Functions

1. **AJC/I/5/2009**

The function f is defined by $\mathrm{f} : x \to \mathrm{e}^{x^2-2x} - 2, \quad x < 0.$

(i) Find f^{-1} in a similar form. [3]

The function g is defined by $\mathrm{g} : x \to \dfrac{x-3}{x-1}, \ x > k, \ x \neq 1.$

(ii) Find the least value of k such that the function $\mathrm{f}^{-1}\mathrm{g}$ exists. [2]

(iii) Hence find the range of $\mathrm{f}^{-1}\mathrm{g}$, giving your answer in exact form. [1]

2. **IJC/I/10**

(a) The function f is defined by

$$\mathrm{f} : x \mapsto x - 2\tan^{-1}(2x), \qquad -\frac{\sqrt{3}}{2} < x < \frac{\sqrt{3}}{2}.$$

(i) Show that f(x) decreases as x increases. [3]

(ii) Find the range of f. [1]

(iii) Explain whether the composite function ff exists. [2]

(iv) The curve C is obtained by reflecting the graph of f in the line $y = x$. Write down the equation of C. [1]

(b) The function h is defined by

$$\mathrm{h} : x \mapsto x^2 + 2, \qquad x \in \mathbb{R}.$$

The function g is such that the composite function gh exists and

$$\mathrm{gh}(x) = x^4 - 2x^2 + 5, \qquad x \in \mathbb{R}.$$

Find an expression for $\mathrm{g}(x)$. [2]

3. **NYJC/I/2**

The functions f and g are defined by

$$\mathrm{f}: x \to x^2 - 1, \qquad x \in \mathbb{R}$$
$$\mathrm{g}: x \to \sqrt{x+4}, \qquad x \in \mathbb{R}^+$$

(i) Show that the composite function fg exists and define fg in similar form. [2]

(ii) If h is a function defined such that h(x) = gfg(x) for all $x > 0$, show that the composite function gfg exists. Hence, find an expression for h(x). [2]

(iii) Solve $\mathrm{h}(x) = \mathrm{g}^{-1}\mathrm{g}(x)$, giving your answer in exact form. [2]

4. **PJC/I/11**

Three functions f, g, h are defined as follows:

$$\mathrm{f}: x \mapsto 2^x, \qquad x \in \mathbb{R}$$
$$\mathrm{g}: x \mapsto 2^{x-1}, \qquad x \in \mathbb{R}$$
$$\mathrm{h}: x \mapsto x^2 + x - 2, \qquad x \in \mathbb{R}$$

(i) Describe a single geometrical transformation which maps the graph of f onto the graph of g. [1]

(ii) Sketch the graph of h and explain why h^{-1} does not exist. [2]

(iii) Write down the value of a such that $\{x \in \mathbb{R},\ x > a\}$ is the largest possible domain of h for which h^{-1} exists, and hence define h^{-1} in a similar form. [4]

(iv) Using the domain of h found in (iii), explain why $\mathrm{h}^{-1}\mathrm{f}$ exists. Define $\mathrm{h}^{-1}\mathrm{f}$ in a similar form and find its range. [4]

5. **YJC/1/8**

The functions f and g are defined as follows :

$$\mathrm{f}: x \to \ln(x+2), \quad x \geq -2,$$
$$\mathrm{g}: x \to \mathrm{f}(|x|), \quad x \in \mathbb{R}.$$

(a) Solve for all the values of x satisfying the equation $\mathrm{fg}(x) = \mathrm{gf}(x)$. [4]

(b) (i) Briefly explain why the inverse function of g cannot be formed. [1]

(ii) The function g has an inverse if its domain is restricted to $x \leq a$. Determine the largest value of a. [1]

(iii) Define completely the inverse function g^{-1} corresponding to the largest value of a determined in part **(b)(ii)**. [3]

6. **MJC/2013/I/6**

The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \lambda - (x+2)^2, \qquad x \in \mathbb{R}, x > -2,$$
$$\mathrm{g}: x \mapsto \ln(1-x), \qquad x \in \mathbb{R}, x < 1,$$

where λ is a constant.

(i) Given that gf exists, find the largest value of λ. [2]

In the rest of the question, use the value of λ found in part (i).

(ii) On the same diagram, sketch the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$, showing clearly the relationship between the graphs. [3]

(iii) Hence find the exact solution of $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

7. **ACJC/2011/I/9**

Sketch the curve given by the equation $y^2 + ax^2 = 5$ for $x \geq 0$ and $y \geq 0$, where a is a positive constant. [1]

The functions f and g are defined by

$$\mathrm{f} : x \mapsto \sqrt{5 - ax^2}\ , x \in \mathbb{R},\quad 0 \leq x \leq \sqrt{\frac{5}{a}}$$
$$\mathrm{g}: x \mapsto 1 + e^{-x}, x \in \mathbb{R},\ x \geq 0.$$

Show that f^{-1} exists and define f^{-1} in a similar form. [4]

Given that $\mathrm{f}^2(x) = x$, for all $x \in \mathbb{R},\quad 0 \leq x \leq \sqrt{\frac{5}{a}}$,

(i) show that $a = 1$ without evaluating $\mathrm{f}^2(x)$, [2]

(ii) using the result in **(i)**, show that fg exists and find its corresponding range. [3]

(iii) Sketch the curve given by the equation $y^2 + ax^2 = 5$ for $x \geq 0$ and $y \geq 0$, where a is a positive constant. [1]

8. **DHS Prelim/2020/02/Q1**

(a) The functions f and g are defined as follows

$$\mathrm{f} : x \mapsto \sqrt{7 - x}, \qquad \text{where } x \in \mathbb{R}, x < 7,$$
$$\mathrm{g} : x \mapsto (x-2)^4 + 2x^2, \quad \text{where } x \in \mathbb{R}, x > -2.$$

(i) Show that the composite function gf exists and define gf in a similar form. [2]

(ii) Find the range of gf. [2]

(b) The functions k and kh are defined by

$$\mathrm{k} : x \mapsto x - 5, \qquad \text{where } x \in \mathbb{R},$$
$$\mathrm{kh} : x \mapsto x^2 + a, \qquad \text{where } x \in \mathbb{R},\ x > \sqrt{5},$$

where a is a positive constant.

Leave your answers for the following parts, **(b)(i)** and **(ii),** in terms of a.

(i) Find h(x). [2]

(ii) Explain why h^{-1} exists. Hence or otherwise, find $\mathrm{h}^{-1}(x)$ and state the domain of h^{-1}. [3]

9. **NYJC/2014/I/1**

The function f is defined by $\mathrm{f}: x \mapsto 2 - (x+1)^2, x \in \mathbb{R}, x \leq k$.

(i) Determine the largest value of k for which the function f^{-1} exists. [1]

With this value of k,

(ii) find f^{-1} in a similar form. [3]

(iii) show algebraically that the x-coordinate of the point of intersection of the curves $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ satisfies the equation $x^2 + 3x - 1 = 0$ and find the value of this x-coordinate, correct to 3 decimal places. [2]

10. **IJC/2013/I/13**

It is given that

$$f(x)=\begin{cases} \dfrac{3}{(2x+3)^2-2} & \text{for } -2\le x<-1, \\ -3 & \text{for } -1\le x<0, \end{cases}$$

and that $f(x)=f(x+2)$ for all real values of x.

(i) Sketch the graph of $y=f(x)$ for $-\frac{5}{2}\le x\le \frac{5}{2}$. [3]

(ii) Find the exact value of $\int_{-\frac{3}{2}}^{1} |f(x)|\,dx$. [4]

The function h is defined by

$$h : x \mapsto \frac{3}{(2x+3)^2-2} \qquad \text{for } -2\le x<a.$$

(iii) Write down the greatest value of a such that h^{-1} exists. [1]

Assume that a takes the value found in part **(iii)**.

(iv) Sketch, on a single diagram, the graphs of $y=h(x)$ and $y=h^{-1}(x)$. [2]

(v) Explain why the x-coordinate of the point of intersection of the curves in part **(iv)** satisfies the equation

$$4x^3+12x^2+7x-3=0,$$

and find the value of this x-coordinate, correct to 4 significant figures. [3]

11. **AJC/2018/I/9**

The functions f and g are defined by

$$f: x \mapsto (x+3)|x-2|, \ x\in\mathbb{R},$$
$$g: x \mapsto \frac{24}{x^2+3}-1, \qquad x\in\mathbb{R}, x>0.$$

(i) Explain why the function f^{-1} does not exist. [1]

In the rest of the question, the domain of f is restricted to [k, 2] where k is the least value such that f^{-1} exists.

(ii) Write down the value of k. Find $f^{-1}(x)$ and state the domain of f^{-1}. [4]

(iii) Sketch the graphs of $y=f(x)$ and $y=f^{-1}(x)$ on the same diagram, showing clearly the relationship between the two graphs. Hence, find the exact solution of the equation $f(x)=f^{-1}(x)$. [4]

(iv) Determine whether $g^{-1}f$ exists. [2]

Show that g is a strictly decreasing function. Hence, without finding g^{-1}, find the range of values of x for which $g^{-1}(f(x))<1$, giving your answer in exact form. [3]

12. **ACJC Prelim/2020/01/Q6**

The function f is defined by

$$\mathrm{f}: x \mapsto \begin{cases} 1-|3x-1| & \text{for } 0 \le x \le 2, \\ (x-2)^2-4 & \text{for } 2 < x \le 4. \end{cases}$$

It is given that $\mathrm{f}(x+4)=\mathrm{f}(x)$ for all real values of x.

(i) Sketch the graph of $y=\mathrm{f}(x)$ for $-1 \le x \le 9$, stating the coordinates of the end points.

Find the range of f. [4]

The function g is defined by

$$\mathrm{g}: x \mapsto \frac{1}{1+(x-1)^2},\ x \in \mathbb{R}, -2 \le x \le 2.$$

The graph of $y=\mathrm{h}(x)$ is obtained by applying the following transformations in succession to the graph of $y=\mathrm{g}(x)$:

(A) Reflection in the y-axis
(B) Stretch with scale factor 2 parallel to the x-axis
(C) Stretch with scale factor 3 parallel to the y-axis

(ii) Find h(x) and state the domain of h. [3]

(iii) Prove that hf(x) exists and find its range. [2]

13. **SAJC/2019/I/Q3**

The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \frac{x-4}{x-1},\ x \in \mathbb{R}, x \neq 1$$

$$\mathrm{g}: x \mapsto x^2+2x+2,\ x \in \mathbb{R}, x > -1$$

(i) Show that f has an inverse. [1]

(ii) Show that $\mathrm{f}=\mathrm{f}^{-1}$ and hence evaluate $\mathrm{f}^{101}(101)$. [5]

(iii) Prove that the composite function fg exists and find its range. [4]

14. **ACJC/2021/I/Q3**

The function f is defined as $\mathrm{f}: x \mapsto \frac{a}{2-x},\ x \in \mathbb{R},\ \ x \neq 0,\ 2.$

(i) Find f^2 and f^{-1} in terms of a, stating their domains clearly. [5]

(ii) Find the value of a such that $\mathrm{f}^2(x)=\mathrm{f}^{-1}(x)$ for all $x \in \mathbb{R},\ \ x \neq 0,\ 2,\ \frac{a}{2}.$ [2]

(iii) Using the value of a found in **(ii)**, find $\mathrm{f}^{2021}(x)$. [2]

15. **ASRJC/2022/I/Q8**

(a) Functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2, \quad x < 0,$$

$$\mathrm{g}: x \mapsto \frac{1}{x}, \quad x > 0.$$

(i) Explain why the composite function gf exists. [1]

(ii) Find the exact value of $\mathrm{f}^{-1}\mathrm{g}^{-1}(3)$. Show your workings clearly. [3]

(b) For real values a, the function h is defined by

$$\mathrm{h}: x \mapsto ax - \frac{1}{x}, \quad x < 0.$$

(i) If a is negative, explain clearly with a well-labelled diagram, why h^{-1} does not exist. [4]

(ii) If $a = 1$, find h^{-1} in similar form. [3]

16. **DHS Prelim 9758/2023/01/Q11**

The function f is given by

$$\mathrm{f}: x \mapsto a + \frac{3a}{(x+3)(x-1)}, \text{ where } a > 0.$$

(a) Using calculus, find the x-coordinate of the stationary point of $y = \mathrm{f}(x)$. [2]

(b) Sketch the graph of $y = \mathrm{f}(x)$, clearly stating the coordinates of the stationary point and equations of any asymptotes. [3]

(c) State the range of values of x for which the graph of $y = \mathrm{f}(x)$ is increasing and concave downwards. [1]

(d) Sketch the graph of $y = \mathrm{f}'(x)$, clearly stating the coordinates of the points where the graph crosses the axes and equations of any asymptotes. Leave your answers in terms of a where appropriate. [3]

It is given that $\mathrm{g}(x) = \mathrm{f}(x)$ for $x \geq 2$. Suppose the composite function gg exists.

(e) Deduce, with a reason, the condition on a. [3]

(f) Find the exact range of gg for $a = 5$. [2]

17. **ACJC Prelim 9758/2024/02/Q5**

The function h is defined by $\mathrm{h}: x \mapsto \left[\ln(x+2)\right]^2 + 1$, $x \in \mathbb{R}$, $x > -2$.

(a) The function h^{-1} exists if the domain of h is restricted to $-2 < x \le k$. State the greatest possible value of k. [1]

The function f is defined by

$$\mathrm{f}(x) = \begin{cases} [\ln(x+2)]^2 + 1, & \text{for } x \in \mathbb{R},\ -2 < x \le -1, \\ \dfrac{1}{x+2}, & \text{for } x \in \mathbb{R},\ x > -1. \end{cases}$$

(b) Sketch the graph of $y = \mathrm{f}(x)$. [1]

(c) Given that f^{-1} exists, find f^{-1} in a form similar to f. [4]

(d) Show that f^2 exists and find its range. [2]

(e) If $\mathrm{f}^2(2) = \mathrm{f}(x)$, find x. [2]

18. **ASRJC Prelim 9758/2024/01/Q8**

The functions f and g are defined by

$$\mathrm{f}: x \mapsto \left|4 + 2x - x^2\right|,\ x \in \mathbb{R}, x \ge 3.5,$$

$$\mathrm{g}: x \mapsto 4 + \mathrm{e}^{ax},\ x \in \mathbb{R}, x \ge -1, \text{ where } a > 0.$$

(a) Find $\mathrm{f}^{-1}(x)$ and state its domain. [3]

(b) Find the value of x for which $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. [2]

(c) Show that the composite function fg exists and express the exact range of fg in the form of $A + B\mathrm{e}^{-a} + C\mathrm{e}^{-2a}$, where A, B and C are real constants. [4]

(d) Without the use of a graphing calculator, solve the inequality $\dfrac{\mathrm{g}(x)}{x^2 - 2x - 2} \ge 0$. Leave your answer in exact form. [3]

19. **SAJC Prelim 9758/2024/02/Q3**

A function g is said to be self-inverse if $g^{-1}(x) = g(x)$ for all x in the domain of g.

The function f is defined by

$$f(x) = \frac{ax+b}{x-2}, x \in \mathbb{R}, x \neq 2,$$

where a and b are real constants.
It is given that f is self-inverse.

(i) Find a and the value(s) that b cannot take. [4]

(ii) Explain why the composite function f^n always exists for all integers $n > 1$. [1]

(iii) Hence find $f^{2025}(1)$ in terms of b. [2]

The function h is defined for all $x \in \mathbb{R}$ by

$$h(x) = \begin{cases} 3x+2 & x \geq 0, \\ 2x^2 & x < 0. \end{cases}$$

(iv) Sketch the graph of h. [2]

(v) Using the value of a found in **(i)** and given that $b = 1$, find $hf\left(\frac{1}{2}\right)$. [1]

20. **VJC Prelim 9758/2024/01/Q3**

The functions f and g are defined by

$$f : x \mapsto \frac{ax-6}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq 3,\ b \neq 9,$$

$$g : x \mapsto e^{-x} \quad \text{for } x \in \mathbb{R},\ x \geq \ln 3.$$

The function f is self-inverse if $f(x) = f^{-1}(x)$ for all values of x in the domain of f.
It is given that f is self-inverse.

(a) Find the value of a. [3]

(b) Find the exact value of $f^6(\pi)$. [1]

(c) Find the exact range of fg. [3]

21. **EJC Prelim 9758/2025/P1/Q9**

(a) (i) State the coordinates of the turning points of the graph of $y = x^3 - 3x^2 - 9x + 5$. [1]

(ii) The function f is defined by

$$\mathrm{f}(x) = x^3 - 3x^2 - 9x + 5,\ x \in \mathbb{R},\ x \le a$$

where a is a constant. Find the largest possible value of a such that f^{-1} exists, and state the domain of f^{-1} in this case. [2]

(b) The function g is defined by

$$\mathrm{g}(x) = \frac{1-2x}{x+2},\ x \in \mathbb{R},\ x \ne -2.$$

(i) Find $\mathrm{g}^2(x)$. [2]

(ii) Hence find the possible expressions of $\mathrm{g}^n(x)$, where n is a positive integer. [2]

(c) The function h is defined by

$$\mathrm{h}(x) = x^2,\ x \in \mathbb{R},\ x \le 0.$$

(i) Explain why the composite function gh exists. [2]

(ii) Find the range of gh. [2]

Answers

1	(i) $\mathrm{f}^{-1}: x \to 1-\sqrt{1+\ln(x+2)}$, $x > -1$ (ii) $k = 2$ (iii) $\left(1-\sqrt{1+\ln 3}, 0\right)$
2	(a) $R_\mathrm{f} = (-1.23, 1.23)$ ff does not exist $x = y - 2\tan^{-1}(2y)$ (b) $\mathrm{g}(x) = (x-3)^2 + 4$
3	(i) fg:$x \to x+3,\quad x \in \Re^+$ (ii) $\mathrm{h}: x \to \sqrt{x+7},\quad x > 0$ (iii) $\frac{1}{2}\left(1+\sqrt{29}\right)$
4	(iii) $a = -\frac{1}{2}$, $\mathrm{h}^{-1}: x \mapsto \frac{-1+\sqrt{4x+9}}{2}, x > -\frac{9}{4}$ (iv) $\mathrm{h}^{-1}\mathrm{f}: x \mapsto \frac{-1+\sqrt{4(2^x)+9}}{2},\quad x \in \mathbb{R}$ $\mathrm{R}_{\mathrm{h}^{-1}\mathrm{f}} = (1, \infty)$
5	(a) $x \ge 0$ or $x = -\sqrt{3}$ (b)(ii) 0 (b)(iii) $\mathrm{g}^{-1}: x \to 2 - e^x$, $x \in [\ln 2, \infty)$
6	(i) $R_\mathrm{f} = (-\infty, \lambda)$ $D_\mathrm{g} = (-\infty, 1)$ Since $R_\mathrm{f} \subseteq D_\mathrm{g}$, largest λ=1. (iii) $\frac{-5+\sqrt{13}}{2}$
7	$\mathrm{f}^{-1}: x \mapsto \sqrt{\frac{5-x^2}{a}},\quad 0 \le x \le \sqrt{5}$ (ii) $[1, 2)$
8	(a)(i) $\mathrm{gf}: x \mapsto \left(\sqrt{7-x}-2\right)^4 + 2(7-x),\ x \in \mathbb{R},\quad x < 7$ (ii) $\mathrm{R}_\mathrm{gf} = [3, \infty)$ (b)(i) $\mathrm{h}(x) = x^2 + 5 + a$ (ii) $\mathrm{h}^{-1}(x) = \sqrt{x-5-a},\quad x \in \mathbb{R},\quad x > 10 + a$
9	(i) $k = -1$ (ii) $\mathrm{f}^{-1}: x \mapsto -1-\sqrt{2-x}$, $x \le 2$ (iii) $x = -3.303$
10	(ii) $3 - \frac{9}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}-1}{1+\sqrt{2}}\right)$ (iii) $-\frac{3}{2}$ (v) $x = -1.781$
11	(ii) Least $k = -\frac{1}{2}$, f^{-1}: $x \mapsto -\frac{1}{2} + \frac{1}{2}\sqrt{25-4x},\quad x \in \mathbb{R},\ 0 \le x \le \frac{25}{4}$.

	(iii) $x=-1+\sqrt{7}$ (v) $-\frac{1}{2}\le x<\frac{-1+\sqrt{5}}{2}$
12	(i) $\mathrm{f}^{-1}:x\to 1-\sqrt{1+\ln\left(x+2\right)}$, $x>-1$ (ii) Least k = 2, $\left(1-\sqrt{1+\ln 3},0\right)$
13	(ii) $\mathrm{f}^{-1}(x)=1-\frac{3}{x-1}=\frac{x-4}{x-1}$, $\frac{97}{100}$ (iii) $\left(-\infty,1\right)$
14	(i) $\mathrm{f}^2:x\mapsto\frac{a(2-x)}{4-a-2x}$, $x\ne 0,2$; $\mathrm{f}^{-1}:x\mapsto 2-\frac{a}{x}$, $x\ne 0,\frac{a}{2}$ (ii) $a=4$ (iii) $\mathrm{f}^{2021}(x)=2-\frac{4}{x}$
15	$\mathrm{h}^{-1}:x\mapsto\frac{x-\sqrt{x^2+4}}{2}$, $x\in\mathbb{R}$
16	(a) -1 (c) $-3<x<-1$ (e) $a\ge 2$ (f) $R_{\mathrm{gg}}=[5\frac{15}{77},5\frac{15}{32})$
17	(a) Greatest possible value of $k=-1$ (c) $\mathrm{f}^{-1}\left(x\right)=\begin{cases}-2+e^{-\sqrt{x-1}}, & \text{for } x\ge 1,\\ -2+\frac{1}{x}, & \text{for } 0<x<1\end{cases}$ (d) $R_{f^2}=\left(0,\frac{1}{2}\right)$ (e) $x=\frac{1}{4}$
18	(a) $\mathrm{f}^{-1}(x)=1+\sqrt{x+5}$, $\mathrm{D}_{\mathrm{f}^{-1}}=[1.25,\infty)$ (b) 4 (c) $\mathrm{R}_{\mathrm{fg}}=\left[4+6\mathrm{e}^{-a}+\mathrm{e}^{-2a},\infty\right)$ (d) $-1\le x<1-\sqrt{3}$ OR $x>1+\sqrt{3}$
19	(i) $a=2$ $b\ne -4$ (iii) $-2-b$ (v) $\frac{32}{9}$
20	(a) $a=3$ (b) π (c) $\left(\frac{15}{8},\ 2\right)$
21	(a)(i) $(-1,10)$ and $(3,-22)$ (a)(ii) $a=-1$, $\left(-\infty,10\right]$ (b)(i) x (b)(ii) $\mathrm{g}^n(x)=\begin{cases}x & \text{if } n \text{ is even}\\ \frac{1-2x}{x+2} & \text{if } n \text{ is odd}\end{cases}$ (c)(ii) $\left(-2,\frac{1}{2}\right]$