Anglo-Chinese School

(Independent)

THE BEST IS YET TO BE

FINAL EXAMINATION 2023

YEAR 5 IB DIPLOMA PROGRAMME

MATHEMATICS : ANALYSIS AND APPROACHES

HIGHER LEVEL

PAPER 1

Friday 22 September 2023

2 hours

Candidate Session Number

0	0	2	3	2	9				

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the boxes provided.
- Section B: answer all questions on the writing paper provided. Fill in your session number on each answer sheet, and attach them to this examination paper using the string provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **Mathematics : Analysis and Approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**

Section A (53 Marks)		Section B (57 Marks)	
Question	**Marks**	**Question**	**Marks**
1		**9**	
2			
3			
4		**10**	
5			
6			
7		**11**	
8			
Subtotal		**Subtotal**	
TOTAL	**/ 110**		

This question paper consists of 12 printed pages including this cover page.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines if necessary.

1. [Maximum mark: 4]

Solve the following equation

$$2^{3x-2} = 5(3^{-x}).$$

Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers.

2. [Maximum mark: 4]

It is given that $cosec\theta = \frac{3}{2}$, where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Find the exact value of $cot\theta$.

3. [Maximum mark: 7]

(a) Show that

$$\sum_{r=1}^{n}(2r-1)=n^2$$

[3]

(b) Hence, find the value of $11+13+15+\cdots+99$. [4]

4. [Maximum mark: 7]

The expansion of $(x+h)^7$, where h is a non-zero rational number, can be written as $x^7 + ax^6 + bx^5 + cx^4 + \cdots + h^7$ where a, b, $c \in R$.

Given that a, b and c are three consecutive terms of an arithmetic sequence, find the possible values of h.

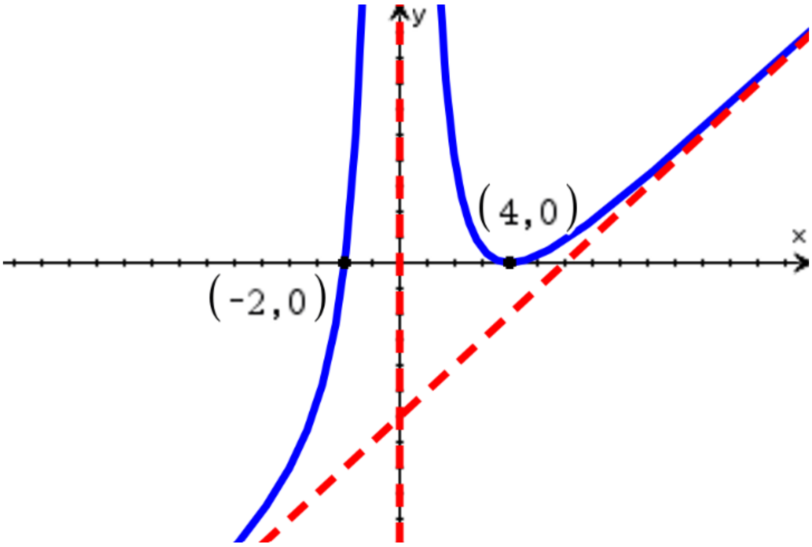
5. [Maximum mark: 9]

(a) Use mathematical induction to prove that for any real number x, $|x|<1$, and any positive integer n, $(1+x)^n \geq 1+nx$. [6]

(b) Hence, show that $0.999^4 > 0.996$. [3]

6. [Maximum mark: 6]

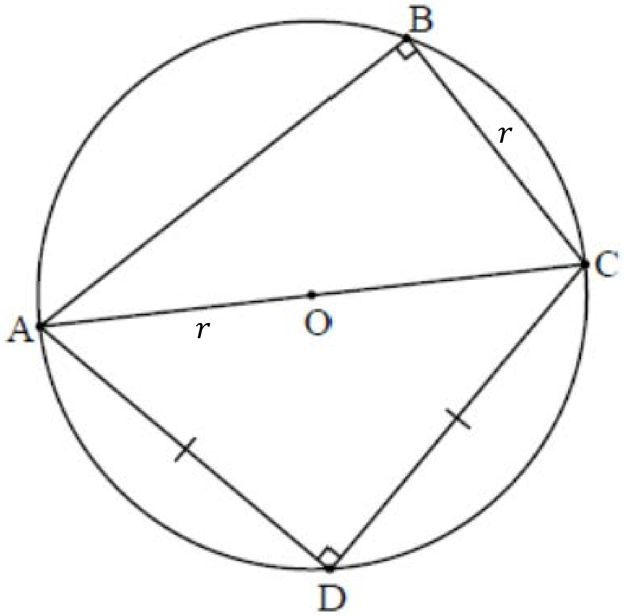
The graph of $y = f(x)$ is shown in the following diagram. The curve intersects the x-axis at $(-2,0)$ and it has a stationary point at $(4,0)$. The equations of the asymptotes are $x = 0$ and $y = 2(x - 6)$.



Sketch the curve $y = \frac{1}{f(x)}$, indicating clearly the equations of any asymptotes, coordinates of any intercepts with the axes and any stationary point(s).

7. [Maximum mark: 10]

In the following diagram, the points A, B, C and D are on the circumference of a circle with centre O and radius r. AC is the diameter of the circle. $BC = r$, $AD = CD$ and $A\hat{B}C = A\widehat{D}C = 90^{\circ}$.



(a) Show that $\cos(B\hat{A}D) = \frac{\sqrt{3}-1}{2\sqrt{2}}$. [5]

(b) Hence or otherwise, show that the area of triangle $ABD = \frac{3+\sqrt{3}}{4}r^2$. [5]

8. [Maximum mark: 6]

Consider the following functions:

$$f(x) = x^2 + ax + b$$

$$g(x) = ax^2 + 2bx - 2 ,$$

where a and b are positive integers.

It is given that the solution of $f(x) \geq g(x)$ is $-5 \leq x \leq 1$, find the values of a and b.

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer sheets provided. Please start each question on a new page.

9. [Maximum mark: 19]

Consider the function $f(x) = \frac{1-2x}{x+1}$, $x \neq -1$.

(a) Sketch the graph of $f(x)$, indicating clearly the equations of any asymptotes and coordinates of any intercepts with the axes. [3]

(b) Given that $g(x) = 3f(x+1)$, find the range and domain of $g(x)$. [2]

(c) Find the inverse function f^{-1}, stating its domain. [4]

(d) Find x where $f(x) = f^{-1}(x)$, giving your answers in the form $\frac{a \pm \sqrt{b}}{2}$, where a and b are integers to be determined. [3]

(e) Sketch the graph of $f(|x|)$. [3]

(f) On a different set of axes, sketch the graph of $y = [f(x)]^2$, indicating clearly the equations of any asymptotes, coordinates of any intercepts with the axes and any stationary point(s). [4]

10. [Maximum mark: 20]

Three points ***A***, ***B*** and ***C*** in three-dimensional space are defined by position vectors $\mathbf{a}$, $\mathbf{b}$ and $3\mathbf{a} - 2\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors.

(a) Given that ***OAB*** is an equilateral triangle, show that $2\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2$. [2]

(b) By considering the vectors $\overrightarrow{\mathbf{AB}}$ and $\overrightarrow{\mathbf{BC}}$, give a geometric interpretation of the relationship between points ***A***, ***B*** and ***C***. [3]

(c) The point ***A*** has coordinates $(2, 0, t)$, where t is a real and positive constant. It is given that the angle between $\overrightarrow{\mathbf{OA}}$ and the x-axis is $arccos\frac{1}{\sqrt{3}}$. Show that $t = 2\sqrt{2}$. [3]

[Turn the page over]

[Question 10 continued]

(d) Consider the lines l_1 and l_2 defined by

$$l_1: \boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} h \\ 2 \\ -2 \end{pmatrix}, \qquad \mu \in R$$

$$l_2: \frac{6-x}{3} = \frac{y-k}{4} = z+1, \qquad \text{where } h \text{ and } k \text{ are constants.}$$

Given the lines are perpendicular, show that the value of $h = 2$. [2]

i) For $k = 3$, find the foot of the perpendicular from the point $\boldsymbol{D}(6,0,-2)$ to l_2. [5]

ii) For $k \neq 7$, prove by contradiction that the lines are skew. [5]

11. [Maximum mark: 18]

The complex numbers u and w satisfy the equations

$$2u^* = u + 1 - 3i$$

$$uw = 7 + i$$

(a) Find u and w, giving your answers in the form $a + ib$, where a and b are real numbers. [5]

(b) Given that $u = 1 + i$ and $w = 4 - 3i$ are the roots of the equation

$$(x^2 + hx + k)(x^2 + mx + p) = 0,$$

find the values of the real numbers h, k, m and p. [3]

Let $z = cos\theta + isin\theta$.

(c) Prove that $z^n + (z^*)^n = 2cos(n\theta)$, $n \in Z^+$. [2]

(d) Write down the binomial expansion of $(z + z^*)^3$ in terms of z. [3]

(e) Hence, show that $cos3\theta = 4cos^3\theta - 3cos\theta$ and find the exact value of $cos135^\circ$. [5]

END OF PAPER