Syllabus Content

- **Sequences and series,** e.g. general terms, sum, limiting behaviours, bounds
- Summation of series by the **method of differences**
- Integration by **Reduction formulae**

1 Method of Differences

Consider the sum $\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]$ for some function f. By writing it as an explicit sum, we can see that many terms cancel when evaluating this:

$$\begin{aligned}\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right] &= [\mathrm{f}(1)-\mathrm{f}(0)\\ &\quad +\mathrm{f}(2)-\mathrm{f}(1)\\ &\quad +\mathrm{f}(3)-\mathrm{f}(2)\\ &\quad +\ldots\\ &\quad +\mathrm{f}(n-2)-\mathrm{f}(n-3)\\ &\quad +\mathrm{f}(n-1)-\mathrm{f}(n-2)\\ &\quad +\mathrm{f}(n)-\mathrm{f}(n-1)]\\ &= \mathrm{f}(n)-\mathrm{f}(0)\end{aligned}$$

This is called the **method of differences** because u_r is a **difference** of two functions in r.

Example 1

Find $\sum_{r=1}^{n}\frac{1}{r^3-r}$ using the method of differences.

2 Integration by Reduction Formulae

Example 2 [2017/PJC/Prelim/Q3]

Let $u_n = \int_0^2 \frac{x^n}{x^3+1}\,\mathrm{d}x$.

(i) Evaluate, in exact form,

(a) $u_0 + u_1$ [3]

(b) $u_0 - u_1 + u_2$ [2]

(c) u_2 [2]

(ii) Deduce the value of u_0 and u_1. [2]

(iii) Show that $u_{n+2} = \frac{2^n}{n} - u_{n-1}$. [3]

Example 3

If $I_n = \int \cos^n x \,\mathrm{d}x$, show that $I_n = \dfrac{1}{n}\sin x \cos^{n-1} x + \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2, n \in \mathbb{Z}^+$.

Hence find (a) $\int \cos^4 x \,\mathrm{d}x$ and (b) $\int_0^{\pi/2} \cos^5 x \,\mathrm{d}x$.

3 Convergence of sequences and series

In university, you will learn the epsilon-delta definitions of limits and convergence. Here, we will content ourselves with an intuitive understanding of convergence. Roughly speaking,

A sequence $u_1, u_2, u_3, \ldots$ converges to a limit *L* if all later terms get arbitrarily close to *L*.
We write this as $\lim_{n \to \infty} u_n = L$, or $u_n \to L$ as $n \to \infty$.
If there does not exist such a limit *L*, we say that the sequence diverges.

Example: $u_k = \frac{1}{2^k}$ does **not** converge to –1 because later terms do **not** get arbitrarily close to –1. In fact, for all values of *k*, $|u_k - (-1)| > 1$. The sequence converges to 0 because $|u_k - 0|$ approaches 0 as $k \to \infty$.

A series $u_1 + u_2 + u_3 + \cdots$ converges to a limit *L* if the sequence of partial sums converges to *L*:
That is, the sequence $u_1, (u_1 + u_2), (u_1 + u_2 + u_3), \ldots$ converges to *L*
We write this as $\sum_{r=1}^{\infty} u_r = L$, or $\sum_{r=1}^{n} u_r \to L$ as $n \to \infty$.

Example: The series $\sum_{i=1}^{\infty} \frac{1}{2^k}$ refers to the sequence of partial sums $\frac{1}{1}, \frac{1}{1} + \frac{1}{2}, \frac{1}{1} + \frac{1}{2} + \frac{1}{4}, \ldots$. Since that sequence converges to 2, we write $\sum_{i=1}^{\infty} \frac{1}{2^k} = 2$.

Algebra of Limits
Assume that (a_n) and (b_n) are sequences such that (a_n) converges to *L* and (b_n) converges to *M*. Then the following hold as $n \to \infty$:

(i) (addition) $a_n + b_n \to L + M$
(ii) (subtraction) $a_n - b_n \to L - M$
(iii) (scalar multiplication) $ca_n \to cL$ for any constant *c*.
(iv) (modulus) $|a_n| \to |L|$
(v) (product) $a_n b_n \to LM$
(vi) (quotient) $\frac{a_n}{b_n} \to \frac{L}{M}$ if $M \neq 0$
(vii) (powers) $a_n{}^k \to L^k$ for any constant *k*.

The following results are useful tests for convergence of a sequence/series:

The first result says that an increasing sequence bounded above is convergent.

If a sequence $u_1, u_2, u_3, \ldots$ satisfies:
- $u_1 \le u_2 \le u_3 \le \ldots$, and
- there is a *k* such that $u_n < k$ for all *n*

then the sequence is convergent.

The second result is a squeeze theorem for sequences:

> Given three sequences $(a_n), (b_n), (c_n)$ such that $a_n \le b_n \le c_n$ for each n,
> if both (a_n) and (c_n) converge to a limit L then (b_n) also converges to L.

The third result is a squeeze theorem for series:

> Given three sequences $(u_n), (v_n), (w_n)$ such that $u_n \le v_n \le w_n$ for each n,
> if both $\sum_{r=1}^{n} u_r$ and $\sum_{r=1}^{n} w_r$ are convergent then $\sum_{r=1}^{n} v_r$ is also convergent.

A corollary is the comparison test:

> Given two series $(a_n), (b_n)$ such that $0 \le a_n \le b_n$ for each n:
> - If $\sum_{r=1}^{\infty} a_r$ diverges, then $\sum_{r=1}^{\infty} b_r$ also diverges.
> - If $\sum_{r=1}^{\infty} b_r$ converges, then $\sum_{r=1}^{\infty} a_r$ also converges.

Two well-known results follow from these. You are expected to be familiar with both the results and their proofs.

Example 4

Prove that the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$ converges.

Example 5

Prove that the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$ does not converge.

Example 6 [2020/RI/Prelim/Q1b]

Let $S_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \cdots + (-1)^n \frac{1}{2n+1}$.

(i) State the sum of $1 - x^2 + x^4 - \cdots + (-1)^n x^{2n}$. [1]

(ii) Hence show that $S_n = A + (-1)^n \int_0^1 \frac{x^{2n+2}}{1+x^2}\, \mathrm{d}x$, where A is a constant to be determined. [3]

(iii) Deduce that $|S_n - A| < \frac{1}{2n+3}$, and hence determine the limit of the sequence S_n. [3]

Tutorial Questions

1. Find $\displaystyle\sum_{r=1}^{n}\frac{2}{4r^2+8r+3}$ in terms of n.

2. By applying the method of differences to the function f defined by $\mathrm{f}(n)=2n^3+3n^2+n$, show that $\displaystyle\sum_{r=1}^{k}r^2=\frac{k(k+1)(2k+1)}{6}$.

3. It is given that $I_n=\displaystyle\int_0^{\alpha}\frac{x^{2n}}{\sqrt{1-x^2}}\,\mathrm{d}x$, $0<\alpha<1$, $n\geq 0$.

Show that $2(n+1)I_{n+1}=(2n+1)I_n-\alpha^{2n+1}\sqrt{1-\alpha^2}$.

4. Differentiate $\dfrac{\sin x}{\cos^{n-1}x}$ with respect to x.

Hence, given that $I_n=\displaystyle\int_0^{t}\frac{1}{\cos^n x}\,\mathrm{d}x$, prove that $(n-1)I_n=\dfrac{\sin t}{\cos^{n-1}t}+(n-2)I_{n-2}$.

5. If $I_n=\int_0^1 x^n\mathrm{e}^{-x^2}\,\mathrm{d}x$, prove that $2I_n=(n-1)I_{n-2}-\mathrm{e}^{-1}$ for $n\geq 2$ and evaluate I_1.

Hence or otherwise, find the exact value of I_9.

By considering the magnitudes of x^n and e^{-x^2} for $0\leq x\leq 1$ and $n\geq 0$, show that $0<I_n<1$ and deduce that $\dfrac{65}{24}<\mathrm{e}<\dfrac{65}{22}$.

6. [**2017/Q2**]

(i) Let y be a differentiable function of x. For any positive integer n, prove that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(xy)=x\frac{\mathrm{d}^n y}{\mathrm{d}x^n}+n\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}$$ [5]

(ii) For any non-negative integer n, define

$$y_n=\mathrm{e}^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x^2}\right).$$

(a) Find y_0, y_1 and y_2. [3]

(b) Prove that $y_{n+2}+2xy_{n+1}+2(n+1)y_n=0$, for $n\geq 0$. [4]

(c) Hence prove that $\dfrac{\mathrm{d}}{\mathrm{d}x}(y_{n+1})=-2(n+1)y_n$, for $n\geq 0$. [3]

7. Determine whether each of the following sequences converge or diverge.

(a) $u_n = \sin\frac{n\pi}{2}$, $n = 1, 2, 3, \dots$

(b) $u_n = \frac{n^3}{3^n}$, $n = 1, 2, 3, \dots$

(c) $u_n = \frac{n^n}{n!}$, $n = 1, 2, 3, \dots$

(d) $u_n = \frac{\sin n}{n}$, $n = 1, 2, 3, \dots$

8. **[N2002/Q5 (S Paper)]**

The geometric progression U has terms $u_1, u_2, u_3, \dots$, with common ratio r, where $|r| < 1$. It is given that $v_i = u_i^2$ for $i = 1, 2, 3, \dots$.

(i) Show that $\sum_{i=1}^{n} v_i = \frac{u_1}{1+r} \sum_{i=1}^{2n} u_i$.

It is given further that $w_i = v_i - v_{i+1}$ for $i = 1, 2, 3, \dots$.

(ii) Show that $\sum_{i=1}^{n} w_i = u_1(1-r) \sum_{i=1}^{2n} u_i$.

Let $S_U = \sum_{i=1}^{\infty} u_i$, $S_V = \sum_{i=1}^{\infty} v_i$ and $S_W = \sum_{i=1}^{\infty} w_i$. Show that

(iii) $\frac{S_u}{S_V} + \frac{1}{S_U} = \frac{2}{u_1}$,

(iv) $S_W = u_1^2$.

9. **(i)** Sketch the graph of $y = \sin x$ and determine the largest constant a and the smallest constant b such that $ax \le \sin x \le bx$ for $0 \le x \le \frac{\pi}{2}$.

(ii) Hence determine for every positive integer k, if the series

$$\sin^k \frac{1}{1} + \sin^k \frac{1}{2} + \sin^k \frac{1}{3} + \dots$$

is convergent or divergent.

10. The r^{th} term u_r of an arithmetic progression U is given by

$$u_r = a + (r-1)d, \quad \text{for } r = 1, 2, 3, \ldots.$$

The progression is such that there exists a term of U equal to a^2. Given that a and d are positive integers, show that

(a) for each term u_r of U, there exists a term of U equal to u_r^2,

(b) for each term u_r of U, there exists a term of U equal to u_r^3,

(c) all the terms of the geometric progression $a, a^2, a^3, \ldots$ are terms of U.

If a and d are not required to be positive integers, show, by giving a counter-example, that the result in part (i) is not necessarily true.

11. [DHS-EJC/2019/Q3(part)]

(a) Let *x*, *y*, *z* be real numbers such that $0 < x < y < z$. Show that x^2, y^2, z^2 are three consecutive terms of an arithmetic sequence if and only if $\dfrac{1}{y+z}$, $\dfrac{1}{x+z}$, $\dfrac{1}{x+y}$ are three consecutive terms of another arithmetic sequence. [3]

(b) An infinite sequence of integers, a_n, $n \in \mathbb{Z}^+$, is defined by $a_1 = a_2 = 1$ and

$$a_{n+1} = \frac{a_n^2 + 2}{a_{n-1}}, \ n \geq 2.$$

(i) Show that $\dfrac{a_{n+2} + a_n}{a_{n+1}} = \dfrac{a_{n+1} + a_{n-1}}{a_n}$ for $n \geq 2$. [2]

(ii) Hence or otherwise, show that $a_{n+1} = 4a_n - a_{n-1}$ and explain why a_n is an odd integer for all $n \in \mathbb{Z}^+$. [2]

12. [2018/Q1]

For any positive integer *n*, the function F_n is defined by

$$\mathrm{F}_n(x) = \sum_{r=1}^{n} \frac{\left(r^2 - r\right)x^2 + \left(2r^2 + 4r - 1\right)x + 1}{r\left(r+1\right)\left(\left(r-1\right)x + 1\right)\left(rx + 1\right)}.$$

(i) Find $\mathrm{F}_n(0)$ and describe the behaviour of $\mathrm{F}_n(0)$ as *n* tends to infinity. [5]

(ii) It is given that the *r*th term of $\mathrm{F}_n(x)$ can be expressed as

$$\frac{1}{r} - \frac{1}{r+1} + \frac{2}{\left(r-1\right)x + 1} - \frac{2}{rx + 1}.$$

(a) Find $\mathrm{F}_n(x)$. [3]

(b) Describe the behaviour of $\mathrm{F}_n(x)$ as *n* tends to infinity. [3]

13. [2019/ASRJC/Prelim/Q7]

Two sequences $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are given by $u_1 = 1,\ v_1 = 1,\ u_{n+1} = u_n + 3v_n,\ v_{n+1} = 2u_n + 7v_n$, for positive integers n. The sequence $r_1, r_2, r_3, \ldots$ is such that $r_n = \dfrac{u_n}{v_n}$ for positive integers n.

(i) Use induction to prove that $2u_n^{\ 2} - 3v_n^{\ 2} + 6u_nv_n = 5$ for all positive integers n. [4]

It is given that as $n \to \infty,\ v_n \to \infty$ and $r_n \to L$ for some real constant L.

(ii) Using the results in **(i)** or otherwise, show that $L = \frac{1}{2}\left(-3 + \sqrt{15}\right)$. [3]

(iii) Describe the behaviour of the sequence $r_1, r_2, r_3, \ldots$, justifying your answer by considering $r_{n+1} - r_n$. Hence, show that $\sqrt{15}$ lies within the interval $\left(\dfrac{15}{2r_n + 3},\ 2r_n + 3\right)$ for all positive integers n. [4]

(iv) Deduce that $\dfrac{213}{55} < \sqrt{15} < \dfrac{275}{71}$. [2]

H3 Sequences and Series Tutorial Suggested Solutions

1.	Find $\displaystyle\sum_{r=1}^{n}\frac{2}{4r^2+8r+3}$ in terms of n.
	$\displaystyle\sum_{r=1}^{n}\frac{2}{4r^2+8r+3}$ $\displaystyle=\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$ $\displaystyle=\frac{1}{3}+\sum_{r=2}^{n}\frac{1}{2r+1}-\sum_{r=1}^{n-1}\frac{1}{2r+3}-\frac{1}{2n+3}$ $\displaystyle=\frac{1}{3}+\sum_{r=2}^{n}\frac{1}{2r+1}-\sum_{s=2}^{n}\frac{1}{2s+1}-\frac{1}{2n+3}$ using substitution $s=r+1$ $\displaystyle=\frac{1}{3}-\frac{1}{2n+3}$ since the sums are identical with a different dummy variable

2.	By applying the method of differences to the function f defined by $\mathrm{f}(n)=2n^3+3n^2+n$, show that $\displaystyle\sum_{r=1}^{k}r^2=\frac{k(k+1)(2k+1)}{6}$.
	Notice that $\mathrm{f}(n-1)=2(n-1)^3+3(n-1)^2+(n-1)=2n^3-3n^2+n$. By method of differences, $\displaystyle\sum_{r=1}^{k}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]=\mathrm{f}(k)-\mathrm{f}(0)$ $\displaystyle\Rightarrow\sum_{r=1}^{k}\left[\left(2r^3+3r^2+r\right)-\left(2r^3-3r^2+r\right)\right]=\left(2k^3+3k^2+k\right)-(0)$ $\displaystyle\Rightarrow\sum_{r=1}^{k}\left(6r^2\right)=2k^3+3k^2+k$ $\displaystyle\Rightarrow\sum_{r=1}^{k}r^2=\frac{2k^3+3k^2+k}{6}=\frac{k\left(2k^2+3k+1\right)}{6}=\frac{k(k+1)(2k+1)}{6}$ (shown)

3.	It is given that $\displaystyle I_n=\int_0^{\alpha}\frac{x^{2n}}{\sqrt{1-x^2}}\,\mathrm{d}x$, $0<\alpha<1$, $n\geq 0$. Show that $2(n+1)I_{n+1}=(2n+1)I_n-\alpha^{2n+1}\sqrt{1-\alpha^2}$.
	We have $\displaystyle I_{n+1}=\int_0^{\alpha}\frac{x^{2n+2}}{\sqrt{1-x^2}}\,\mathrm{d}x$

	$=\int_0^{\alpha}\frac{x}{\sqrt{1-x^2}}\left(x^{2n+1}\right)\mathrm{d}x$ $=\left[\left(-\sqrt{1-x^2}\right)\left(x^{2n+1}\right)\right]_0^{\alpha}-\int_0^{\alpha}\left(-\sqrt{1-x^2}\right)(2n+1)\left(x^{2n}\right)\mathrm{d}x$ using integration by parts $=\left(-\sqrt{1-\alpha^2}\right)\left(\alpha^{2n+1}\right)+(2n+1)\int_0^{\alpha}\frac{\left(1-x^2\right)\left(x^{2n}\right)}{\sqrt{1-x^2}}\mathrm{d}x$ $=-\alpha^{2n+1}\sqrt{1-\alpha^2}+(2n+1)\int_0^{\alpha}\frac{x^{2n}-x^{2n+2}}{\sqrt{1-x^2}}\mathrm{d}x$ $=-\alpha^{2n+1}\sqrt{1-\alpha^2}+(2n+1)I_n-(2n+1)I_{n+1}$ Rearranging, $(2n+2)I_{n+1}=(2n+1)I_n-\alpha^{2n+1}\sqrt{1-\alpha^2}$ as required.

4.	Differentiate $\frac{\sin x}{\cos^{n-1}x}$ with respect to x. Hence, given that $I_n=\int_0^t\frac{1}{\cos^n x}\mathrm{d}x$, prove that $(n-1)I_n=\frac{\sin t}{\cos^{n-1}t}+(n-2)I_{n-2}$.
	$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos^{n-1}x}=\frac{\cos x}{\cos^{n-1}x}+(\sin x)\frac{-(n-1)(-\sin x)}{\cos^n x}$ by product rule $=\frac{1}{\cos^{n-2}x}+(n-1)\frac{\sin^2 x}{\cos^n x}$ $=\frac{1}{\cos^{n-2}x}+(n-1)\frac{1-\cos^2 x}{\cos^n x}$ $=(n-1)\frac{1}{\cos^n x}+(2-n)\frac{1}{\cos^{n-2}x}$ Integrating both sides with respect to x from $x=0$ to $x=t$, $\left[\frac{\sin x}{\cos^{n-1}x}\right]_0^t=(n-1)\int_0^t\frac{1}{\cos^n x}\mathrm{d}x+(2-n)\int_0^t\frac{1}{\cos^{n-2}x}\mathrm{d}x$ $\frac{\sin t}{\cos^{n-1}t}=(n-1)I_n+(2-n)I_{n-2}$ $\frac{\sin t}{\cos^{n-1}t}+(n-2)I_{n-2}=(n-1)I_n$ (shown)

5.	If $I_n=\int_0^1 x^n\mathrm{e}^{-x^2}\,\mathrm{d}x$, prove that $2I_n=(n-1)I_{n-2}-\mathrm{e}^{-1}$ for $n\geq 2$ and evaluate I_1. Hence or otherwise, find the exact value of I_9.

	By considering the magnitudes of x^n and e^{-x^2} for $0 \le x \le 1$ and $n \ge 0$, show that $0 < I_n < 1$ and deduce that $\frac{65}{24} < e < \frac{65}{22}$.

$$I_n = \int_0^1 x^n e^{-x^2}\ dx$$
$$= -\frac{1}{2}\int_0^1 \left(x^{n-1}\right)\left(-2xe^{-x^2}\right) dx$$
$$= \left(-\frac{1}{2}\right)\left[\left(x^{n-1}\right)\left(e^{-x^2}\right)\right]_0^1 + \frac{1}{2}\int_0^1 (n-1)\left(x^{n-2}\right)\left(e^{-x^2}\right) dx$$
$$= -\frac{1}{2e} + \frac{n-1}{2} I_{n-2}$$

Rearranging,

$$2I_n = (n-1)I_{n-2} - e^{-1} \quad \text{(shown)}$$

$$I_1 = \int_0^1 xe^{-x^2}\ dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2} - \frac{1}{2e}$$

Using the result repeatedly,

$$I_3 = \frac{1}{2}\left[2I_1 - \frac{1}{e}\right] = \frac{1}{2} - \frac{1}{e},$$
$$I_5 = \frac{1}{2}\left[4I_3 - \frac{1}{e}\right] = 1 - \frac{5}{2e},$$
$$I_7 = \frac{1}{2}\left[6I_5 - \frac{1}{e}\right] = 3 - \frac{8}{e},$$
$$I_9 = \frac{1}{2}\left[8I_7 - \frac{1}{e}\right] = 12 - \frac{65}{2e}$$

For $0 < x < 1$, we have $0 < x^n < 1$ and $0 < e^{-x^2} < 1$, so the integrand lies strictly between 0 and 1 in this interval. Hence $0 = \int_0^1 0\ dx < \int_0^1 x^n e^{-x^2}\ dx < \int_0^1 1\ dx = 1$ as required.

Thus, $0 < I_9 < 1$

$$0 < 12 - \frac{65}{2e} < 1$$
$$12 - 1 < \frac{65}{2e} < 12$$
$$\frac{65}{12} < 2e < \frac{65}{12-1}$$
$$\frac{65}{24} < e < \frac{65}{22} \quad \text{(shown)}$$

6.	[**2017/Q2**] (i) Let y be a differentiable function of x. For any positive integer n, prove that $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(xy) = x\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + n\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} \qquad [5]$$ (ii) For any non-negative integer n, define $$y_n = \mathrm{e}^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x^2}\right).$$ (a) Find y_0, y_1 and y_2. [3] (b) Prove that $y_{n+2} + 2xy_{n+1} + 2(n+1)y_n = 0$, for $n \geq 0$. [4] (c) Hence prove that $\frac{\mathrm{d}}{\mathrm{d}x}(y_{n+1}) = -2(n+1)y_n$, for $n \geq 0$. [3]
	(i) We use mathematical induction. Let P_n be the statement $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(xy) = x\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + n\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}$ for positive integers n. Base case: Since $\frac{\mathrm{d}}{\mathrm{d}x}(xy) = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$ by product rule (implicit differentiation), P_1 is true. Suppose P_k is true for some positive integer k. Then considering P_{k+1}, we have $LHS = \frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(xy) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}(xy)\right]$ $= \frac{\mathrm{d}}{\mathrm{d}x}\left[x\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + k\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}}\right]$ using induction hypothesis $= \left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + x\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}\right) + k\frac{\mathrm{d}^k y}{\mathrm{d}x^k}$ $= x\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} + (k+1)\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = RHS$ so P_{k+1} is true. Since P_1 is true, and P_k implies P_{k+1}, by mathematical induction, P_n is true for all positive integer n.
	(ii)(a) $y_0 = 1$, $y_1 = -2x$, $y_2 = 4x^2 - 2$
	(ii)(b) $y_{n+2} = \mathrm{e}^{x^2}\frac{\mathrm{d}^{n+2}}{\mathrm{d}x^{n+2}}\left(\mathrm{e}^{-x^2}\right) = \mathrm{e}^{x^2}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(-2x\mathrm{e}^{-x^2}\right)$ $= -2\mathrm{e}^{x^2}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(x\mathrm{e}^{-x^2}\right)$ $= -2\mathrm{e}^{x^2}\left[x\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(\mathrm{e}^{-x^2}\right) + (n+1)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x^2}\right)\right]$ using part (a)

	$= -2xe^{x^2}\dfrac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) - 2(n+1)e^{x^2}\dfrac{d^n}{dx^n}\left(e^{-x^2}\right)$ $= -2xy_{n+1} - 2(n+1)y_n$ hence $y_{n+2} + 2xy_{n+1} + 2(n+1)y_n = 0$ (shown)
	(ii)(c) We have $\dfrac{d}{dx}(y_{n+1}) = \dfrac{d}{dx}\left[e^{x^2}\dfrac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)\right]$ $= 2xe^{x^2}\dfrac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) + e^{x^2}\dfrac{d^{n+2}}{dx^{n+2}}\left(e^{-x^2}\right)$ $= 2xy_{n+1} + y_{n+2}$ But from the result of part (ii)(b), this is equal to $-2(n+1)y_n$. (shown)

7.	Determine whether each of the following sequences converge or diverge. **(a)** $u_n = \sin\dfrac{n\pi}{2}$, $n = 1, 2, 3, \ldots$ **(b)** $u_n = \dfrac{n^3}{3^n}$, $n = 1, 2, 3, \ldots$ **(c)** $u_n = \dfrac{n^n}{n!}$, $n = 1, 2, 3, \ldots$ **(d)** $u_n = \dfrac{\sin n}{n}$, $n = 1, 2, 3, \ldots$
	(a) This sequence is $1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots$ which is divergent. (Periodic, period 4)
	(b) Discussion: A result that we (should) know is that "exponential growth dominates polynomial growth", and so this sequence converges to zero. The intuition is that exponential functions are like infinite polynomials. This can be proved using Maclaurin series as follows: $u_n = \dfrac{n^3}{3^n} = \dfrac{n^3}{e^{(\ln 3)n}} = \dfrac{n^3}{1 + an + bn + \ldots}$ (using Maclaurin series of $e^{(\ln 3)n}$) $< \dfrac{n^3}{cn^4}$ (delete all other terms, note that all coeff. >0) $\to 0$ as $n \to \infty$ (since n^3 is dominated by n^4)

	This means that $u_3, u_4, u_5, u_6, \ldots$ is a decreasing sequence of positive numbers. This must converge to a limit (say L). But it is easy to see that the original sequence $u_1, u_2, u_3, u_4, \ldots$ must also have the same limit L, so it converges.
	(c) Note that $u_n = \dfrac{n \times n \times n \times \ldots \times n}{1 \times 2 \times 3 \times \ldots \times n}$ $\geq \dfrac{n}{1}$ (this is obviously true for $n=1$, and for $n \geq 2$ we can "cancel" each denominator term with a term that is at least as large from the numerator) Since $u_n \geq n$ for all n, this is clearly divergent.
	(d) Define $a_n = -\dfrac{1}{n}$, $b_n = \dfrac{1}{n}$. Then we have $a_n \leq u_n \leq b_n$, and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$, so u_n is convergent (with limit 0).

8.	**[N2002/Q5 (S Paper)]** The geometric progression U has terms $u_1, u_2, u_3, \ldots$, with common ratio r, where $\lvert r \rvert < 1$. It is given that $v_i = u_i^2$ for $i = 1, 2, 3, \ldots$. **(i)** Show that $\sum_{i=1}^{n} v_i = \dfrac{u_1}{1+r} \sum_{i=1}^{2n} u_i$. It is given further that $w_i = v_i - v_{i+1}$ for $i = 1, 2, 3, \ldots$. **(ii)** Show that $\sum_{i=1}^{n} w_i = u_1(1-r) \sum_{i=1}^{2n} u_i$. Let $S_U = \sum_{i=1}^{\infty} u_i$, $S_V = \sum_{i=1}^{\infty} v_i$ and $S_W = \sum_{i=1}^{\infty} w_i$. Show that **(iii)** $\dfrac{S_u}{S_V} + \dfrac{1}{S_U} = \dfrac{2}{u_1}$, **(iv)** $S_W = u_1^2$.
	(i) Note that

	$v_i = u_i^{\ 2} = \left(u_1 r^{i-1}\right)^2 = u_1^{\ 2}\left(r^2\right)^{i-1}$ i.e. the v_i form a geometric progression with first term $u_1^{\ 2}$ and common ratio r^2, so $RHS = \dfrac{u_1}{1+r}\displaystyle\sum_{i=1}^{2n} u_i$ $= \dfrac{u_1}{1+r} \times \dfrac{u_1\left(1-r^{2n}\right)}{1-r}$ $= \dfrac{u_1^{\ 2}\left(1-\left(r^2\right)^n\right)}{1-r^2}$ $= \displaystyle\sum_{i=1}^{n} v_i = LHS$ (shown)
	(ii) Using method of differences, $\displaystyle\sum_{i=1}^{n} w_i = \sum_{i=1}^{n}\left(v_i - v_{i+1}\right) = v_1 - v_{n+1}$ $= u_1^{\ 2} - u_1^{\ 2} r^{2n}$ $= u_1^{\ 2}\left(1-r^{2n}\right)$ $= u_1\left(1-r\right) \times \dfrac{u_1\left(1-r^{2n}\right)}{1-r}$ $= u_1\left(1-r\right)\displaystyle\sum_{i=1}^{2n} u_i$ (shown) Alternatively, $\displaystyle\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} v_i - \sum_{i=1}^{n} v_{i+1}$ $= \displaystyle\sum_{i=1}^{n} v_i - r^2 \sum_{i=1}^{n} v_i$ $= \left(1-r^2\right)\displaystyle\sum_{i=1}^{n} v_i$

	$=\left(1-r^2\right)\times\dfrac{u_1}{1+r}\sum_{i=1}^{2n}u_i$ $=u_1\left(1-r\right)\sum_{i=1}^{2n}u_i$
	(iii) Clearly $S_U=\dfrac{u_1}{1-r}$ using the formula for sum to infinity of GP. Furthermore, taking $n\to\infty$ in part (i) we get $S_V=\dfrac{u_1}{1+r}S_U$, i.e. $\dfrac{S_U}{S_V}=\dfrac{1+r}{u_1}$. Then $LHS=\dfrac{S_U}{S_V}+\dfrac{1}{S_U}=\dfrac{1+r}{u_1}+\dfrac{1-r}{u_1}=\dfrac{2}{u_1}=RHS$ (shown)
	(iv) Taking $n\to\infty$ in part (ii) we get $S_W=u_1\left(1-r\right)S_U=u_1\left(1-r\right)\times\dfrac{u_1}{1-r}=u_1^{\ 2}$. Alternatively, since $\sum_{i=1}^{n}w_i=\sum_{i=1}^{n}\left(v_i-v_{i+1}\right)=v_1-v_{n+1}$, taking $n\to\infty$ we immediately get $\sum_{i=1}^{\infty}w_i=v_1=u_1^{\ 2}$.

9.	**(i)** Sketch the graph of $y=\sin x$ and determine the largest constant *a* and the smallest constant *b* such that $ax\le\sin x\le bx$ for $0\le x\le\dfrac{\pi}{2}$. **(ii)** Hence determine for every positive integer *k*, if the series $$\sin^k\frac{1}{1}+\sin^k\frac{1}{2}+\sin^k\frac{1}{3}+\ldots$$ is convergent or divergent.
	(i) <graph omitted> $a=\dfrac{2}{\pi}$, $b=1$
	(ii) The series diverges if $k=1$, and converges if $k\ge 2$. When $k=1$, since $\sin x\ge\dfrac{2}{\pi}x$, we have $\sum_{r=1}^{n}\sin\dfrac{1}{r}\ge\dfrac{2}{\pi}\sum_{r=1}^{n}\dfrac{1}{r}$ but the latter tends to infinity as $n\to\infty$. So, by comparison test, we must also have $\sum_{r=1}^{n}\sin\dfrac{1}{r}$ tends to infinity, i.e. diverges. When $k\ge 2$, we have

	$\sin^k x = (\sin x)^k \le x^k$ (using part (i)) Therefore $\sum_{r=1}^{\infty} \sin^k \frac{1}{r} \le \sum_{r=1}^{\infty} \left(\frac{1}{r^k}\right) \le \sum_{r=1}^{\infty} \left(\frac{1}{r^2}\right)$. By comparison test, since the right-hand series converges, the given series must also converge.

10.	The r^{th} term u_r of an arithmetic progression U is given by $$u_r = a + (r-1)d, \quad \text{for } r = 1, 2, 3, \ldots.$$ The progression is such that there exists a term of U equal to a^2. Given that a and d are positive integers, show that **(a)** for each term u_r of U, there exists a term of U equal to u_r^2, **(b)** for each term u_r of U, there exists a term of U equal to u_r^3, **(c)** all the terms of the geometric progression $a, a^2, a^3, \ldots$ are terms of U. If a and d are not required to be positive integers, show, by giving a counter-example, that the result in part (a) is not necessarily true.
	By definition of *U*, *x* is a term of *U* if and only if $x = a + kd$ for some non-negative integer *k*. Since a^2 is a term of *U*, we can set $a^2 = a + bd$ for some non-negative integer *b*. We will use these for each of the three parts.
	(a) We can write $u_r{}^2 = \left[a + (r-1)d\right]^2 = a^2 + 2a(r-1)d + (r-1)^2 d^2 = a + \left[b + 2a(r-1) + (r-1)^2 d\right]d$ which is in the form $a + kd$. The term in the square brackets is a sum of three non-negative integers, so $u_r{}^2$ is a term of *U*.
	(b) Let $u_r{}^2 = a + c_r d$ for some non-negative integer c_r. Then $u_r{}^3 = \left[a + (r-1)d\right]\left[a + c_r d\right]$ $= a^2 + a(c_r + r - 1)d + c_r(r-1)d^2$ $= a + \left[b + a(c_r + r - 1) + c_r(r-1)d\right]d$ where again the term in the square bracket is a sum of three non-negative integers. So $u_r{}^3$ is a term of *U*.
	(c) We have $a = a$ $a^2 = a + bd$ $a^3 = a(a + bd) = a^2 + abd = a + bd + abd$ $a^4 = a(a + bd + abd) = a^2 + abd + a^2bd = a + bd + abd + a^2bd$ and in general it can be shown inductively that $a^n = a + b\left(\sum_{r=0}^{n-2} a^r\right)d$ Since this is precisely in the form $a + kd$, a^n is a term of *U*.

	For a counter-example, set $a=1$ and $d=-1$. Then $a^2=1$ is a term of *U*, so the conditions for the sequence are satisfied. But ${u_4}^2=(-2)^2=4$ is not a term of *U*, so the result in part (a) is false.

11.	**[DHS-EJC/2019/Q3(part)]** (a) Let *x*, *y*, *z* be real numbers such that $0<x<y<z$. Show that x^2, y^2, z^2 are three consecutive terms of an arithmetic sequence if and only if $\frac{1}{y+z}$, $\frac{1}{x+z}$, $\frac{1}{x+y}$ are three consecutive terms of another arithmetic sequence. [3] (b) An infinite sequence of integers, a_n, $n\in\mathbb{Z}^+$, is defined by $a_1=a_2=1$ and $$a_{n+1}=\frac{a_n^2+2}{a_{n-1}},\ n\geq 2.$$ (i) Show that $\frac{a_{n+2}+a_n}{a_{n+1}}=\frac{a_{n+1}+a_{n-1}}{a_n}$ for $n\geq 2$. [2] (ii) Hence or otherwise, show that $a_{n+1}=4a_n-a_{n-1}$ and explain why a_n is an odd integer for all $n\in\mathbb{Z}^+$. [2]
(a)	$\frac{1}{y+z}, \frac{1}{x+z}, \frac{1}{x+y}$ are three consecutive numbers of an AP $\Leftrightarrow \frac{1}{x+z}-\frac{1}{y+z}=\frac{1}{x+y}-\frac{1}{x+z}$ $\Leftrightarrow \frac{(y+z)-(x+z)}{(x+z)(y+z)}=\frac{(x+z)-(x+y)}{(x+y)(x+z)}$ $\Leftrightarrow \frac{y-x}{(x+z)(y+z)}=\frac{z-y}{(x+y)(x+z)}$ $\Leftrightarrow \frac{(y-x)(x+y)}{(x+y)(x+z)(y+z)}=\frac{(z-y)(y+z)}{(x+y)(x+z)(y+z)}$ $\Leftrightarrow \frac{y^2-x^2}{(x+y)(x+z)(y+z)}=\frac{z^2-y^2}{(x+y)(x+z)(y+z)}$ $\Leftrightarrow y^2-x^2=z^2-y^2$ $\Leftrightarrow x^2, y^2, z^2$ are three consecutive numbers of an AP

(b) (i)	$$\frac{a_{n+2}+a_n}{a_{n+1}}=\frac{\frac{a_{n+1}^2+2}{a_n}+a_n}{a_{n+1}}$$ $$=\frac{a_{n+1}^2+2+a_n^2}{a_{n+1}a_n}$$ $$=\frac{a_{n+1}+\frac{a_n^2+2}{a_{n+1}}}{a_n}$$ $$=\frac{a_{n+1}+a_{n-1}}{a_n},\ n\geq 2$$
(b) (ii)	$a_3=\frac{a_2^2+2}{a_1}=\frac{1^2+2}{1}=3$ $\frac{a_{n+1}+a_{n-1}}{a_n}=\ldots=\frac{a_3+a_1}{a_2}=\frac{3+1}{1}=4$ $a_{n+1}=4a_n-a_{n-1},\ n\geq 2.$ Therefore a_{n+1} has the same parity as a_{n-1} since $4a_n$ is even. Since both a_1 and a_2 are both odd, by induction, a_n is an odd integer for all $n\in\mathbb{Z}^+$.

12.	**[2018/Q1]** For any positive integer *n*, the function F_n is defined by $$\mathrm{F}_n(x)=\sum_{r=1}^{n}\frac{\left(r^2-r\right)x^2+\left(2r^2+4r-1\right)x+1}{r\left(r+1\right)\left(\left(r-1\right)x+1\right)\left(rx+1\right)}.$$ (i) Find $\mathrm{F}_n(0)$ and describe the behaviour of $\mathrm{F}_n(0)$ as *n* tends to infinity. [5] (ii) It is given that the *r*th term of $\mathrm{F}_n(x)$ can be expressed as $$\frac{1}{r}-\frac{1}{r+1}+\frac{2}{\left(r-1\right)x+1}-\frac{2}{rx+1}.$$ (a) Find $\mathrm{F}_n(x)$. [3] (b) Describe the behaviour of $\mathrm{F}_n(x)$ as *n* tends to infinity. [3]
	(i) $\mathrm{F}_n\left(0\right)=\sum_{r=1}^{n}\frac{\left(0\right)+\left(0\right)+1}{r\left(r+1\right)\left(1\right)\left(1\right)}=\sum_{r=1}^{n}\frac{1}{r\left(r+1\right)}=\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)=\sum_{r=1}^{n}\frac{1}{r}-\sum_{r=2}^{n+1}\frac{1}{r}=\frac{n}{n+1}$ As $n\to\infty$, $\mathrm{F}_n\left(0\right)\to 1$

(ii) **(a)** $$\mathrm{F}_n(x) = \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) + \sum_{r=1}^{n}\left(\frac{2}{(r-1)x+1} - \frac{2}{rx+1}\right)$$
$$= \frac{n}{n+1} + \sum_{r=0}^{n-1}\frac{2}{rx+1} - \sum_{r=1}^{n}\frac{2}{rx+1}$$
$$= \frac{n}{n+1} + 2 - \frac{2}{nx+1}$$
$$= \frac{3n+2}{n+1} - \frac{2}{nx+1}$$

(b) For all $x \neq 0$, as $n \to \infty$ we have $nx+1 \to \pm\infty$ and thus $\dfrac{2}{nx+1} \to 0$.

Therefore, $\mathrm{F}_n(x) \to \begin{cases} 1, & x = 0 \\ 3, & x \neq 0 \end{cases}$

13. **[2019/ASRJC/Prelim/Q7]**

Two sequences $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are given by $u_1 = 1,\ v_1 = 1,\ u_{n+1} = u_n + 3v_n,\ v_{n+1} = 2u_n + 7v_n,$ for positive integers *n*. The sequence $r_1, r_2, r_3, \ldots$ is such that $r_n = \dfrac{u_n}{v_n}$ for positive integers *n*.

(i) Use induction to prove that $2{u_n}^2 - 3{v_n}^2 + 6u_nv_n = 5$ for all positive integers *n*. [4]

It is given that as $n \to \infty,\ v_n \to \infty$ and $r_n \to L$ for some real constant *L*.

(ii) Using the results in **(i)** or otherwise, show that $L = \dfrac{1}{2}\left(-3+\sqrt{15}\right)$. [3]

(iii) Describe the behaviour of the sequence $r_1, r_2, r_3, \ldots$, justifying your answer by considering $r_{n+1} - r_n$. Hence, show that $\sqrt{15}$ lies within the interval $\left(\dfrac{15}{2r_n+3},\ 2r_n+3\right)$ for all positive integers *n*. [4]

(iv) Deduce that $\dfrac{213}{55} < \sqrt{15} < \dfrac{275}{71}$. [2]

(i) Let P_n be the statement $2{u_n}^2 - 3{v_n}^2 + 6u_nv_n = 5$ for $n \in \mathbb{Z}^+$.

For *n* = 1, $\text{LHS} = 2{u_1}^2 - 3{u_1}^2 + 6u_1v_1 = 2(1) - 3(1) + 6 = 5 = \text{RHS}$. So P_1 is true.

Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e. $2{u_k}^2 - 3{v_k}^2 + 6u_kv_k = 5$.

LHS of $P_{k+1} = 2u_{k+1}^{\ 2} - 3v_{k+1}^{\ 2} + 6u_{k+1}v_{k+1}$

$= 2\left(u_k^{\ 2} + 6u_kv_k + 9v_k^{\ 2}\right) - 3\left(4u_k^{\ 2} + 28u_kv_k + 49v_k^{\ 2}\right) + 6\left(2u_k^{\ 2} + 13u_kv_k + 21v_k^{\ 2}\right)$

$= 2u_k^{\ 2} - 3v_k^{\ 2} + 6u_kv_k$

$= 5.$ So P_k is true $\Rightarrow$ P_{k+1} is true.

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by the Principle of Mathematical Induction, P_n is true for all $n \in \mathbb{Z}^+$.

(ii) For $2u_n^{\ 2} - 3v_n^{\ 2} + 6u_nv_n = 5$, dividing throughout by $v_n^{\ 2}$ yields $2r_n^{\ 2} - 3 + 6r_n = \dfrac{5}{v_n^{\ 2}}$.

Since $\lim_{n\to\infty}(v_n) = \infty$ and $\lim_{n\to\infty}(r_n) = L$,

$2L^2 - 3 + 6L = 0 \quad \Rightarrow \quad L = \frac{1}{2}\left(-3 + \sqrt{15}\right)$ or $\frac{1}{2}\left(-3 - \sqrt{15}\right)$.

However, as $r_n > 0$, its limit L must be non-negative. Therefore, $L = \frac{1}{2}\left(-3 + \sqrt{15}\right)$.

(iii)
$$\begin{aligned} r_{n+1} - r_n &= \frac{u_n + 3v_n}{2u_n + 7v_n} - \frac{u_n}{v_n} \\ &= \frac{u_nv_n + 3v_n^{\ 2} - 2u_n^{\ 2} - 7u_nv_n}{v_n(2u_n + 7v_n)} \\ &= -\frac{2u_n + 6u_nv_n - 3v_n^{\ 2}}{v_n(2u_n + 7v_n)} \\ &= -\frac{5}{v_n(2u_n + 7v_n)} < 0 \quad \text{(since } u_n,\ v_n > 0\text{)}. \end{aligned}$$

So the sequence $\{r_n\}$ is strictly decreasing and tends to the limit $\frac{1}{2}\left(-3 + \sqrt{15}\right)$.

$\therefore\ r_n > \frac{1}{2}\left(-3 + \sqrt{15}\right)$

$\Rightarrow 2r_n + 3 > \sqrt{15}$ ---- (2)

$\Rightarrow \dfrac{1}{2r_n + 3} < \dfrac{1}{\sqrt{15}}$

$\Rightarrow \dfrac{15}{2r_n + 3} < \sqrt{15}$ ---- (3)

By taking intersection of (2) and (3), we have the result.

(iv) $r_3 = \dfrac{u_3}{v_3} = \dfrac{31}{71}$. So $\dfrac{15}{2r_3 + 3} < \sqrt{15} < 2r_3 + 3 \quad \Rightarrow \dfrac{213}{55} < \sqrt{15} < \dfrac{275}{71}$.