Section A: Pure Mathematics [44 marks]

	Solution	
2.	(a) $\det(\mathbf{A}) \neq 0 \Rightarrow \mathbf{A}$ is invertible but $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is non-invertible. <u>OR</u> $\det\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not in the set. Since the set does not contain the zero matrix, it is not a subspace of $\mathbf{M}_{2\times2}(\mathbb{R})$. (b) Let U denote the set of all 2×2 matrices $\mathbf{B}$ such that $\mathbf{B}^T = -\mathbf{B}$. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ Hence $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in U$ and U is non-empty. Consider any two elements $\mathbf{B}$, $\mathbf{C} \in U$. $(\mathbf{B}+\mathbf{C})^T = \mathbf{B}^T + \mathbf{C}^T = -\mathbf{B} + (-\mathbf{C}) = -(\mathbf{B}+\mathbf{C})$ $\therefore \mathbf{B}+\mathbf{C} \in U$ (Closure under addition) Consider any element $\mathbf{B} \in U$ and $k \in \mathbb{R}$. $(k\mathbf{B})^T = k\mathbf{B}^T = k(-\mathbf{B}) = -(k\mathbf{B})$ $\therefore k\mathbf{B} \in U$ (Closure under scalar multiplication) Hence U is a subspace of $\mathbf{M}_{2\times2}(\mathbb{R})$.	

	Solution	
3.	$x = \sec\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec\theta\tan\theta.$ $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}x}\sec\theta\tan\theta.$	

	$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)\sec\theta\tan\theta+\frac{\mathrm{d}y}{\mathrm{d}x}\left(\sec\theta\tan^2\theta+\sec^3\theta\right)$$ $$=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\sec^2\theta\tan^2\theta+\frac{\mathrm{d}y}{\mathrm{d}x}\left(\sec\theta\tan^2\theta+\sec^3\theta\right)$$ $$=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\sec^2\theta\left(\sec^2\theta-1\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\sec\theta\left(\sec^2\theta-1+\sec^2\theta\right)$$ $$=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(\sec^4\theta-\sec^2\theta\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\sec\theta\left(2\sec^2\theta-1\right)$$ $$=\left(x^4-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\left(2x^2-1\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$ $$\frac{1}{x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}\right)=\left(x^3-x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(2x^2-1\right)\frac{\mathrm{d}y}{\mathrm{d}x}.$$ Substituting into the differential equation in x and y: $$\frac{1}{x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}\right)+\frac{ky}{x}=\frac{2}{x^3}$$ $$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}+ky=\frac{2}{x^2}$$ $$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}+ky=\frac{2}{\sec^2\theta}$$ $$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2}+ky=2\cos^2\theta\ \text{ (shown)}$$ Characteristic equation: $$m^2+k=0$$ $$m=\pm\sqrt{k}\mathrm{i}\quad(\because k>0)$$ Complementary function: $$y_c=A\cos\sqrt{k}\theta+B\sin\sqrt{k}\theta$$ $$=A\cos\sqrt{k}\left(\sec^{-1}x\right)+B\sin\sqrt{k}\left(\sec^{-1}x\right).$$ Since $2\cos^2\theta=\cos2\theta+1$, particular integral: $$y_p=C\cos2\theta+D\sin2\theta+E$$ $$y_p'=-2C\sin2\theta+2D\cos2\theta$$ $$y_p''=-4C\cos2\theta-4D\sin2\theta.$$ Substituting into the new differential equation: $$-4C\cos2\theta-4D\sin2\theta+k\left(C\cos2\theta+D\sin2\theta+E\right)=\cos2\theta+1$$ $$C\left(k-4\right)\cos2\theta+D\left(k-4\right)\sin2\theta+kE=\cos2\theta+1$$ Comparing coefficients:	

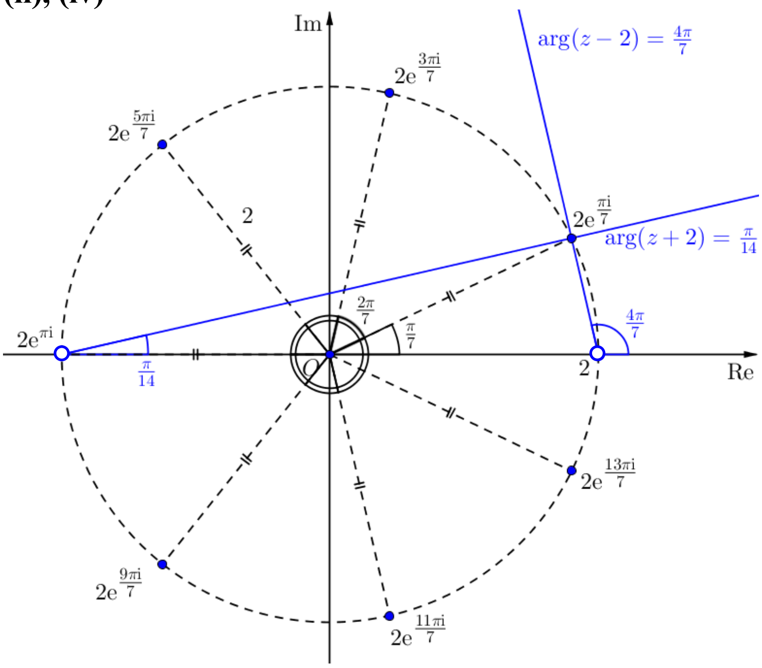
	$kE=1$ $\quad C(k-4)=1$ $\quad$ and $\quad D(k-4)=0$ $E=\frac{1}{k}$, $\quad C=\frac{1}{k-4}$, $\quad D=0 \quad (\because k\neq 4)$. $$\therefore y_p=\frac{1}{k-4}\cos 2\theta+\frac{1}{k}.$$ Thus, $y=A\cos\sqrt{k}\theta+B\sin\sqrt{k}\theta+\frac{1}{k-4}\cos 2\theta+\frac{1}{k}$. Now, $\cos 2\theta=2\cos^2\theta-1$ $=\frac{2}{\sec^2\theta}-1$ $=\frac{2}{x^2}-1=\frac{2-x^2}{x^2}$. Thus, the general solution is $$y=A\cos\sqrt{k}\left(\sec^{-1}x\right)+B\sin\sqrt{k}\left(\sec^{-1}x\right)+\frac{2-x^2}{(k-4)x^2}+\frac{1}{k}$$ where A and B are arbitrary constants.	

	Source: TMJC Promo 2025 Qn 4
(a)	$\mathrm{f}(x,y)=\sqrt{x^2+y^2}$ $\mathrm{f}_x=\frac{x}{\sqrt{x^2+y^2}}$ $\mathrm{f}_y=\frac{y}{\sqrt{x^2+y^2}}$ $\nabla\mathrm{f}=\begin{pmatrix}\frac{x}{\sqrt{x^2+y^2}}\\ \frac{y}{\sqrt{x^2+y^2}}\end{pmatrix}$ At $x=1, y=2$, $\nabla\mathrm{f}=\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$ Directional derivative in the direction of greatest ascent is in the direction $\mathbf{u}=\begin{pmatrix}1\\2\end{pmatrix}$ $D_{\mathbf{u}}\mathrm{f}=\nabla\mathrm{f}\bullet\mathbf{u}=\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}\bullet\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}=1$
(b)	Equation of tangent plane at $x=x_0, y=y_0$: $z=\mathrm{f}(x_0,y_0)+\mathrm{f}_x(x_0,y_0)(x-x_0)+\mathrm{f}_y(x_0,y_0)(y-y_0)$ $z=\sqrt{x_0{}^2+y_0{}^2}+\frac{x_0}{\sqrt{x_0{}^2+y_0{}^2}}(x-x_0)+\frac{y_0}{\sqrt{x_0{}^2+y_0{}^2}}(y-y_0)$

	When $x=0, y=0$, $z=\sqrt{x_0^{\,2}+y_0^{\,2}}+\frac{x_0}{\sqrt{x_0^{\,2}+y_0^{\,2}}}\left(-x_0\right)+\frac{y_0}{\sqrt{x_0^{\,2}+y_0^{\,2}}}\left(-y_0\right)$ $=\sqrt{x_0^{\,2}+y_0^{\,2}}-\frac{x_0^{\,2}+y_0^{\,2}}{\sqrt{x_0^{\,2}+y_0^{\,2}}}$ $=0$ Hence, $(0,0,0)$ lies on all tangent planes. Hence, all tangent planes to the surface pass through the origin.

(c)	Equation of normal to the plane at $x=x_0, y=y_0$: $\mathbf{r}=\begin{pmatrix} x_0 \\ y_0 \\ \sqrt{x_0^{\,2}+y_0^{\,2}} \end{pmatrix}+\lambda\begin{pmatrix} \frac{x_0}{\sqrt{x_0^{\,2}+y_0^{\,2}}} \\ \frac{y_0}{\sqrt{x_0^{\,2}+y_0^{\,2}}} \\ -1 \end{pmatrix}, \lambda\in\mathbb{R}.$ In parametric form: $x=x_0+\frac{\lambda x_0}{\sqrt{x_0^{\,2}+y_0^{\,2}}}=x_0\left(1+\frac{\lambda}{\sqrt{x_0^{\,2}+y_0^{\,2}}}\right)$ $y=y_0+\frac{\lambda y_0}{\sqrt{x_0^{\,2}+y_0^{\,2}}}=y_0\left(1+\frac{\lambda}{\sqrt{x_0^{\,2}+y_0^{\,2}}}\right)$ $z=\sqrt{x_0^{\,2}+y_0^{\,2}}-\lambda$ Sub into equation of surface $z=\mathrm{f}(x,y)=\sqrt{x^2+y^2}$, we have: $\sqrt{x_0^{\,2}+y_0^{\,2}}-\lambda=\sqrt{\left(x_0\left(1+\frac{\lambda}{\sqrt{x_0^{\,2}+y_0^{\,2}}}\right)\right)^2+\left(y_0\left(1+\frac{\lambda}{\sqrt{x_0^{\,2}+y_0^{\,2}}}\right)\right)^2}$ $=\sqrt{\left(x_0^{\,2}+y_0^{\,2}\right)\left(1+\frac{\lambda}{\sqrt{x_0^{\,2}+y_0^{\,2}}}\right)^2}$ $=\sqrt{x_0^{\,2}+y_0^{\,2}}\left\|\left(1+\frac{\lambda}{\sqrt{x_0^{\,2}+y_0^{\,2}}}\right)\right\|$ $=\left\|\sqrt{x_0^{\,2}+y_0^{\,2}}+\lambda\right\|$

	$\sqrt{{x_0}^2+{y_0}^2}-\lambda=\pm\left(\sqrt{{x_0}^2+{y_0}^2}+\lambda\right)$ $\sqrt{{x_0}^2+{y_0}^2}-\lambda=\sqrt{{x_0}^2+{y_0}^2}+\lambda$ or $\sqrt{{x_0}^2+{y_0}^2}-\lambda=-\left(\sqrt{{x_0}^2+{y_0}^2}+\lambda\right)$ $-\lambda=\lambda$ or $1=-1$ (n.a.) $\lambda=0$ When $\lambda=0$, we get the point $\left(x_0,y_0,\sqrt{{x_0}^2+{y_0}^2}\right)$ so the normal only intersects the surface at $\left(x_0,y_0,\sqrt{{x_0}^2+{y_0}^2}\right)$ and does not intersect the surface again.

	Solution	
5.	**(i)** $z^7+128=0$ $z^7=-128$ $z^7=128\mathrm{e}^{\mathrm{i}(2k+1)\pi},\ k\in\mathbb{Z}$ $z=2\mathrm{e}^{\mathrm{i}\left(\frac{2k+1}{7}\right)\pi},\ k=0,1,2,3,4,5,6$ $=2\mathrm{e}^{\mathrm{i}\frac{\pi}{7}},\ 2\mathrm{e}^{\mathrm{i}\frac{3\pi}{7}},\ 2\mathrm{e}^{\mathrm{i}\frac{5\pi}{7}},\ 2\mathrm{e}^{\mathrm{i}\pi},\ 2\mathrm{e}^{\mathrm{i}\frac{9\pi}{7}},\ 2\mathrm{e}^{\mathrm{i}\frac{11\pi}{7}},\ 2\mathrm{e}^{\mathrm{i}\frac{13\pi}{7}}$ **(ii), (iv)**  **(iii)** Considering sum of roots of the equation $z^7+128=0$ and taking real part,	

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}+\cos\pi+\cos\frac{9\pi}{7}+\cos\frac{11\pi}{7}+\cos\frac{13\pi}{7}=0$$

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}+(-1)+\cos\frac{5\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{\pi}{7}=0$$

$$2\left(\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}\right)-1=0$$

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$$

$$1-2\sin^2\left(\frac{\pi}{14}\right)+1-2\sin^2\left(\frac{3\pi}{14}\right)+1-2\sin^2\left(\frac{5\pi}{14}\right)=\frac{1}{2}$$

$$3-2\left[\sin^2\left(\frac{\pi}{14}\right)+\sin^2\left(\frac{3\pi}{14}\right)+\sin^2\left(\frac{5\pi}{14}\right)\right]=\frac{1}{2}$$

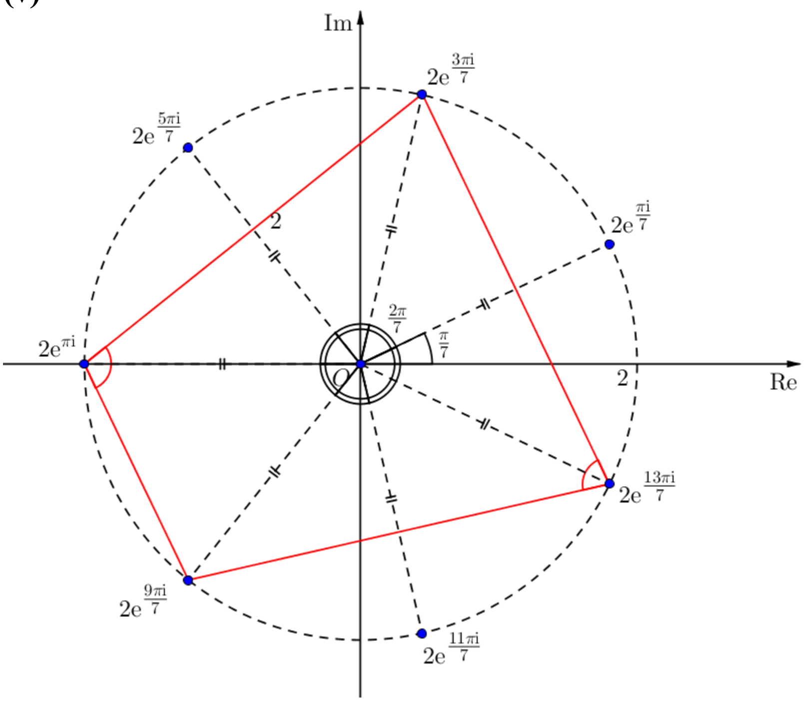
$$\therefore \sin^2\left(\frac{\pi}{14}\right)+\sin^2\left(\frac{3\pi}{14}\right)+\sin^2\left(\frac{5\pi}{14}\right)=\frac{3-\frac{1}{2}}{2}=\frac{5}{4}$$

(iv)
Area

$$=\frac{1}{2}(2)^2\left(\sin\left(3\times\frac{2\pi}{7}\right)+\sin\frac{\pi}{7}\right) \text{ or } \frac{1}{2}\times 4\times 2\sin\frac{\pi}{7}$$

$$=4\sin\frac{\pi}{7} \text{ or } 1.74 \text{ units}^2$$

(v)



$$\arg\left(\frac{(z_1-z_2)(z_3-z_4)}{(z_3-z_2)(z_1-z_4)}\right)$$

$$=\left(\arg(z_1-z_2)-\arg(z_3-z_2)\right)+\left(\arg(z_3-z_4)-\arg(z_1-z_4)\right)$$

This argument represents the sum of two opposite angles in a cyclic quadrilateral (or equivalently angles in opposite segments) and thus they must add up to π radians (shown).

(Note that since z_1, z_2, z_3 and z_4 are any 4 distinct roots, the quadrilateral shown above is not unique.)

Section B: Statistics [50 marks]

	Solution	

6. **(i)** H_0 : The colour preference is independent of gender

H_1 : The colour preference is dependent on gender

Under H_0, the expected frequencies are

$$e_i = \frac{\text{row total} \times \text{column total}}{\text{grand total}}$$

	White	**Green**	**Black**	**Blue**	**Red**	**Others**
Male	92	75.3	65.3	53.3	84	30
Female	46	37.7	32.7	26.7	42	15

Degree of freedom $= (6-1)\times(2-1) = 5$

From GC, $\chi^2_{\text{test}} = 14.02682385$ and $p\text{-value} = 0.0154399223$

Since p-value < 0.05, we reject H_0 at the 5% significance level and conclude that there is sufficient evidence to claim that preferences for colours of gaming keyboards differ between male and female gamers.

(ii) The table of contributions are as follows:

	White	**Green**	**Black**	**Blue**	**Red**	**Others**
M	0.0435	0.00147	2.08	0.00208	2.01	0.533
F	0.0870	0.00295	4.17	0.00417	4.02	1.07

The largest contributions are from the "Black" and "Red" columns. More male prefer black while fewer male prefer red than expected, and more female prefer red while fewer female prefer black than expected.

	Solution	

7. **(i)** To carry out a t-test on the data will require the condition that the scores (or difference between the scores) to be normally distributed. This condition might not be met by the data and hence it is not appropriate to use the t-test.

(ii) Let X = score after module – score before module

Assume that the distribution of the population of the differences is symmetrical.

H_0 : Population median of X is 0

H_1 : Population median of X is greater than 0

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*
1st	71	70	65	42	74	81	56	69	57	77	81	55
2nd	51	72	46	93	95	31	81	84	88	67	70	82
X	−20	2	−19	51	21	−50	25	15	31	−10	−11	27
R	6	1	5	12	7	11	8	4	10	2	3	9

From table, the sum of negative ranks, $Q = 27$.

From MF26, at 1% level of significance with $n = 12$,

One-tail critical value of $T \le 9 \quad (\text{or } \ge 69)$

Since 27 falls beyond the critical region, we do not reject H_0 and conclude at 1% level of significance that there is insufficient evidence to claim that the population median for the scores in the writing test has improved (i.e. insufficient evidence to conclude that the module is effective).

	Solution	
8.	**(i)** For a p.d.f., $\int_0^1 \frac{k}{x^2 - x + 1}\,dx = 1$ $\Rightarrow k\int_0^1 \frac{1}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx = 1$ $\Rightarrow k\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1 = 1$ $\Rightarrow \frac{2k}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1 = 1$ $\Rightarrow \frac{2k}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right] = 1$ $\Rightarrow \frac{2k}{\sqrt{3}}\left[\frac{\pi}{6} + \frac{\pi}{6}\right] = 1$ $\Rightarrow k = \frac{3\sqrt{3}}{2\pi}$ **(ii)** From symmetry, Mean = Median = Mode = 0.5 **(iii)** For $0 \le x \le 1$, $\mathrm{P}(X < x) = \frac{3\sqrt{3}}{2\pi}\int_0^x \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dt$ $= \frac{3}{\pi}\left[\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)\right]_0^x$ $= \frac{3}{\pi}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + \frac{1}{2}$	

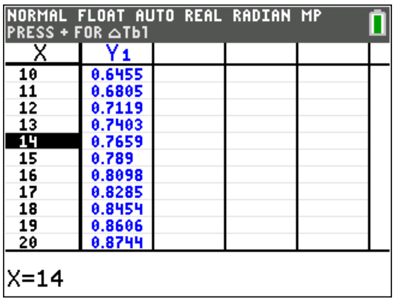
	Hence $\mathrm{F}(x)=\begin{cases}0 & \text{for } x<0,\\ \dfrac{3}{\pi}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right)+\dfrac{1}{2} & \text{for } 0\le x\le 1,\\ 1 & \text{for } x>1.\end{cases}$ **(iv)** $\mathrm{E}(Y)=\mathrm{E}\left(50\tan^{-1}X\right)=\dfrac{3\sqrt{3}}{2\pi}\displaystyle\int_0^1\dfrac{50\tan^{-1}x}{x^2-x+1}\mathrm{d}x$ $=22.03293348\approx 22.0$ (3 s.f.) **(v)** $\mathrm{P}(Y>25)=\mathrm{P}\left(50\tan^{-1}X>25\right)=\mathrm{P}\left(X>\tan\left(\dfrac{1}{2}\right)\right)$ $=\dfrac{3\sqrt{3}}{2\pi}\displaystyle\int_{\tan(1/2)}^1\dfrac{1}{x^2-x+1}\mathrm{d}x=0.4489963784\approx 0.449$ (3 s.f.)	

	Solution	

	Solution	
10.	**(i)** We need to assume that (1) the arrivals of customers occur independently, and (2) the mean number of customer arrivals over any 10-min period remains constant. **(ii)** Assume that the number of customers arriving and the number of food delivery orders received are independent. Let S be the random variable denoting the sum of the number of customers arriving and the number of food delivery orders received in a 10-minute interval. Then $S\sim\mathrm{Po}(2+5)=\mathrm{Po}(7)$. $\mathrm{P}(S\le 10)=0.901479206\approx 0.901$ (3 s.f.) **(iii)** The conditions necessary are (1) the event that each successive interval is busy is independent of all other intervals,and (2) the probability of each interval being busy is constant. The conditions are addressed by the independent nature of the Poisson distribution. **(iv)** $\mathrm{P}(S>10)=0.098521$ From (b), $B\sim\mathrm{Geo}(0.098521)$ $\mathrm{E}(B)=\dfrac{1}{0.098521}=10.1501415\approx 10.2$ (3 s.f.)	

$$\mathrm{Var}(B)=\frac{1-0.098521}{0.098521^2}=92.87523101\approx 92.9 \text{ (3 s.f.)}$$

(v) $\mathrm{P}(B>m)<0.25 \quad \Rightarrow \quad \mathrm{P}(B\le m)>0.75$



NORMAL FLOAT AUTO REAL RADIAN MP
PRESS + FOR △Tbl

X	Y1
10	0.6455
11	0.6805
12	0.7119
13	0.7403
14	0.7659
15	0.789
16	0.8098
17	0.8285
18	0.8454
19	0.8606
20	0.8744

X=14

From GC, least m = 14

END OF PAPER