RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Tutorial 7C: Maclaurin Series

Section A (Basic Questions)

1 It is given that $y = e^{2x}\cos x$.

(a) Show that $\dfrac{d^2y}{dx^2} = 4\dfrac{dy}{dx} - 5y$. [3]

(b) Find the Maclaurin series for y up to the term in x^2. [2]

(c) Hence, show that the Maclaurin series for $e^x\sqrt{\cos x}$ is $1 + x + \dfrac{1}{4}x^2$, up to the term in x^2. [2]

[(b) $1 + 2x + \dfrac{3}{2}x^2 + ...$**]**

2 It is given that $f(x) = \dfrac{\sqrt{4-x}}{1+2x}$.

(i) Using standard series from the List of Formulae (MF 27), find the series expansion for $f(x)$, up to and including the term in x^2. Give the coefficients as exact fractions in their simplest form. [4]

(ii) Hence, by using $x = -\dfrac{1}{6}$, find an estimation for $\sqrt{6}$. Leave your answer as an exact fraction in its simplest form. [2]

[(i) $2 - \dfrac{17}{4}x + \dfrac{543}{64}x^2 + ...$ **(ii)** $\dfrac{5760}{2261}$ or $\dfrac{2261}{960}$**]**

3 In triangle ABC, angle A is $\left(\dfrac{\pi}{4} + \theta\right)$ radians and angle B is $\dfrac{1}{3}\pi$ radians. Show that when θ is sufficiently small for terms in θ^3 and higher powers of θ to be neglected,

$$\frac{AC}{BC} \approx \frac{\sqrt{6}}{2}\left(1 - \theta + k\theta^2\right)$$

where k is a constant to be found. [6]

[$k = \dfrac{3}{2}$ **]**

Section B (Discussion Questions)

1 **(i)** Given that $\mathrm{f}(x)=\ln(1+2\sin x)$, find $\mathrm{f}(0),\ \mathrm{f}'(0),\ \mathrm{f}''(0)$ and $\mathrm{f}'''(0)$. Hence write down the first three non-zero terms in the Maclaurin series for f(x). [7]

(ii) The first two non-zero terms in the Maclaurin series for f(x) are equal to the first two non-zero terms in the series expansion of $\mathrm{e}^{ax}\sin nx$. Using appropriate expansions from the List of Formulae, find the constants a and n. Hence find the third non-zero term of the series expansion of $\mathrm{e}^{ax}\sin nx$ for these values of a and n. [5]

[(i) 0, 2, –4, 14 ; $2x-2x^2+\frac{7}{3}x^3$ **(ii)** $a=-1$; $n=2$; $-\frac{1}{3}x^3$ **]**

2 It is given that $y=\ln\left(2-\mathrm{e}^{-2x}\right)$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=4\mathrm{e}^{-y}-2$. [2]

(b) Hence find the Maclaurin series for y, up to and including the term in x^2. [3]

[(b) $2x-4x^2+\ldots$ **]**

3 It is given that $y=\sin\left[\ln\left(1+2x\right)\right]$.

(i) Show that $\left(1+2x\right)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(1+2x\right)\frac{\mathrm{d}y}{\mathrm{d}x}+ky=0$, where k is a constant to be found. Hence, find the first three non-zero terms of the Maclaurin expansion for y. [6]

(ii) Using standard series from the List of Formulae, verify the correctness of your result from part **(i)** up to and including the term in x^3. [2]

Explain why the expansion is not valid when $x=-\frac{1}{2}$. [1]

[(i) 4 ; $2x-2x^2+\frac{4}{3}x^3+\ldots$ **]**

4 **(a)** A curve has equation $y=\sqrt{1+\tan^{-1}x}$. Show that $y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-\frac{x}{\left(1+x^2\right)^2}$.

By further differentiation, obtain the Maclaurin series for y, up to and including the term in x^3.
State the equation of the tangent to the curve at $x=0$. [6]

(b) Given that x is a sufficiently small angle, show that $\sqrt{1+2\cos x}\approx\sqrt{3}\left(1-\frac{x^2}{6}\right)$.

It is also given that $1-10\sin x=\sqrt{1+2\cos x}$, form a quadratic equation in x and solve it. [4]

[(a) $1+\frac{1}{2}x-\frac{1}{8}x^2-\frac{5}{48}x^3$; $y=1+\frac{1}{2}x$

(b) $\frac{\sqrt{3}}{6}x^2-10x+1-\sqrt{3}=0$; $x=-0.0731$**]**

5 Let $\mathrm{f}(x) = \mathrm{e}^x \sin x$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \le x \le 3$. [2]

(ii) Find the series expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^3. [3]

Denote the answer to part **(ii)** by $\mathrm{g}(x)$.

(iii) On the same diagram as in part **(i)**, sketch the graph of $y = \mathrm{g}(x)$. Label the two graphs clearly. [1]

(iv) Find, for $-3 \le x \le 3$, the set of values of x for which the value of $\mathrm{g}(x)$ is within ± 0.5 of the value of $\mathrm{f}(x)$. [3]

[(ii) $x + x^2 + \frac{1}{3}x^3 + \ldots$ **(iv)** $(-1.96, 1.56)$**]**

6 **(a)** Expand $(1+y)^8$ in ascending powers of y, up to and including the term in y^3. [1]

In the expansion of $(1+x+kx^2)^8$ in ascending powers of x, the coefficient of x^3 is zero. Find the value of the constant k. [3]

(b) Find the first two terms in the expansion of $\dfrac{(9+x)^{\frac{1}{2}}}{1+2x}$ in ascending powers of x. [4]

(c) Find the set of values of x for which the expansion in part **(b)** is valid. [2]

[(a) $1 + 8y + 28y^2 + 56y^3 + \ldots + y^8$**;** $k = -1$**;**

(b) $3 - \frac{35}{6}x + \ldots$ **(c)** $-\frac{1}{2} < x < \frac{1}{2}$ **]**

7 It is given that $y = \mathrm{f}(x)$ is a function such that $\mathrm{f}(0) = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$. Find the first three non-zero terms in the series expansion of $\dfrac{\mathrm{f}(x)}{\sqrt{9-x}}$. [4]

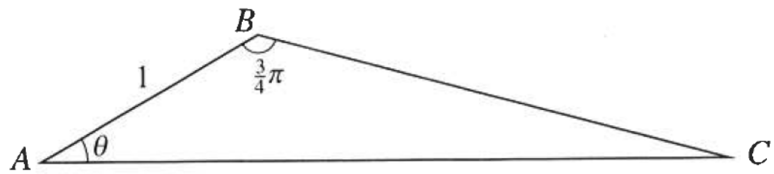
[$\frac{1}{3}\left(1 + \frac{37}{18}x + \frac{457}{216}x^2\right)$ **]**

8 **(a)** In the triangle ABC, $AB = 1$, $BC = 3$ and angle $ABC = \theta$ radians. Given that θ is a sufficiently small angle, show that $AC \approx (4+3\theta^2)^{\frac{1}{2}} \approx a + b\theta^2$, for constants a and b to be determined. [5]

(b) Given that x is small, show that $\tan x - \sqrt{2}\cos\left(\dfrac{\pi}{4} - 2x\right) \approx 2x^2 - x - 1$ [3]

[(a) $a = 2,\ b = \frac{3}{4}$**]**

9



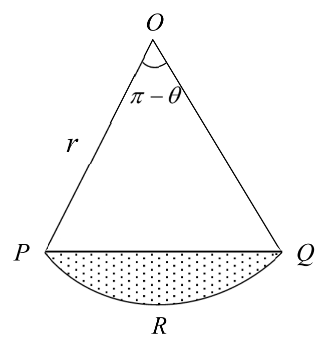
In the triangle ABC, $AB = 1$, angle $BAC = \theta$ radians and angle $ABC = \dfrac{3\pi}{4}$ radians.

(i) Show that $AC = \dfrac{1}{\cos\theta - \sin\theta}$. [4]

(ii) Given that θ is a sufficiently small angle, show that $AC \approx 1 + a\theta + b\theta^2$, for constants a and b to be determined. [4]

[(ii) $a = 1$, $b = 3/2$ **]**

10 An arc PQ of a circle of radius r subtends an angle $(\pi - \theta)$ radians at the centre O. The length of chord PQ is L units and the area of the shaded segment PRQ is A square units.



(i) Show that $L^2 = 2r^2(1+\cos\theta)$. [1]

(ii) Given that θ is small and $A = kL^2$, show that $2k\theta^2 - 2\theta + \pi - 8k \approx 0$. [4]