NANYANG JUNIOR COLLEGE

DEPARTMENT OF MATHEMATICS

2026 J2 H2 FM

Linear Algebra (Set 2)

Instruction: *Attempt the questions marked with *.*

1* The matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as follows.

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \text{ with } \theta \geq 0 \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The transformations T_1 and T_2 from $\mathbb{R}^2$ to $\mathbb{R}^2$ are defined by

$$\mathrm{T}_1 : \begin{pmatrix} x \\ y \end{pmatrix} \to \mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \mathrm{T}_2 : \begin{pmatrix} x \\ y \end{pmatrix} \to \mathbf{B}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) By considering $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}$ for $r > 0$ and $\alpha \in [0, 2\pi]$, describe the geometrical transformation T_1, explaining your answer. [2]

(b) Describe T_2 geometrically. [1]

Let matrix $\mathbf{M}$ be $\begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix}$.

(c) Without using a calculator, state the smallest positive integer value of k such that $\mathbf{M}^k$ gives the identity matrix and explain your answer. [2]

(d) Given that $\mathbf{C} = \mathbf{M}\mathbf{B}\mathbf{M}^{-1}$, find the Cartesian equations of the two lines through the origin which are invariant under the transformation given by the matrix $\mathbf{C}$. [3]

2* **(a)** The matrix $\mathbf{A}$ has eigenvectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ with corresponding eigenvalues -1, 3 and 1 respectively. Find $\mathbf{A}$. [2]

(b) The vectors $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ are defined as:

$$\mathbf{y}_1 = \begin{pmatrix} 7 \\ -4 \\ 5 \end{pmatrix}, \mathbf{y}_2 = \begin{pmatrix} -1 \\ 8 \\ 13 \end{pmatrix}, \mathbf{y}_3 = \begin{pmatrix} 66 \\ -60 \\ 6 \end{pmatrix}.$$

Given that $\mathbf{Q}$ is a 3 x 3 matrix, determine whether or not the vectors $\mathbf{Q}\mathbf{y}_1$, $\mathbf{Q}\mathbf{y}_2$, $\mathbf{Q}\mathbf{y}_3$ are linearly independent, justifying your conclusion. [2]

(c) **(i)** Show that the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} -4 & \frac{1}{a} & 3 \\ 2a & 1 & -a \\ -6 & \frac{2}{a} & 5 \end{pmatrix}$$

are independent of a, where a is a non-zero constant. [3]

(ii) Obtain, in terms of a, an eigenvector corresponding to each eigenvalue. [3]

(iii) Hence find a matrix **P** and a diagonal matrix **D** such that $(\mathbf{A}^{-1} + 2\mathbf{I})^3 = \mathbf{PDP}^{-1}$. [2]

3 Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x + y \\ x - y + z \end{pmatrix} \text{ for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

(a) Find a basis for the range of T and explain why the dimension of the kernel of T is 1. [3]

(b) Find a basis $\mathbf{v}_1$ for the kernel of T, where $\mathbf{v}_1$ is in the form $\begin{pmatrix} x_1 \\ y_1 \\ 6 \end{pmatrix}$ and $x_1, y_1 \in \mathbb{R}$ are constants to be determined. Hence, find a general solution $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$. [3]

(c) Let $\mathbf{v}_2 = \begin{pmatrix} a \\ -a + 3b^2 \\ 3 - 2a \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} a^2 \\ -a^2 - 9 \\ -2a^2 - 9 \end{pmatrix}$. Determine the conditions $a, b \in \mathbb{R}$ such that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$, explaining your answers clearly. [4]