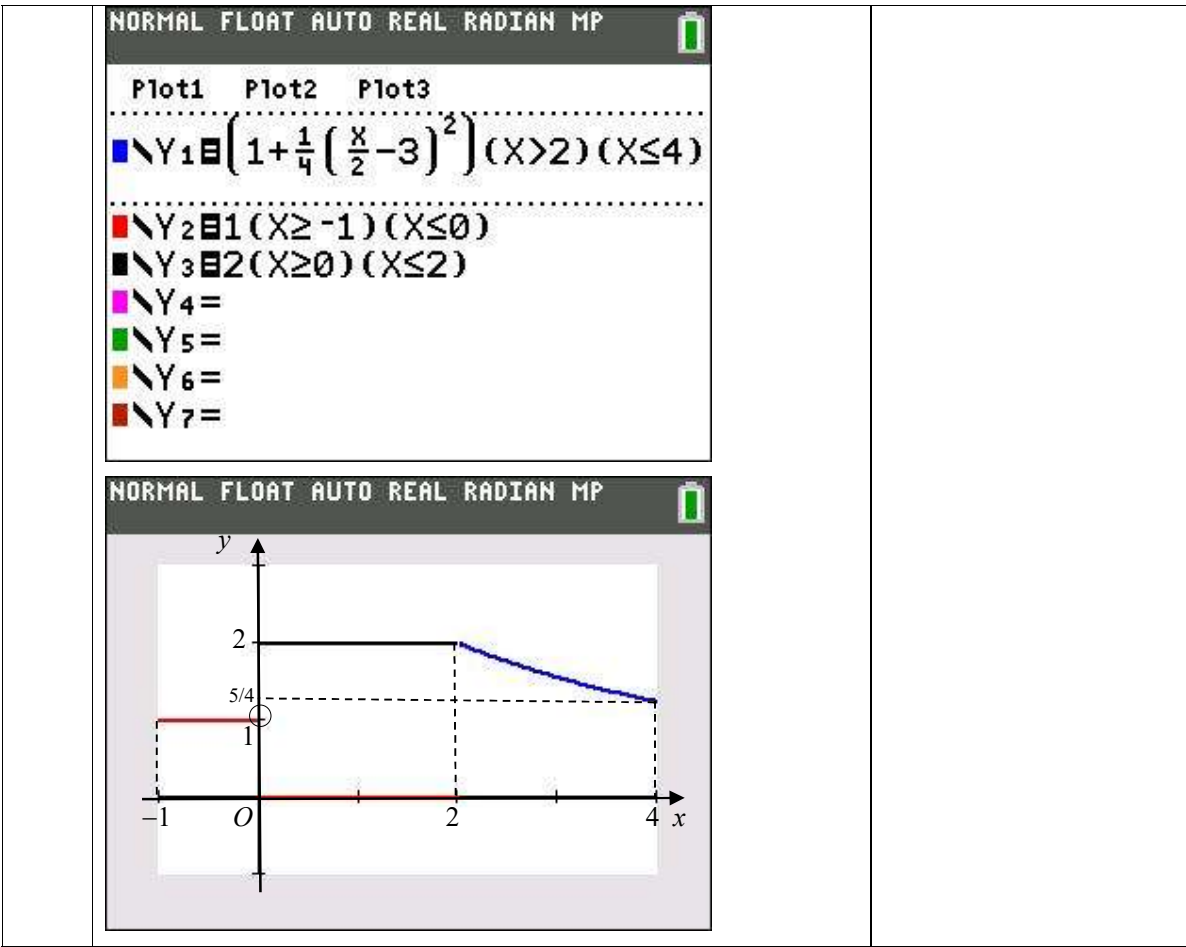
No.	Suggested Solution	Remarks for Student
(i)	 Note that $\frac{1}{n} \left\{ f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right\}$ is the sum of area of the n rectangles in the diagram. As $n \rightarrow \infty$, the sum will approach the exact area under the curve. Hence $\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right\} = \int_0^1 f(x) \, dx$	Since f is any continuous function, we can use $y = x^2$ for convenience.
(ii)	$\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \frac{\sqrt[3]{1} + \sqrt[3]{2} + \dots + \sqrt[3]{n}}{\sqrt[3]{n}} \right\} = \int_0^1 \sqrt[3]{x} \, dx = \left[\frac{3}{4} x^{\frac{4}{3}} \right]_0^1 = \frac{3}{4}$	

Question 4

No.	Suggested Solution	Remarks for Student
	Length of rectangle $= 2x + 2y$ Length of semi-circle $= d - (2x + 2y)$ But length of semi-circle also $= \frac{1}{2}(2\pi x) + 2x = \pi x + 2x = (\pi + 2)x$ Thus, $d - (2x + 2y) = (\pi + 2)x$ $\Rightarrow 2x + 2y = d - 2x - \pi x$ $\Rightarrow y = \frac{1}{2}d - \frac{1}{2}\pi x - 2x$ Total area, $A = xy + \frac{1}{2}\pi x^2$ $= x\left(\frac{1}{2}d - \frac{1}{2}\pi x - 2x\right) + \frac{1}{2}\pi x^2$ $= \frac{1}{2}xd - 2x^2$ $= \frac{1}{2}x(d - 4x)$ This is an quadratic expression with max value attained when $x = \frac{d}{8}$, that is the mid-point of the 2 roots 0 and $\frac{d}{4}$. Thus, max value of $A = \frac{1}{2}\left(\frac{d}{8}\right)\left(d - 4\left(\frac{d}{8}\right)\right) = \frac{1}{32}d^2 \text{ m}^2$ So, $k = \frac{1}{32}$	

Question 5

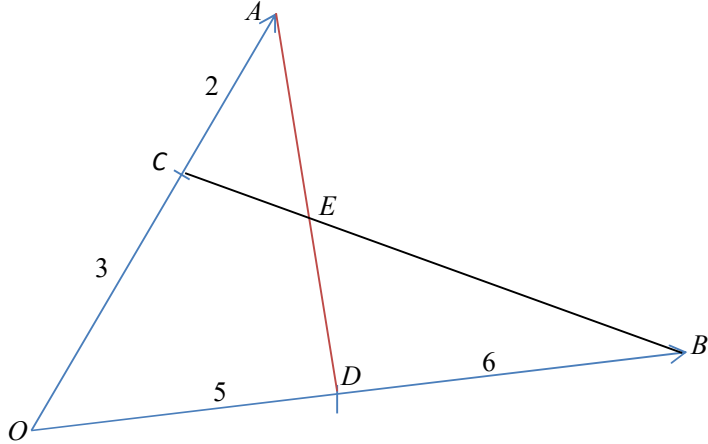
No.	Suggested Solution	Remarks for Student
(i)	$y = x^2$ ↓ replace x by $(x-3)$ $y = (x-3)^2$ ↓ replace y by $y/\frac{1}{4}$ $y = \frac{1}{4}(x-3)^2$ Translate 3 units in the positive x directions followed by scaling of factor $\frac{1}{4}$ parallel to y-axis.	
(ii)		
(iii)		



Question 6

No.	Suggested Solution	Remarks for Student
(i)	$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \dots \approx 2x - 2x^2 + \frac{8}{3}x^3$	
(ii)	$ax(1+bx)^c = ax \left[1 + bcx + \frac{c(c-1)}{2!}(bx)^2 + \frac{c(c-1)(c-2)}{3!}(bx)^3 \dots \right]$ $= ax + abcx^2 + \frac{ab^2c(c-1)}{2}x^3 + \frac{ab^3c(c-1)(c-2)}{6}x^4 \dots$ Given that $ax + abcx^2 + \frac{ab^2c(c-1)}{2}x^3 = 2x - 2x^2 + \frac{8}{3}x^3$ Thus, $a = 2$ $bc = -1$ $-b(c-1) = \frac{8}{3} \Rightarrow -bc + b = \frac{8}{3} \Rightarrow b = \frac{5}{3}$ $c = -\frac{3}{5}$ Coefficient of x^4 is $\frac{ab^3c(c-1)(c-2)}{6} = \frac{2\left(\frac{5}{3}\right)^3\left(-\frac{3}{5}\right)\left(-\frac{3}{5}-1\right)\left(-\frac{3}{5}-2\right)}{6} = -\frac{104}{27}$	

Question 7

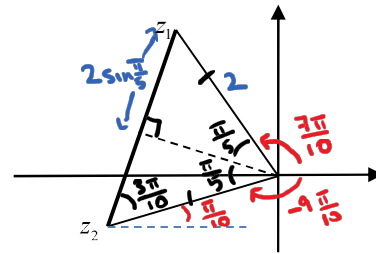
No.	Suggested Solution	Remarks for Student
		
(i)	$\overrightarrow{OC} = \frac{3}{5}\underline{a}, \quad \overrightarrow{OD} = \frac{5}{11}\underline{b}$	
(ii)	$\begin{aligned} l_{BC} : \underline{r} &= \underline{b} + \lambda \overrightarrow{BC}, \lambda \in \mathbb{R} \\ &= \underline{b} + \lambda \left(\frac{3}{5}\underline{a} - \underline{b} \right) \\ &= \frac{3}{5}\lambda \underline{a} + (1 - \lambda)\underline{b} \end{aligned}$ $l_{AD} : \underline{r} = \frac{5}{11}\mu \underline{b} + (1 - \mu)\underline{a}, \mu \in \mathbb{R}$	
(iii)	$\begin{aligned} \frac{3}{5}\lambda \underline{a} + (1 - \lambda)\underline{b} &= \frac{5}{11}\mu \underline{b} + (1 - \mu)\underline{a} \\ \frac{3}{5}\lambda &= 1 - \mu \Rightarrow \frac{3}{5}\lambda + \mu = 1 \\ 1 - \lambda &= \frac{5}{11}\mu \Rightarrow \lambda + \frac{5}{11}\mu = 1 \\ \text{Using GC, } \lambda &= \frac{3}{4}, \mu = \frac{11}{20} \end{aligned}$ $\text{Thus, } \overrightarrow{OE} = \frac{3}{5} \left(\frac{3}{4} \right) \underline{a} + \left(1 - \frac{3}{4} \right) \underline{b} = \frac{9}{20}\underline{a} + \frac{1}{4}\underline{b}$	

$\overline{AE} = \frac{9}{20}a + \frac{1}{4}b - a = \frac{1}{4}b - \frac{11}{20}a$ $\overline{ED} = \frac{5}{11}b - \left(\frac{9}{20}a + \frac{1}{4}b \right) = \frac{9}{44}b - \frac{9}{20}a = \frac{9}{11} \left(\frac{1}{4}b - \frac{11}{20}a \right) = \frac{9}{11} \overline{AE}$ Thus, $AE : ED = 11 : 9$	
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Question 8

No.	Suggested Solution	Remarks for Student
	$1.5h = 1.5 \times 60 \times 60 = 5400 \text{ s}$ $1.75h = 1.75 \times 60 \times 60 = 6300 \text{ s}$	
(i)	$5400 \leq \frac{50}{2} [2T + (50-1)(2)] \leq 6300$ $5400 \leq 50T + 2450 \leq 6300$ $\therefore 59 \leq T \leq 77$ Set of values of T is $[59, 77]$	
(ii)	$5400 \leq \frac{t(1.02^{50} - 1)}{1.02 - 1} \leq 6300$ $108 \leq t(1.02^{50} - 1) \leq 126$ $\therefore 63.845 \leq t \leq 74.486$ Set of values of t is $[63.9, 74.4] \text{ (3 s.f.)}$	Examiner report does not accept $[63.8, 74.5] \text{ (3 s.f.)}$
(iii)	$63.845(1.02)^{50-1} - [59 + (50-1)(2)] = 11.475 \approx 11$	

Question 9

No.	Suggested Solution	Remarks for Student
(a)	$\frac{w^2}{w^*} = \frac{(a+ib)^2}{a-ib}$ $= \frac{(a+ib)^3}{a^2+b^2}$ $= \frac{a^3+3ia^2b-3ab^2-ib^3}{a^2+b^2}$ $= \frac{(a^3-3ab^2)+i(3a^2b-b^3)}{a^2+b^2}$ $\frac{w^2}{w^*}$ is purely imaginary $\operatorname{Re}\left(\frac{w^2}{w^*}\right) = 0 \Rightarrow \frac{(a^3-3ab^2)}{a^2+b^2} = 0$ $\Rightarrow a(a^2-3b^2) = 0$ $\Rightarrow a = 0 \text{ (rejected, since } a \neq 0) \quad \text{or} \quad b = \pm \frac{a}{\sqrt{3}}$ Thus, $w = a + i\frac{a}{\sqrt{3}}$ or $w = a - i\frac{a}{\sqrt{3}}$.	
b(i)	$z^5 = -32i = 2^5 e^{i\left(-\frac{\pi}{2}\right)} = 2^5 e^{i\left(-\frac{\pi}{2}+2k\pi\right)} = 2^5 e^{i\left(\frac{4k\pi-\pi}{2}\right)}$ $\therefore z = 2e^{i\frac{(4k-1)\pi}{10}}, \quad k = -2, -1, 0, 1, 2$	You may want to list down all values if you have time.
(ii)	 $\arg(z_1) = \frac{7\pi}{10}, \quad \arg(z_2) = -\frac{9\pi}{10}$ $ z_1 - z_2 = 2 \times 2 \sin \frac{\pi}{5}$ $\arg(z_1 - z_2) = \frac{3\pi}{10} + \frac{\pi}{10} = \frac{2\pi}{5}$	

Alternatively,

$$z_1 = 2e^{i\frac{7\pi}{10}}, z_2 = 2e^{i\frac{-9\pi}{10}}$$

$$\begin{aligned} z_1 - z_2 &= 2e^{i\frac{7\pi}{10}} - 2e^{i\frac{-9\pi}{10}} \\ &= 2e^{i\left(\frac{-\pi}{10}\right)} \left(e^{i\frac{8\pi}{10}} - e^{i\frac{-8\pi}{10}} \right) \\ &= 2e^{i\left(\frac{-\pi}{10}\right)} \left(2i \sin \frac{8\pi}{10} \right) \\ &= \left(4 \sin \frac{4\pi}{5} \right) e^{i\left(\frac{-\pi}{10}\right)} (i) \\ &= \left(4 \sin \frac{\pi}{5} \right) e^{i\left(\frac{-\pi}{10}\right)} \left(e^{i\frac{\pi}{2}} \right), \quad \text{since } \sin \frac{4\pi}{5} = \sin \left(\pi - \frac{\pi}{5} \right) = \sin \frac{\pi}{5} \\ &= \left(4 \sin \frac{\pi}{5} \right) e^{i\left(\frac{2\pi}{5}\right)} \end{aligned}$$

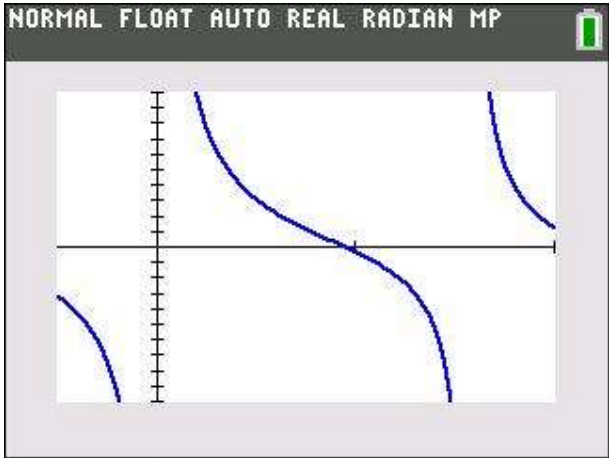
$$\text{Thus, } \arg(z_1 - z_2) = \frac{2\pi}{5} \text{ and } |z_1 - z_2| = 4 \sin \frac{\pi}{5}$$

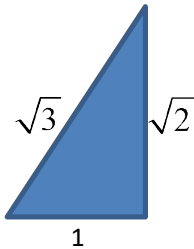
Question 10

No.	Suggested Solution	Remarks for Student
(i)	$A_1 = \int_0^{\frac{\pi}{4}} \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$ $= [-\cos x]_0^{\frac{\pi}{4}} + [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $= \left[-\frac{\sqrt{2}}{2} - (-1) \right] + \left(1 - \frac{\sqrt{2}}{2} \right)$ $= 2 - \sqrt{2}$ $A_2 = \int_0^{\frac{\pi}{4}} \cos x \, dx - \int_0^{\frac{\pi}{4}} \sin x \, dx$ $= [\sin x]_0^{\frac{\pi}{4}} + [\cos x]_0^{\frac{\pi}{4}}$ $= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1$ $= \sqrt{2} - 1$ $= \frac{1}{\sqrt{2}} (2 - \sqrt{2})$ $= \frac{1}{\sqrt{2}} A_1$ $\therefore \frac{A_1}{A_2} = \sqrt{2}$	
(ii)	$y = \sin x \Rightarrow x = \sin^{-1} y$ $\text{Required volume} = \pi \int_0^{\frac{\sqrt{2}}{2}} x^2 \, dy = \pi \int_0^{\frac{\sqrt{2}}{2}} (\sin^{-1} y)^2 \, dy$	
(iii)	$y = \sin u \Rightarrow \frac{dy}{du} = \cos u$ $y = 0 \Rightarrow u = 0$ $y = \frac{\sqrt{2}}{2} \Rightarrow u = \frac{\pi}{4}$	

	$\pi \int_0^{\frac{\sqrt{2}}{2}} \left(\sin^{-1} y \right)^2 dy = \pi \int_0^{\frac{\pi}{4}} u^2 \cos u du \quad (\text{shown})$ $= \pi \left\{ \left[u^2 \sin u \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2u \sin u du \right\}$ $= \pi \left\{ \frac{\pi^2 \sqrt{2}}{32} - \left(\left[-2u \cos u \right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} 2 \cos u du \right) \right\}$ $= \pi \left\{ \frac{\pi^2 \sqrt{2}}{32} + \frac{\pi \sqrt{2}}{4} - \left[2 \sin u \right]_0^{\frac{\pi}{4}} \right\}$ $= \pi \left\{ \frac{\pi^2 \sqrt{2}}{32} + \frac{\pi \sqrt{2}}{4} - \sqrt{2} \right\}$	
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Question 11

No.	Suggested Solution	Remarks for Student
(i)	$x = \sin^3 \theta, \quad y = 3 \sin^2 \theta \cos \theta$ $\frac{dx}{d\theta} = 3 \sin^2 \theta \cos \theta \quad \frac{dy}{d\theta} = 6 \sin \theta \cos^2 \theta - 3 \sin^3 \theta$ $\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ $= \frac{6 \sin \theta \cos^2 \theta - 3 \sin^3 \theta}{3 \sin^2 \theta \cos \theta}$ $= \frac{6 \sin \theta \cos^2 \theta}{3 \sin^2 \theta \cos \theta} - \frac{3 \sin^3 \theta}{3 \sin^2 \theta \cos \theta}$ $= 2 \cot \theta - \tan \theta \text{ (shown)}$	
(ii)	$\frac{dy}{dx} = 0 \Rightarrow 2 \cot \theta - \tan \theta = 0$ $\Rightarrow 2 \left(\frac{1}{\tan \theta} \right) - \tan \theta = 0$ $\Rightarrow \tan^2 \theta = 2$ $\therefore \tan \theta = \sqrt{2} \quad (\text{reject } \tan \theta = -\sqrt{2} \text{ as } 0 \leq \theta \leq \frac{\pi}{2})$ Thus, $k = 2$. C has only one stationary point when $\tan \theta = \sqrt{2}$, for $0 \leq \theta \leq \frac{\pi}{2}$. Graph of $\frac{dy}{d\theta} = 2 \cot \theta - \tan \theta$  Gradient graph indicates there is only one change in sign for $0 \leq \theta \leq \frac{\pi}{2}$. Thus, the stationary point is a turning point.	

	Note that the change in sign is from positive to negative. Thus, first derivative test shows that the turning point is a maximum. A right-angled triangle with $\tan \theta = \sqrt{2}$  Thus, $x = \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^3 = \frac{2\sqrt{2}}{3\sqrt{3}}$, $y = 3\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 \left(\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}$ Coordinates = $\left(\frac{2\sqrt{2}}{3\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$	
(iii)	Note that x increases from 0 to 1 for $0 \leq \theta \leq \frac{\pi}{2}$ $\begin{aligned} \text{Area} &= \int_0^1 y \, dx \\ &= \int_0^{\frac{\pi}{2}} (3 \sin^2 \theta \cos \theta) \left(\frac{dx}{d\theta}\right) d\theta \\ &= \int_0^{\frac{\pi}{2}} (3 \sin^2 \theta \cos \theta) (3 \sin^2 \theta \cos \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} 9 \sin^4 \theta \cos^2 \theta \, d\theta \quad (\text{shown}) \\ &= 0.884 \quad (3 \text{ d.p.}) \end{aligned}$	
(iv)	$\begin{aligned} y = ax &\Rightarrow 3 \sin^2 \theta \cos \theta = a \sin^3 \theta \\ &\Rightarrow \frac{3}{a} = \tan \theta \quad (\text{shown}) \\ &\Rightarrow \frac{3}{a} = \sqrt{2} \\ \therefore a &= \frac{3}{\sqrt{2}} \end{aligned}$	