RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Chapter 1: Basics

CONTENT FROM O LEVEL ADDITIONAL MATHEMATICS

The following content is stated as assumed knowledge in your 9758 H2 Mathematics syllabus document:

ALGEBRA

A1, A2	Quadratic functions; equations and inequalities • Finding the maximum or minimum value of a quadratic function using the method of completing the square • conditions for a quadratic equation to have: – two real roots, two equal roots, no real roots • conditions for $ax^2 + bx + c$ to be always positive (or always negative) • solving simultaneous equations in two variables by substitution, with one of the equations being a linear equation
A3	Surds • four operations on surds, including rationalising the denominator • solving equations involving surds

A4	Polynomials and partial fractions
	• multiplication and division of polynomials
	• use of remainder and factor theorems
	• partial fractions with cases where the denominator is not more complicated than:
	  – $(ax + b)(cx + d)$
	  – $(ax + b)(cx + d)^2$
	  – $(ax + b)(x^2 + c^2)$

A6	Exponential and Logarithmic functions
	• Exponential and logarithmic functions a^x, e^x, $\log_a x$, $\ln x$ and their graphs, including
	  – laws of logarithms
	  – equivalence of $y = a^x$ and $x = \log_a y$
	  – change of base of logarithms
	• Simplifying expressions and solving simple equations involving exponential and logarithmic functions

GEOMETRY AND TRIGONOMETRY

G1	Trigonometric functions, identities and equations • six trigonometric functions, and principal values of the inverses of sine, cosine and tangent (which will be discussed in the chapter on Functions) • trigonometric equations and identities (see Formulae List) • expression of $a\cos\theta + b\sin\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$.
G2	Coordinate geometry in two dimensions • coordinate geometry of the circle with the equation in the form $(x-a)^2 + (y-b)^2 = r^2$ or $x^2 + y^2 + 2gx + 2fy + c = 0$

CALCULUS (We will revisit this section in Chapters 5 and 8)

C1	Differentiation and Integration • derivative of $\mathrm{f}(x)$ as the gradient of the tangent to the graph of $y = \mathrm{f}(x)$ at a point • derivative as rate of change • derivatives of x^n for any rational n, $\sin x$, $\cos x$, $\tan x$, e^x and $\ln x$, together with constant multiples, sums and differences • use of Chain Rule • derivatives of products and quotients of functions • increasing and decreasing functions • stationary points (maximum and minimum turning points and points of inflexion) • use of second derivative test to discriminate between maxima and minima • connected rates of change • maxima and minima problems • integration as the reverse of differentiation • integration of x^n for any rational n, e^x, $\sin x$, $\cos x$, $\sec^2 x$ and their constant multiples, sums and differences • integration of $(ax+b)^n$ for any rational n, $\sin(ax+b)$, $\cos(ax+b)$ and e^{ax+b}

The objective of this chapter is to familiarize you with content above (although you should already be familiar) that is often required for solving exercises and problems in H2 Mathematics.

CONTENT

1 Set Language and Notation

1.1 Sets and Elements

A set is a collection of objects such as numbers, letters, etc.

Each object in a set is known as an **element** (or **member**) of the set.

We usually use **capital letters** such as A, B, C, etc to denote a set, **small letters** such as a, b, c, etc to denote the elements of a set.

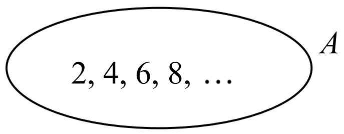
We write $a \in A$ to mean that the element *a belongs to* the set A, or *a is an element of A*.

We write $b \notin A$ to mean that *b is not an element of A*.

Example: Suppose the set A contains all positive even integers 2, 4, 6, 8, ….

There are two ways in general to represent the set A, namely:

1) Venn diagram



2) Set-builder notation
   (i) by listing the elements of the set within curly brackets,
   i.e. $A = \{2, 4, 6, 8, \dots\}$.

   (ii) by using a mathematical relationship to describe the elements of the set,
   i.e. $A = \{x = 2k : k \text{ is a positive integer}\}$.

Some examples of commonly occurring sets:

Notation	Description
$\varnothing$ or $\{\ \}$	The empty set or null set. This set has no elements. Eg, The set of all integers who are odd and divisible by 2 = $\varnothing$
$\mathbb{N}$	the set of natural numbers = $\{1, 2, 3, \dots\}$.
$\mathbb{Z}$	the set of integers = $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
$\mathbb{Z}^+$	the set of positive integers = $\{1, 2, 3, \dots\} = \mathbb{N}$
$\mathbb{Z}^-$	the set of negative integers = $\{\dots, -3, -2, -1\}$
$\mathbb{Z}_0^+$	the set of nonnegative integers = $\{0, 1, 2, 3, \dots\}$
$\mathbb{Z}_0^-$	the set of nonpositive integers = $\{\dots, -3, -2, -1, 0\}$

$\mathbb{Q}$	the set of rational numbers of the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$, e.g. $\frac{1}{2}, \frac{4}{3}, -\frac{5}{6}, 7, -0.89898989, \ldots$, etc
$\mathbb{R}$	the set of real numbers You should be familiar with $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}_0^+$, $\mathbb{R}_0^-$.

1.2 Subsets, union and intersection of sets

a) A set A is a **subset** of a set B, written $A \subseteq B$, if every element of A is an element of B.

Mathematically, $A \subseteq B$ if for all $a \in A$, then $a \in B$. This can be shown on a Venn diagram as depicted in Fig. 1.

Also, we write $A \not\subseteq B$ if the set A is **not a subset** of the set B.

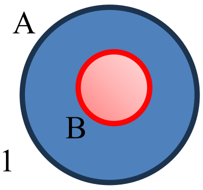


Fig. 1

Below are some properties involving subsets.

1) $\varnothing \subseteq A$, i.e. the empty set is a subset of any set.

2) $A \subseteq A$, i.e. any set is a subset of itself.

3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

If $A \subseteq B$ and $B \subseteq A$, then the two sets A and B are said to be **equal**, written $A = B$.

If $A \subseteq B$ and $A \neq B$, then the set A is a **proper subset** of the set B, written $A \subset B$.

Note: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

b) The **union** of two sets A and B, denoted by $A \cup B$, is the set of elements which *belongs to the set A <u>or</u> the set B <u>or</u> both.*

Mathematically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
In Fig. 2, the shaded region represents $A \cup B$ in the Venn diagram.

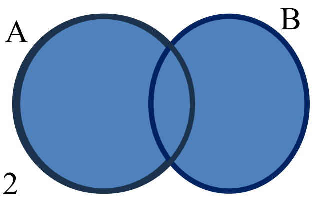


Fig.2

c) The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of elements which *belongs to <u>both</u> the set A <u>and</u> the set B.*

Mathematically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

In Fig. 3, the shaded region represents $A \cap B$ in the Venn diagram.

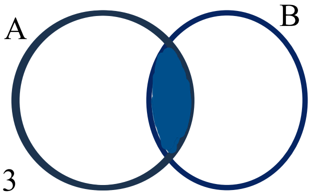
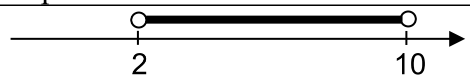
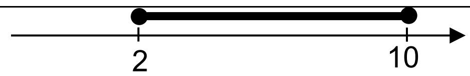
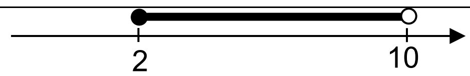
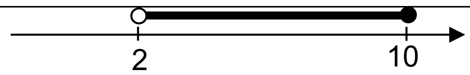
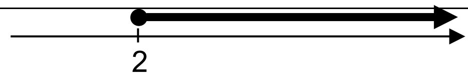
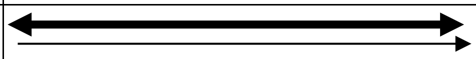
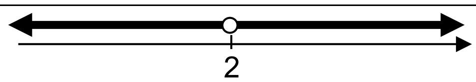


Fig.3

1.3 Interval notation

A very useful and convenient notation for representing sets that involves a range of real values is the **interval notation**, which uses the *round* and/or *square brackets*.
In general,
(a,b) represents the open interval $\{x \in \mathbb{R} : a < x < b\}$,
$[a,b]$ represents the open interval $\{x \in \mathbb{R} : a \le x \le b\}$,
$(a,b]$ represents the interval $\{x \in \mathbb{R} : a < x \le b\}$, and
$[a,b)$ represents the interval $\{x \in \mathbb{R} : a \le x < b\}$.

Example:

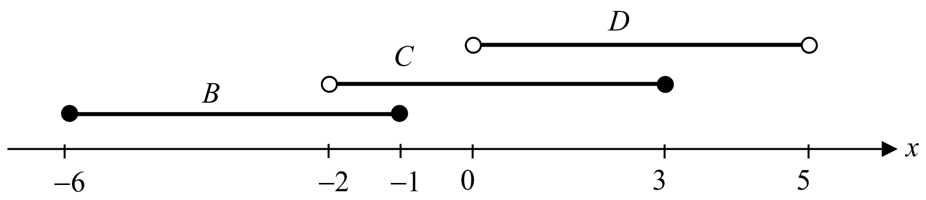
	Set	Interval Notation	Representation on number line
(a)	$\{x \in \mathbb{R} : 2 < x < 10\}$	(2, 10)	
(b)	$\{x \in \mathbb{R} : 2 \le x \le 10\}$	$[2,10]$	
(c)	$\{x \in \mathbb{R} : 2 \le x < 10\}$	$[2,10)$	
(d)	$\{x \in \mathbb{R} : 2 < x \le 10\}$	$(2,10]$	
(e)	$\{x \in \mathbb{R} : x \ge 2\}$	$[2,\infty)$	
(f)	$\mathbb{R}$	$(-\infty,\infty)$	
(g)	$\{x \in \mathbb{R} : x \ne 2\}$ Or $\mathbb{R}\setminus\{2\}$ Note: The notation "\" means "to exclude", so $\mathbb{R}\setminus\{2\}$ reads as the set of real numbers excluding 2.	$(-\infty,2)\cup(2,\infty)$	

Example 1

Given that $B = [-6,-1]$, $C = (-2,3]$ and $D = (0,5)$, write down

(i) $B \cup C$, (ii) $B \cap C$, (iii) $B \cap D$, (iv) $B \cup D$, (v) $C \cup \mathbb{R}_0^+$.

Solution



Referring to the number line above, it is clear that

(i) $B \cup C = [-6,3]$.

(ii) $B \cap C = (-2,-1]$.

(iii) $B \cup D = [-6,-1] \cup (0,5)$.

(iv) $B \cap D = \varnothing$.

(v) $C \cup \mathbb{R}_0^+ = (-2,\infty)$.

2 Algebra

2.1 Quadratic Expressions

In Chapter 3 on Equations and Inequalities, you need to know how to complete the square given a quadratic expression, as this will be required of you to show that certain quadratic expressions are strictly positive or negative. You should also know the conditions for which a quadratic equation has two, one or no real roots via the use of the discriminant. In fact, properties and results of quadratic expression/curve/equations are the basics of many H2 Math topics.

Note:
Given a quadratic equation $ax^2+bx+c=0$ with roots α and β and $a \neq 0$, it can be proven that

(1) sum of the roots, $\alpha+\beta=-\dfrac{b}{a}$,

(2) product of roots, $\alpha\beta=\dfrac{c}{a}$.

Example 2
By completing the square,

(i) show that $2x^2-3x+2$ is always positive,

(ii) find the minimum value of $2x^2+2x+3$, hence deduce the range of values of $\dfrac{1}{2x^2+2x+3}$.

Solution

(i)	$2x^2-3x+2$ $=2\left(x^2-\frac{3}{2}x\right)+2$ $=2\left[x^2-2\left(\frac{3}{4}\right)x+\left(\frac{3}{4}\right)^2\right]-2\left(\frac{3}{4}\right)^2+2$ $=2\left(x-\frac{3}{4}\right)^2+\frac{7}{8}$ $\because \left(x-\frac{3}{4}\right)^2 \geq 0$ for all real x, $\therefore 2x^2-3x+2$ $=2\left(x-\frac{3}{4}\right)^2+\frac{7}{8}\geq\frac{7}{8}>0$ for all real x.	**(ii)**	$2x^2+2x+3$ $=2\left(x^2+x\right)+3$ $=2\left[x^2+2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2\right]-2\left(\frac{1}{2}\right)^2+3$ $=2\left(x+\frac{1}{2}\right)^2+\frac{5}{2}\geq\frac{5}{2}>0$ Minimum value of $2x^2+2x+3$ is $\frac{5}{2}$. $2x^2+2x+3\geq\frac{5}{2}\Rightarrow 0<\frac{1}{2x^2+2x+3}\leq\frac{2}{5}$ Does "$a<b\Rightarrow\frac{1}{a}>\frac{1}{b}$" in general? No! Only if a and b have the same sign.

Example 3

(i) Find the values of *k* if the roots of $3x^2+kx+12=0$ are equal.

(ii) Show that the roots of $ax^2+\left(a+b\right)x+b=0$ are real for all non-zero values of *a* and *b*.

Solution

(i)	$3x^2+kx+12=0$ To determine nature of roots of a quadratic equation, use "Discriminant". Discriminant $=k^2-4\left(3\right)\left(12\right)=0$ $k^2=144$ $k=\pm 12$	**(ii)**	$a\neq 0,\ ax^2+\left(a+b\right)x+b=0$ Discriminant $=\left(a+b\right)^2-4ab$ $=a^2+b^2-2ab$ $=\left(a-b\right)^2\geq 0$ for all a and b. $\therefore$ roots are real for all non-zero a and b

2.2 Polynomials and Partial Fractions

In the chapter on Complex Numbers, an important skill you will need to have is to be able to carry out long division.

Example 4

(a) Using long division, find the quotient and remainder when x^3+3x^2-6x+2 is divided by $x+2$.

(b) Factorise x^3-2x+1.

Solution

(a)	x^2+x-8 $x+2\,\overline{)\,x^3+3x^2-6x+2}$ $\underline{-(x^3+2x^2)}$ x^2-6x $\underline{-(x^2+2x)}$ $-8x+2$ $\underline{-(-8x-16)}$ 18 The quotient is x^2+x-8. The remainder is 18. Note : By remainder theorem, let $\mathrm{f}\left(x\right)=x^3+3x^2-6x+2$. remainder$=\mathrm{f}\left(-2\right)=18$	**(b)**	Let $\mathrm{f}\left(x\right)=x^3-2x+1$. $\mathrm{f}(1)=1^3-2+1=0$ $\Rightarrow \left(x-1\right)$ is a factor We can apply long division or compare coefficients to further factorise f(x). x^2+x-1 $x-1\,\overline{)\,x^3+0x^2-2x+1}$ $\underline{-(x^3-x^2)}$ x^2-2x $\underline{-(x^2-x)}$ $-x+1$ $\underline{-(-x+1)}$ 0 Or $x^3-2x+1=\left(x-1\right)\left(x^2+ax-1\right)$ Comparing coefficient of x^2 : $0=a-1\Rightarrow a=1$

			$\therefore \mathrm{f}\left(x\right)=x^3-2x+1$ $=\left(x-1\right)\left(x^2+x-1\right)$ $=\left(x-1\right)\left[\left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2\right]$ $=\left(x-1\right)\left(x+\frac{1+\sqrt{5}}{2}\right)\left(x+\frac{1-\sqrt{5}}{2}\right)$

There are also certain algebraic expressions that have standard factorizations, which you should be able to recognize immediately (You should verify them by expanding the RHS or using long division by dividing the LHS by $a \pm b$.):

$$a^2-b^2=(a+b)(a-b)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$
$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

Example 5
Factorize the following expressions completely:
(i) x^3+8 **;**
(ii) x^6-64 .

Solution

(i) $x^3+8=x^3+2^3=(x+2)(x^2-2x+4)$.

(ii) $x^6-64=(x^3-8)(x^3+8)=(x-2)(x^2+2x+4)(x+2)(x^2-2x+4)$.

Partial fractions are especially important when we are dealing with integration problems which involve rational functions, i.e. expressions of the form $\frac{\mathbf{P}(x)}{\mathbf{Q}(x)}$, where P($x$) and Q($x$) are polynomials.

Example 6

Express the following in partial fractions

(i) $\dfrac{2x}{x^2-1}$, **(ii)** $\dfrac{3x^2+3x+2}{(x+1)(x^2+1)}$,

(iii) $\dfrac{11-x-x^2}{(x+2)(x-1)^2}$, **(iv)** $\dfrac{2x^2+6x-5}{(x-1)(x+2)}$.

(i)	$\dfrac{2x}{x^2-1}=\dfrac{2x}{(x-1)(x+1)}\equiv\dfrac{A}{x-1}+\dfrac{B}{x+1}$ $2x=A(x+1)+B(x-1)$ $x=1,\quad 2=2A\quad\Rightarrow A=1$ $x=-1,-2=-2B\Rightarrow B=1$ $\dfrac{2x}{x^2-1}=\dfrac{1}{x-1}+\dfrac{1}{x+1}$	**(ii)**	$\dfrac{3x^2+3x+2}{(x+1)(x^2+1)}\equiv\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}$ $3x^2+3x+2=A(x^2+1)+(Bx+C)(x+1)$ $x=-1,\ 3-3+2=2A\Rightarrow A=1$ $x=0,\quad 2=A+C\Rightarrow C=1$ $x=1,\quad 3+3+2=2+(B+1)(2)\Rightarrow B=2$ $\dfrac{3x^2+3x+2}{(x+1)(x^2+1)}=\dfrac{1}{x+1}+\dfrac{2x+1}{x^2+1}$
(iii)	$\dfrac{11-x-x^2}{(x+2)(x-1)^2}\equiv\dfrac{A}{x+2}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$ $11-x-x^2=A(x-1)^2+B(x+2)(x-1)+C(x+2)$ $x=1,\quad 11-1-1=3C\Rightarrow C=3$ $x=-2,\ 11+2-4=9A\Rightarrow A=1$ $x=0,\quad 11=1-2B+3(2)\Rightarrow B=-2$ $\dfrac{11-x-x^2}{(x+2)(x-1)^2}=\dfrac{1}{x+2}-\dfrac{2}{x-1}+\dfrac{3}{(x-1)^2}$	**(iv)**	$\dfrac{2x^2+6x-5}{(x-1)(x+2)}\equiv 2+\dfrac{A}{x-1}+\dfrac{B}{x+2}$ $2x^2+6x-5=2(x-1)(x+2)+A(x+2)+B(x-1)$ $x=1,\quad 2+6-5=3A\Rightarrow A=1$ $x=-2,\ 8-12-5=-3B\Rightarrow B=3$ $\dfrac{2x^2+6x-5}{(x-1)(x+2)}=2+\dfrac{1}{x-1}+\dfrac{3}{x+2}$

2.3 Power, Exponential and Logarithmic Functions

The ability to solve basic logarithmic and exponential equations is essential, together with being able to sketch their graphs, with clear knowledge of finding asymptotes and intercepts.

$$\boxed{\text{For } a>0,\ a\neq 1,\qquad y=a^x \Leftrightarrow x=\log_a y}$$

In particular, $y=a^{\log_a y}$ and $x=\log_a a^x$.

Let $a=\mathrm{e}$, we have $x=\mathrm{e}^{\ln x}$.

Example 7
Solve the following equations. Express your answer in exact form.

(i) $\log_x(3x-2)=2$, **(ii)** $(3x)^{\ln 3}=(4x)^{\ln 4}$.

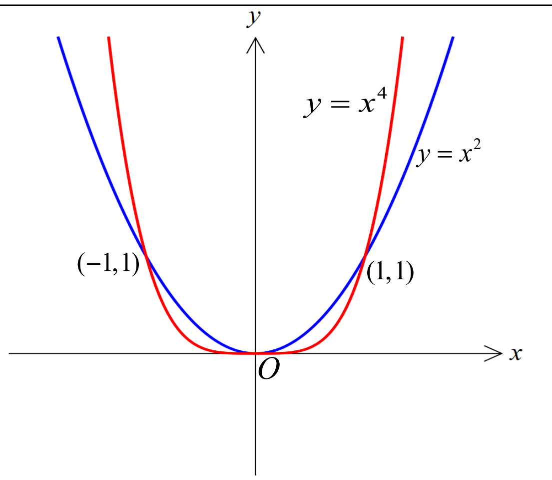
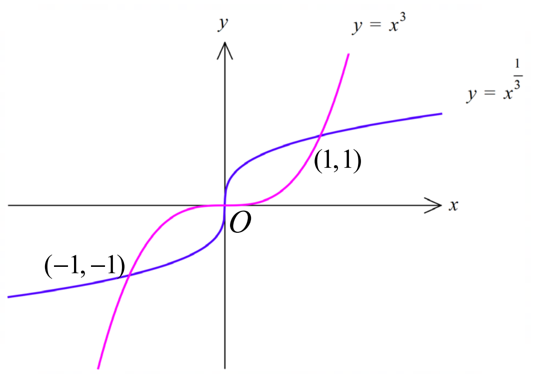
Solution

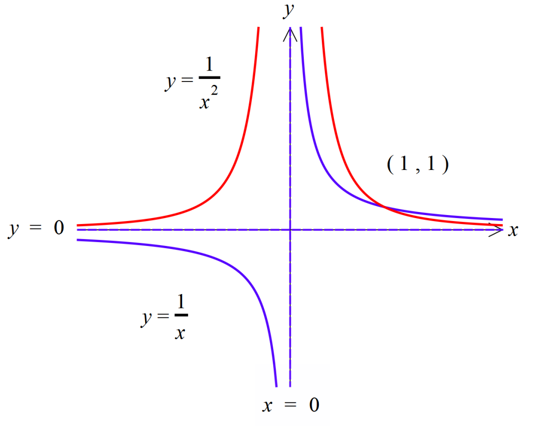
(i)	"implicit assumption", $x>0,\ x\neq 1$ $\log_x(3x-2)=2$ $x^2=3x-2$ $x^2-3x+2=0$ $(x-2)(x-1)=0$ $x=2$ or $x=1$ (NA) Why ?	**(ii)**	$(3x)^{\ln 3}=(4x)^{\ln 4}\cdots(*)$ For $x>0$, $\ln(3x)^{\ln 3}=\ln(4x)^{\ln 4}$ $(\ln 3)(\ln 3+\ln x)=(\ln 4)(\ln 4+\ln x)$ $(\ln 3-\ln 4)\ln x=(\ln 4)^2-(\ln 3)^2$ $\ln x=\dfrac{(\ln 4-\ln 3)(\ln 4+\ln 3)}{(\ln 3-\ln 4)}$ $\ln x=-\ln 12$ $x=\dfrac{1}{12}$ Note: by "definition" $b>0,\ 0^b=0$ so x=0 is also a solution to (*).

Example 8
Sketch on the same diagram the graphs of

(a) $y=x^2$ and $y=x^4$, **(b)** $y=x^{\frac{1}{3}}$ and $y=x^3$, **(c)** $y=\dfrac{1}{x}$ and $y=\dfrac{1}{x^2}$.

Solution

(a)		**(b)**	

<table>
<tr>
<td>(c)</td>
<td>
</td>
<td></td>
<td>Remarks:
Notice the behavior for each graph for values of x between 0 and 1, and for values of x greater than 1.</td>
</tr>
</table>

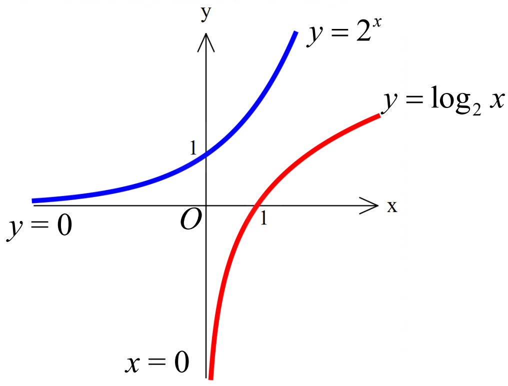
Example 9

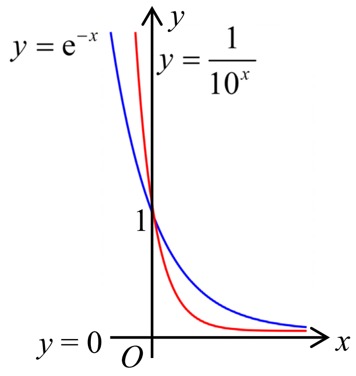
Sketch on the same diagram the graphs of

(a) $y = 2^x$ and $y = \log_2 x$, **(b)** $y = \mathrm{e}^{-x}$ and $y = \dfrac{1}{10^x}$,

indicating in each case the axial intercepts and equations of asymptotes.

Solution

<table>
<tr>
<td>(a)</td>
<td>

Note that $x > 0,\ 2^{\log_2 x} = x$</td>
<td>(b)</td>
<td>
</td>
</tr>
</table>

3 Geometry and Trigonometry

You will need to be familiar with the following geometric and trigonometric tools, as they can be used in almost any contexts, such as applications of differentiation (finding minima/maxima), solving inequalities or complex numbers.

Let us first start off with the special values of the trigonometric functions which you need to know.

Example 10

Complete the following table:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

Example 11

Simplify the following expressions, leaving your answer in a non-trigonometric fraction with integral denominator:

(a) $\dfrac{\cos\frac{3\pi}{4}+\sin\frac{7\pi}{4}}{\tan\frac{5\pi}{3}+\cot\frac{11\pi}{3}}$ **(b)** $\tan\dfrac{7\pi}{12}$

Solution

(a) $$\frac{\cos\frac{3\pi}{4}+\sin\frac{7\pi}{4}}{\tan\frac{5\pi}{3}+\cot\frac{11\pi}{3}}=\frac{-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}{-\sqrt{3}-\frac{1}{\sqrt{3}}}=\frac{\sqrt{2}}{\frac{3+1}{\sqrt{3}}}=\frac{\sqrt{6}}{4}.$$

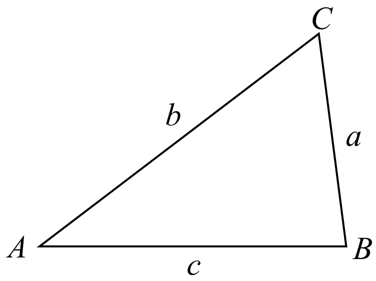
(b) $$\tan\frac{7\pi}{12}=\tan\left(\frac{\pi}{4}+\frac{\pi}{3}\right)=\frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{3}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{3}}=\frac{1+\sqrt{3}}{1-\sqrt{3}}=-\frac{4+2\sqrt{3}}{2}=-2-\sqrt{3}.$$

3.1 Sine Rule and Cosine Rule

Recall that the sine rule is useful when a pair of angles and opposite side is known. Together with another side (or angle), we can use it to find the corresponding opposite angle (or side).

The Sine Rule

$$\boxed{\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}}$$

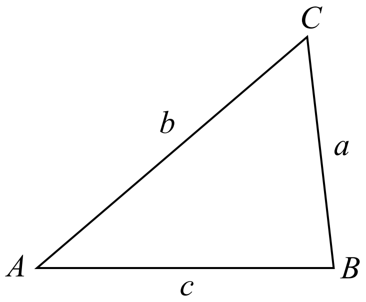


The cosine rule is useful when 2 sides and the included angle is given, and we are required to find the side opposite the included angle. We can also use the cosine rule to find the angles of a triangle if all 3 sides are given.

The Cosine Rule

For any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



By rearranging the formula, we can express the cosine of the angle in terms of the lengths of the sides.

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \quad \text{or} \quad \cos B = \frac{a^2+c^2-b^2}{2ac} \quad \text{or} \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

The Area of a Triangle

When the base and height of a triangle are not given, we can appeal to trigonometry, where we use half of the product of two sides multiplied by the sine of the included angle, that is,

$$\boxed{\text{Area of triangle} = \frac{1}{2}ab\sin C}$$

We will re-visit this formula in the chapter on Vectors.

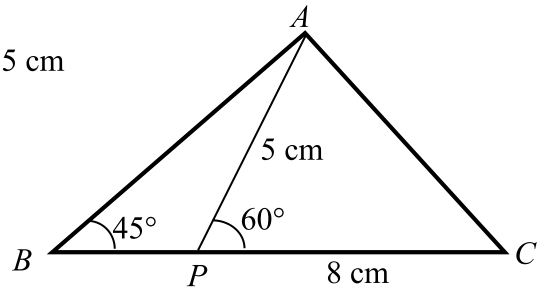
Example 12

In the diagram, given that $\angle ABP = 45°$, $\angle APC = 60°$, $AP = 5$ cm and $CP = 8$ cm, calculate

(a) AB,

(b) AC.

(c) the area of triangle ABC.



Solution

(a) $\angle APB = 180^\circ - 60^\circ = 120^\circ$ $\dfrac{AB}{\sin 120^\circ} = \dfrac{5}{\sin 45^\circ}$ $AB = \dfrac{5\sin 120^\circ}{\sin 45^\circ} = \dfrac{5\times\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = \dfrac{5\sqrt{6}}{2}$ cm (b) $AC^2 = 8^2 + 5^2 - 2(8)(5)\cos 60^\circ$ $= 49$ $AC = 7$ cm	(c) $\angle PAB = 180° - 120° - 45° = 15°$ Hence area of triangle ABC $=$ area of triangle $PAB +$ area of triangle APC $= \dfrac{1}{2}(5)\left(\dfrac{5\sqrt{6}}{2}\right)\sin 15^\circ + \dfrac{1}{2}(5)(8)\sin 60^\circ$ $= 21.3\text{cm}^2$ (3 s.f)

3.2 Trigonometric Functions, Identities and Equations

The addition and double angle formulae are required in H2 Math topics such as Complex Numbers and Integrations. While the formulae are given in the list of Formulae, you need to know how to apply them.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin^2 A + \cos^2 A = 1$$
$$\sin 2A = 2\sin A \cos A$$
$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Example 13

Solve the following equations for x between **0 and 2π**.

(i) $\left(\sin x+1\right)\left(\cos x+2\right)=0$, **(ii)** $\sec x=4\cos x$,

(iii) $\cos 2x+\sin x=0$, **(iv)** $\cos x+\sqrt{3}\sin x=\sqrt{3}$.

Solution

(i)	$\left(\sin x+1\right)\left(\cos x+2\right)=0$ $\sin x=-1$ or $\cos x=-2$ (rej, since $\lvert\cos x\rvert\le 1$) $\therefore x=\dfrac{3\pi}{2}$	**(ii)**	$\sec x=4\cos x$ $\cos^2 x=\dfrac{1}{4}$ $\cos x=\pm\dfrac{1}{2}$ $\therefore x=\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$ or $\dfrac{4\pi}{3}$ or $\dfrac{5\pi}{3}$
(iii)	$\cos 2x+\sin x=0$ $1-2\sin^2 x+\sin x=0$ $2\sin^2 x-\sin x-1=0$ $\left(2\sin x+1\right)\left(\sin x-1\right)=0$ $\sin x=-\dfrac{1}{2}$ or $\sin x=1$ $x=\dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$ or $\dfrac{\pi}{2}$	**(iv)**	$\cos x+\sqrt{3}\sin x=\sqrt{3}$ Let $\cos x+\sqrt{3}\sin x=R\cos(x-\alpha)$ $=R\cos x\cos\alpha+R\sin x\sin\alpha$ where $R>0$ and $0<\alpha<\dfrac{\pi}{2}$ Comparing, $R\cos\alpha=1$ and $R\sin\alpha=\sqrt{3}$. Hence $R=\sqrt{1+3}=2$ $\tan\alpha=\sqrt{3},\alpha=\dfrac{\pi}{3}$ $\therefore 2\cos(x-\dfrac{\pi}{3})=\sqrt{3}$ $x-\dfrac{\pi}{3}=\cos^{-1}\dfrac{\sqrt{3}}{2}=\dfrac{\pi}{6}$ or $-\dfrac{\pi}{6}$ $x=\dfrac{\pi}{2}$ or $\dfrac{\pi}{6}$

Example 14

Prove the following identities.

(i) $\dfrac{\sin x}{1+\cos x} \equiv \dfrac{1-\cos x}{\sin x}$, **(ii)** $\dfrac{\sin 2x}{1+\cos 2x} \equiv \tan x$,

(iii) $\cos 2x \equiv \dfrac{1-\tan^2 x}{1+\tan^2 x}$, **(iv)** $\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$.

Solution

(i)	LHS $= \dfrac{\sin x}{1+\cos x} \times \dfrac{1-\cos x}{1-\cos x}$ $= \dfrac{\sin x(1-\cos x)}{1-\cos^2 x}$ $= \dfrac{\sin x(1-\cos x)}{\sin^2 x}$ $= \dfrac{1-\cos x}{\sin x}$ = RHS	**(ii)**	LHS $= \dfrac{\sin 2x}{1+\cos 2x}$ $= \dfrac{2\sin x\cos x}{1+\left(2\cos^2 x - 1\right)}$ $= \dfrac{\sin x\cos x}{\cos^2 x}$ $= \dfrac{\sin x}{\cos x}$ $= \tan x$ = RHS
(iii)	RHS $= \dfrac{1-\tan^2 x}{1+\tan^2 x}$ $= \dfrac{1-\tan^2 x}{\sec^2 x}$ $= \dfrac{1-\dfrac{\sin^2 x}{\cos^2 x}}{\dfrac{1}{\cos^2 x}}$ $= \cos^2 x - \sin^2 x$ $= \cos 2x$ = LHS **OR**	**(iv)**	$\cos P + \cos Q \equiv 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$ Let $A = \dfrac{P+Q}{2}$, $B = \dfrac{P-Q}{2}$. $P = A + B$ and $Q = A - B$ LHS $= (\cos A\cos B - \sin A\sin B)$ $+ (\cos A\cos B + \sin A\sin B)$ $= 2\cos A\cos B + 0$ =RHS (Shown)

	LHS $= \cos 2x$ $= \cos^2 x - \sin^2 x$ $= \dfrac{1}{\sec^2 x} - \dfrac{\sin^2 x \sec^2 x}{\sec^2 x}$ $= \dfrac{1 - \tan^2 x}{\sec^2 x}$ $= \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$ $=$ RHS		

4 Vectors

In the real world, we tend to associate things with its magnitude. For instance, the mass of a body and the time taken for a person to complete a 100m sprint are quantities that can be completely described by magnitude alone. Such quantities are called scalars. However, there are certain phenomena whose direction must be specified and which cannot be described completely by magnitude alone. Such quantities are called vectors. In this section, we shall review some basic properties and results of vectors.

4.1 Scalars and Vectors

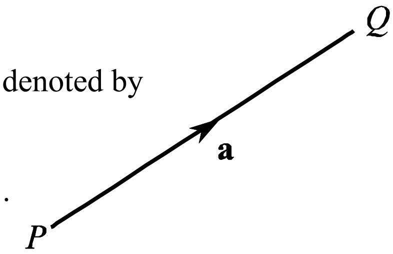
Scalars are quantities that have magnitude but no associated direction.
Vectors are quantities that have both <u>magnitude</u> and <u>direction</u>.

Identify the following quantities as either scalars or vectors:
Distance, Displacement, Time, Speed, Velocity, Temperature, Mass, Weight, Force, Area, Volume

Scalars		Vectors
Distance	Mass	Displacement
Time	Area	Velocity
Speed	Volume	Weight
Temperature		Force

4.2 Notation and Geometrical Representation

A vector, denoted by $\mathbf{a}$ (in print) or $\underset{\sim}{a}$ (in written form), can be represented by a directed line segment $\overrightarrow{PQ}$, where



- the magnitude of $\mathbf{a}$ is the length of the line segment PQ, denoted by $|\mathbf{a}|$, $|\overrightarrow{PQ}|$ or simply PQ, and
- the direction of $\mathbf{a}$ is the direction of the line from P to Q.

So we write $\mathbf{a} = \overrightarrow{PQ}$.
We call P the initial point and Q the terminal point.

4.3 Position & Free (Displacement) Vectors

In general, there are two types of vectors – position vectors and free vectors.

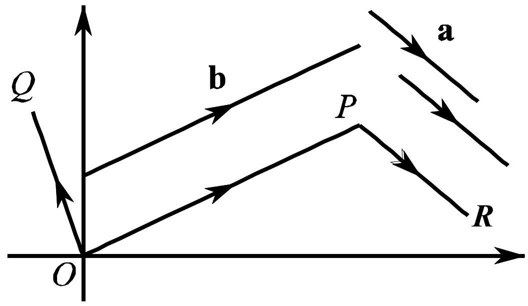
A <u>position vector</u> is used to denote the location of a point. It defines the position of one point relative to another point.

A <u>free vector</u> (also known as displacement vector) is a vector with no associated position.

$\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are the position vectors of points P and Q with respect to the origin O.

On the other hand, the vectors $\mathbf{a}$ and $\mathbf{b}$ are free vectors.

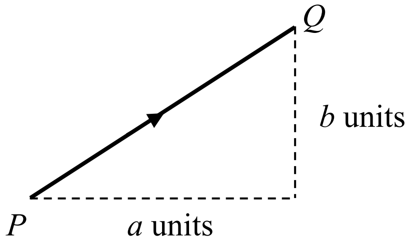
In fact, $\mathbf{a}$ can be represented by $\overrightarrow{PR}$, or any directed line segment of length $|\mathbf{a}|$ and having the same direction as $\mathbf{a}$.



4.4 Vectors in Two Dimensions (2D Vectors)

Any vector parallel to the 2-dimensional xy-plane can also be represented in terms of components in the directions of the x- and y- axes.

Any vector $\overrightarrow{PQ}$ can also be represented in a column vector form as $\overrightarrow{PQ} = \begin{pmatrix} a \\ b \end{pmatrix}$ where a and b are known as the horizontal and vertical **components** of $\overrightarrow{PQ}$ respectively.

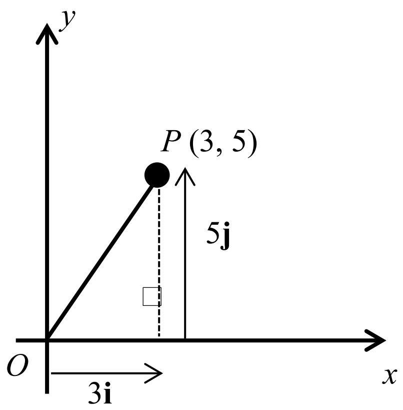


Magnitude of $\overrightarrow{PQ}$ is represented by $\left|\overrightarrow{PQ}\right| = \sqrt{a^2 + b^2}$.

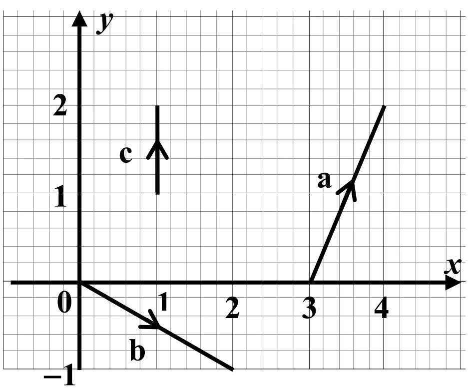
Referring to the figure on the right, the vector $\overrightarrow{OP}$ is 3 units in the x direction and 5 units in the y direction.

We write $\overrightarrow{OP} = 3\mathbf{i} + 5\mathbf{j}$, where **i** and **j** are vectors of magnitude 1 in the x and y directions respectively.

In column vector form, $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\left|\overrightarrow{OP}\right| = \sqrt{3^2 + 5^2} = 34$.



Example 15
State the corresponding column vectors for **a**, **b** and **c**.



Solution

$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example 16

Find the magnitude of each of the following vectors, leaving your answer in exact form.

(a) $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ **(b)** $\begin{pmatrix} -78 \\ -30 \end{pmatrix}$

Solution

(a)

$$\left|\begin{pmatrix} 4 \\ -5 \end{pmatrix}\right| = \sqrt{4^2 + (-5)^2}$$
$$= \sqrt{16+25}$$
$$= \sqrt{41} \text{ units}$$

(b)

$$\begin{pmatrix} -78 \\ -30 \end{pmatrix} = 6\begin{pmatrix} -13 \\ -5 \end{pmatrix}$$
$$\left|\begin{pmatrix} -78 \\ -30 \end{pmatrix}\right| = 6\left|\begin{pmatrix} -13 \\ -5 \end{pmatrix}\right|$$
$$= 6\sqrt{(-13)^2 + (-5)^2}$$
$$= 6\sqrt{194} \text{ units}$$

Alternatively,

$$\left|\begin{pmatrix} -78 \\ -30 \end{pmatrix}\right| = \sqrt{\left(-78\right)^2 + \left(-30\right)^2} = 6\sqrt{194}$$

Example 17

Given that $\overrightarrow{AB} = \begin{pmatrix} -18 \\ 12 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} k \\ -3 \end{pmatrix}$, find the possible values of k if $\left|\overrightarrow{AB}\right| = 4\left|\overrightarrow{CD}\right|$.

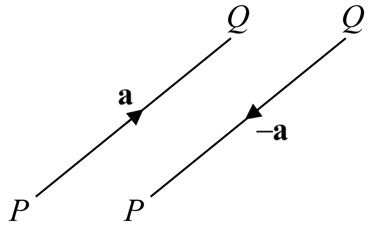
Solution

$$\overrightarrow{AB} = \begin{pmatrix} -18 \\ 12 \end{pmatrix}$$
$$= 6\begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = 4\left|\overrightarrow{CD}\right|$$
$$6\sqrt{(-3)^2 + 2^2} = 4\sqrt{k^2 + (-3)^2}$$
$$3\sqrt{13} = 2\sqrt{k^2 + 9}$$
$$117 = 4(k^2 + 9)$$
$$k^2 = \frac{81}{4}$$
$$k = \pm\frac{9}{2}$$

4.5 Negative Vectors

The negative of vector $\mathbf{a}$, denoted by $-\mathbf{a}$, has the same magnitude as $\mathbf{a}$ but the opposite direction.
Geometrically, if $\mathbf{a}$ is represented by $\overrightarrow{PQ}$,
then $-\mathbf{a}$ is represented by $\overrightarrow{QP}$.



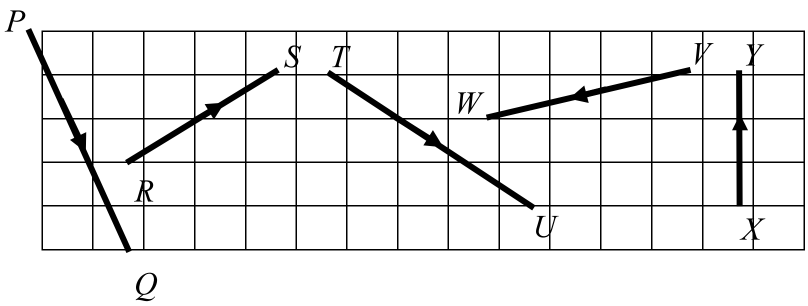
If $\overrightarrow{PQ} = \begin{pmatrix} a \\ b \end{pmatrix}$, then $\overrightarrow{QP} = \begin{pmatrix} -a \\ -b \end{pmatrix}$.

Note: $|\overrightarrow{PQ}| = |\overrightarrow{QP}|$

Example 18

Given that $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ in the diagram below. Express each of the following as a column vector and then write down its **negative** vector.

(a) $\overrightarrow{RS}$ **(b)** $\overrightarrow{TU}$ **(c)** $\overrightarrow{VW}$ **(d)** $\overrightarrow{XY}$



Solution

(a) $\overrightarrow{RS} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. The negative of $\overrightarrow{RS}$ is $\overrightarrow{SR} = -\overrightarrow{RS} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

(b) $\overrightarrow{TU} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. The negative of $\overrightarrow{TU}$ is $\overrightarrow{UT} = -\overrightarrow{TU} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

(c) $\overrightarrow{VW} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$. The negative of $\overrightarrow{VW}$ is $\overrightarrow{WV} = -\overrightarrow{VW} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

(d) $\overrightarrow{XY} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$. The negative of $\overrightarrow{XY}$ is $\overrightarrow{YX} = -\overrightarrow{XY} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

4.6 The Zero Vector

The zero vector (or the null vector), denoted by $\mathbf{0}$ or $\underset{\sim}{0}$, is the vector with zero magnitude and no direction. For any point P, $\overrightarrow{PP}=\mathbf{0}$. In 2D, the zero vector is $\begin{pmatrix}0\\0\end{pmatrix}$.

For any vector $\mathbf{a}$, $\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$ and $\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$.

4.7 Vector Addition and Subtraction

<u>Algebraic Definition</u>

In 2D, $\begin{pmatrix}a\\b\end{pmatrix}\pm\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}a\pm c\\b\pm d\end{pmatrix}$

Example 19

If $\mathbf{a}=\begin{pmatrix}-1\\-4\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}7\\4\end{pmatrix}$, $\mathbf{c}=\begin{pmatrix}-5\\2\end{pmatrix}$, find

(a) $\mathbf{a}+2\mathbf{b}-3\mathbf{c}$,

(b) $|2\mathbf{a}-\mathbf{b}|$,

(c) the vector $\mathbf{d}$ if $\mathbf{b}-2\mathbf{d}=5\mathbf{c}$.

Solution

(a) $\mathbf{a}+2\mathbf{b}-3\mathbf{c}=\begin{pmatrix}-1\\-4\end{pmatrix}+2\begin{pmatrix}7\\4\end{pmatrix}-3\begin{pmatrix}-5\\2\end{pmatrix}=\begin{pmatrix}28\\-2\end{pmatrix}$

(b) $2\mathbf{a}-\mathbf{b}=2\begin{pmatrix}-1\\-4\end{pmatrix}-\begin{pmatrix}7\\4\end{pmatrix}=\begin{pmatrix}-9\\-12\end{pmatrix}=-3\begin{pmatrix}3\\4\end{pmatrix}$

$|2\mathbf{a}-\mathbf{b}|=\left|-3\begin{pmatrix}3\\4\end{pmatrix}\right|=\left|3\begin{pmatrix}3\\4\end{pmatrix}\right|=3\sqrt{3^2+4^2}=3\times5=15$ units

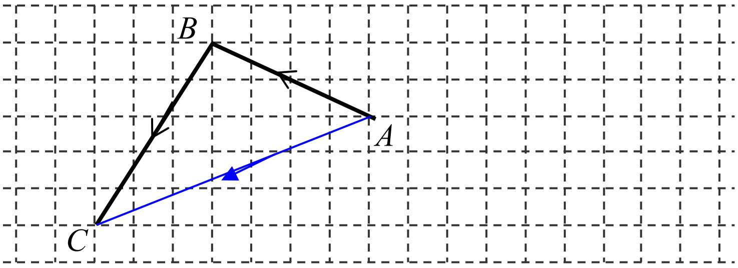
(c) $\mathbf{b}-2\mathbf{d}=5\mathbf{c}$

$\mathbf{d}=\frac{1}{2}(\mathbf{b}-5\mathbf{c})=\frac{1}{2}\left[\begin{pmatrix}7\\4\end{pmatrix}-5\begin{pmatrix}-5\\2\end{pmatrix}\right]=\begin{pmatrix}16\\-3\end{pmatrix}$

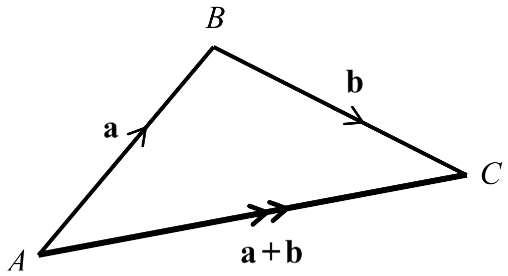
Geometric Interpretation (Triangle law of addition)

Consider the vectors $\overrightarrow{AB}=\begin{pmatrix}-4\\2\end{pmatrix}$ and $\overrightarrow{BC}=\begin{pmatrix}-3\\-5\end{pmatrix}$, find $\begin{pmatrix}-4\\2\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}$

In the grid, represent the resultant vector of $\overrightarrow{AB}+\overrightarrow{BC}$



Notice that $\overrightarrow{AC}=\begin{pmatrix}-7\\-3\end{pmatrix}$, and we can express $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ as $\begin{pmatrix}-4\\2\end{pmatrix}+\begin{pmatrix}-3\\-5\end{pmatrix}=\begin{pmatrix}-7\\-3\end{pmatrix}$.



If two vectors $\mathbf{a}$ and $\mathbf{b}$ are represented by the sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of a triangle, then $\mathbf{a}+\mathbf{b}$ is represented by the third side $\overrightarrow{AC}$.

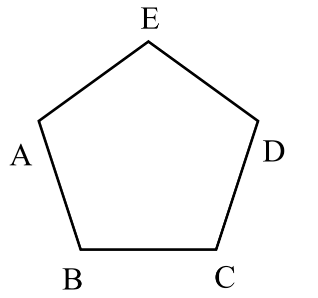
$$\boxed{\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}}$$

Notes:

1. Remember to place vectors "head to tail" to faciliatate the addition of each pair of vectors.
2. Take note of the intermediate letter B.

Example 20

ABCDE is a regular pentagon. Simplify the following:



(a) $\overrightarrow{AB}+\overrightarrow{BE}+\overrightarrow{EC}$

(b) $\overrightarrow{BA}+\overrightarrow{EC}+\overrightarrow{AE}+\overrightarrow{CD}$

(c) $\overrightarrow{CB}-\overrightarrow{AB}+\overrightarrow{AE}-\overrightarrow{DE}$

(d) $\overrightarrow{AB}-\overrightarrow{EC}+\overrightarrow{BC}-\overrightarrow{DE}$

Solution

(a) $\overrightarrow{AB}+\overrightarrow{BE}+\overrightarrow{EC}=\overrightarrow{AC}$

(b) $\overrightarrow{BA}+\overrightarrow{EC}+\overrightarrow{AE}+\overrightarrow{CD}=\overrightarrow{BD}$

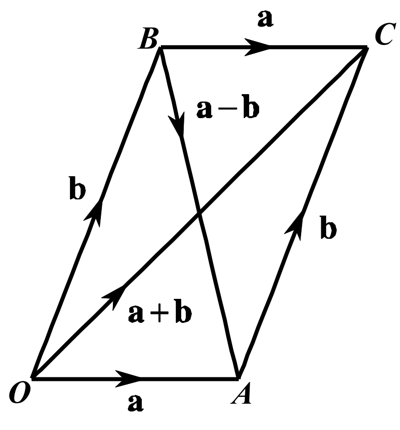
(c) $\overrightarrow{CB}-\overrightarrow{AB}+\overrightarrow{AE}-\overrightarrow{DE}=\overrightarrow{CB}+\overrightarrow{BA}+\overrightarrow{AE}+\overrightarrow{ED}=\overrightarrow{CD}$

(d) $\overrightarrow{AB}-\overrightarrow{EC}+\overrightarrow{BC}-\overrightarrow{DE}=\overrightarrow{AB}+\overrightarrow{CE}+\overrightarrow{BC}+\overrightarrow{ED}=\overrightarrow{AD}$

<u>Parallelogram Law of Addition</u>

Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of points A and B respectively, relative to the origin O.

The vector $\mathbf{a}+\mathbf{b}$ represents the position vector of point *C*:



$$\mathbf{a}+\mathbf{b}=\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}$$

The vector $\mathbf{a}-\mathbf{b}$, defined to be $\mathbf{a}+(-\mathbf{b})$, represents the vector $\overrightarrow{BA}$:

$$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})=\overrightarrow{BO}+\overrightarrow{OA}=\overrightarrow{BA}$$

Any displacement vector can be written in terms of the position vectors of its end points.
For example, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$.

The vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ represent the <u>diagonals</u> of the parallelogram $OACB$. This is known as the parallelogram law of addition for vectors.

4.8 Scalar Multiplication and parallel vectors

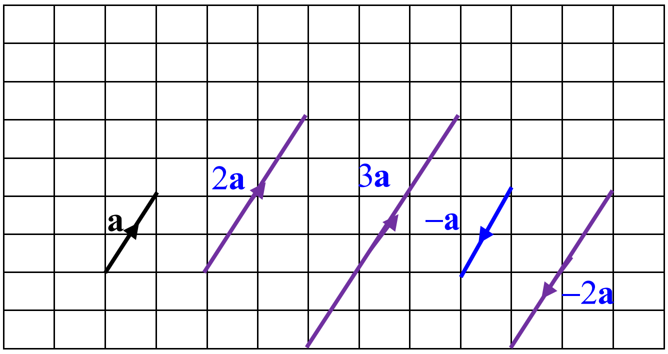
Let $\mathbf{a}$ be a non-zero vector and $\lambda \in \mathbb{R}$ be a scalar.

The scalar multiple of $\mathbf{a}$ by λ, denoted by $\lambda\mathbf{a}$, is a vector whose direction depends on the sign of λ as follows:

- If $\lambda > 0$, then $\lambda\mathbf{a}$ and $\mathbf{a}$ are in the same direction.
- If $\lambda = 0$, then $\lambda\mathbf{a}$ is the zero vector, i.e. $\lambda\mathbf{a} = \mathbf{0}$.
- If $\lambda < 0$, then $\lambda\mathbf{a}$ and $\mathbf{a}$ are in opposite directions.

In 2D, if k is a scalar, then $k\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ka \\ kb \end{pmatrix}$.

Consider $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and draw vectors $2\mathbf{a}, 3\mathbf{a}, -\mathbf{a}$ and $-2\mathbf{a}$.



What do you observe? **All the vectors are parallel.**

For any vector $\mathbf{a}$, the vector $k\mathbf{a}$ has magnitude $|k||\mathbf{a}|$.

What can you say if

(i) if $k > 0$? **It is parallel to $\mathbf{a}$ and in the same direction**

(ii) if $k < 0$? **It is parallel to $\mathbf{a}$ and in the opposite direction of $\mathbf{a}$**

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-zero vectors.

In general, we say that $\mathbf{a}$ is parallel to $\mathbf{b}$, denoted by $\mathbf{a} \mathbin{//} \mathbf{b}$, if and only if $\mathbf{a} = \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$. Essentially two non-zero vectors are parallel if one is a scalar multiple of another.

Example 21

The vector $\mathbf{u}$ is represented by $\begin{pmatrix} 15 \\ 8 \end{pmatrix}$. If $\mathbf{u}$ and $\mathbf{v}$ are parallel in the same direction and $|\mathbf{v}| = 51$, find $\mathbf{v}$.

Solution

$|\mathbf{v}| = 51 \Rightarrow \sqrt{a^2 + b^2} = 51$

$\mathbf{u}$ and $\mathbf{v}$ are parallel and in the same direction $\Rightarrow$ $\mathbf{u} = k\mathbf{v}$ for some k>0.

$$\Rightarrow |\mathbf{u}| = k|\mathbf{v}|$$

$$\Rightarrow \left|\begin{pmatrix} 15 \\ 8 \end{pmatrix}\right| = 51k$$

$$\Rightarrow k = \frac{\sqrt{15^2 + 8^2}}{51} = \frac{17}{51} = \frac{1}{3}$$

$\mathbf{u} = \frac{1}{3}\mathbf{v}$

$\mathbf{v} = 3\mathbf{u} = 3\begin{pmatrix} 15 \\ 8 \end{pmatrix} = \begin{pmatrix} 45 \\ 24 \end{pmatrix}$

4.9 Laws of Vector Algebra

In this section, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors while λ and μ are scalars, i.e. $\lambda, \mu \in \mathbb{R}$.
The operations of vector addition and scalar multiplication satisfy the following laws:

Commutative Laws:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ 2. $\lambda\mathbf{a} = \mathbf{a}\lambda$

Associative Laws:

1. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ 2. $(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a})$

Distributive Laws:

1. $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$ 2. $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$

Notice that the above laws are similar to that we are used to when operating with addition and multiplication with real numbers.

4.10 Equal Vectors

Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, what can you say about $\mathbf{a}$ and $\mathbf{b}$?

They are equal vectors.

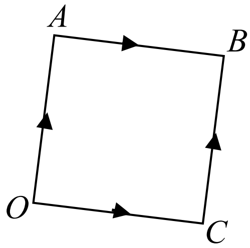
In general, if $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} r \\ s \end{pmatrix}$ and $\mathbf{a} = \mathbf{b}$, then $p = r$ and $q = s$.

If $OABC$ is a square, then

$\overrightarrow{OA} = \overrightarrow{CB}$ and $\overrightarrow{OC} = \overrightarrow{AB}$.

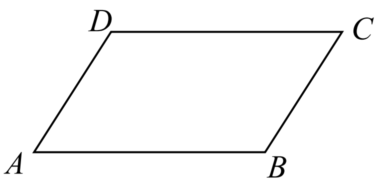
We can say that $OA = OC$ but $\overrightarrow{OA} \neq \overrightarrow{OC}$. Why?

They are in different directions.



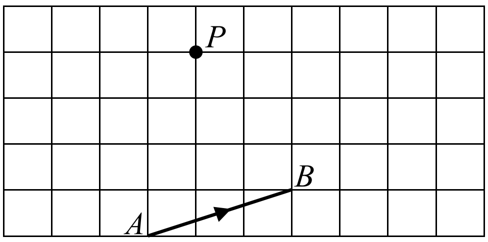
What if ABCD is a parallelogram?

Then the opposite sides are equal vectors, i.e. $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.



Example 22

The figure shows the positions of the points *P*, *A* and *B* where $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.



(a) Express $\overrightarrow{PB}$ as a column vector.

(b) Q is a point such that $ABQP$ is a parallelogram. Express $\overrightarrow{QB}$ as a column vector.

(c) R is a point such that $ABPR$ is a parallelogram. Express $\overrightarrow{PR}$ as a column vector.

(d) Do the two vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ have the same magnitude?
Is $\overrightarrow{PQ} = \overrightarrow{PR}$? Explain your answer.

Solution

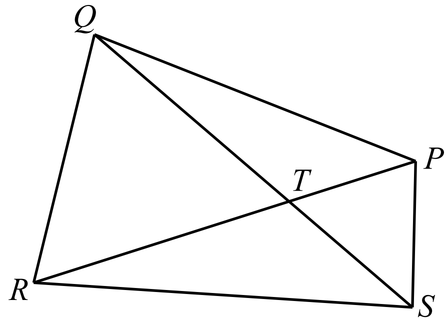
(a) $\overrightarrow{PB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **(b)** $\overrightarrow{QB} = \overrightarrow{PA} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$ **(c)** $\overrightarrow{PR} = \overrightarrow{BA} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

(d) Yes, they have the same magnitude. They are not equal vectors as they are in opposite directions.

Example 23

In the diagram, T is the point of intersection of the diagonals of the quadrilateral $PQRS$.

$\overrightarrow{PR} = 3\overrightarrow{PT}$, $\overrightarrow{PS} = 5\mathbf{b}$, $\overrightarrow{PQ} = 4\mathbf{a} + \mathbf{b}$ and $\overrightarrow{PR} = 3\mathbf{a} + 12\mathbf{b}$.



(a) Express, as simply as possible, in terms of **a** and **b**,

(i) $\overrightarrow{RS}$,

(ii) $\overrightarrow{RT}$,

(iii) $\overrightarrow{RQ}$.

(b) Show that $\overrightarrow{QT} = 3(\mathbf{b} - \mathbf{a})$.

(c) Express $\overrightarrow{QS}$ as simply as possible, in terms of **a** and **b**.

(d) Calculate the value of

(i) $\dfrac{QT}{QS}$,

(ii) $\dfrac{\text{area of } \Delta PQT}{\text{area of } \Delta PQS}$,

(iii) $\dfrac{\text{area of } \Delta PQT}{\text{area of } \Delta RQT}$.

Solution

(a)(i)
$$\begin{aligned}\overrightarrow{RS} &= \overrightarrow{PS} - \overrightarrow{PR} \\ &= 5\mathbf{b} - (3\mathbf{a} + 12\mathbf{b}) \\ &= -3\mathbf{a} - 7\mathbf{b}\end{aligned}$$

(ii)
$$\begin{aligned}\overrightarrow{RT} &= \frac{2}{3}\overrightarrow{RP} \\ &= \frac{2}{3}(-3\mathbf{a} - 12\mathbf{b}) \\ &= -2\mathbf{a} - 8\mathbf{b}\end{aligned}$$

(iii)
$$\begin{aligned}\overrightarrow{RQ} &= \overrightarrow{PQ} - \overrightarrow{PR} \\ &= 4\mathbf{a} + \mathbf{b} - (3\mathbf{a} + 12\mathbf{b}) \\ &= \mathbf{a} - 11\mathbf{b}\end{aligned}$$

(b)
$$\begin{aligned}\overrightarrow{QT} &= \overrightarrow{QR} + \overrightarrow{RT} \\ &= (-\mathbf{a} + 11\mathbf{b}) + (-2\mathbf{a} - 8\mathbf{b}) \\ &= -3\mathbf{a} + 3\mathbf{b} \\ &= 3(\mathbf{b} - \mathbf{a}) \text{ (shown)}\end{aligned}$$

(c)
$$\begin{aligned}\overrightarrow{QS} &= \overrightarrow{PS} - \overrightarrow{PQ} \\ &= 5\mathbf{b} - (4\mathbf{a} + \mathbf{b}) \\ &= 4\mathbf{b} - 4\mathbf{a} \\ &= 4(\mathbf{b} - \mathbf{a})\end{aligned}$$

(d)(i)
$$\overrightarrow{QT} = 3(\mathbf{b} - \mathbf{a}) = \frac{3}{4} \times 4(\mathbf{b} - \mathbf{a}) = \frac{3}{4}\overrightarrow{QS}$$

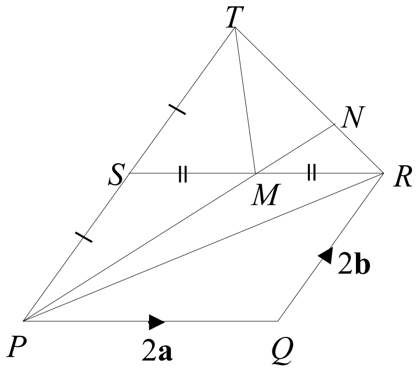
$$\frac{QT}{QS} = \frac{3}{4}$$

(ii)
$$\begin{aligned}\frac{\text{area of } \Delta PQT}{\text{area of } \Delta PQS} &= \frac{\frac{1}{2}(QT)h}{\frac{1}{2}(QS)h} \\ &= \frac{QT}{QS} \\ &= \frac{3}{4}\end{aligned}$$

(iii)
$$\begin{aligned}\frac{\text{area of } \Delta PQT}{\text{area of } \Delta RQT} &= \frac{\frac{1}{2}(PT)h'}{\frac{1}{2}(RT)h'} \\ &= \frac{PT}{RT} \\ &= \frac{1}{2}\end{aligned}$$

Example 24

PQRS is a parallelogram and *M* is the midpoint of *SR*. *PS* is produced to *T* so that $PS = ST$. *PM* is produced to *N* and $RT = 3RN$. $\overrightarrow{PQ} = 2\mathbf{a}$ and $\overrightarrow{QR} = 2\mathbf{b}$.



(a) Express, as simply as possible, in terms of **a** and/or **b**,

(i) $\overrightarrow{ST}$, **(ii)** $\overrightarrow{TR}$, **(iii)** $\overrightarrow{TN}$.

(b) Show that $\overrightarrow{PN} = \frac{4}{3}(\mathbf{a} + 2\mathbf{b})$.

(c) Express $\overrightarrow{PM}$ as simply as possible, in terms of **a** and **b**.

(d)

Calculate the value of

(i) $\frac{PM}{PN}$, **(ii)** $\frac{\text{the area of } \Delta MTP}{\text{the area of } \Delta MTN}$.

Solution

(a)(i) $\overrightarrow{ST} = 2\mathbf{b}$

(ii) $\overrightarrow{TR} = \overrightarrow{TS} + \overrightarrow{SR}$

$= -2\mathbf{b} + 2\mathbf{a}$

(iii) $\overrightarrow{TN} = \frac{2}{3}\overrightarrow{TR} = \frac{2}{3} \times (-2\mathbf{b} + 2\mathbf{a})$

$= \frac{4}{3}\mathbf{a} - \frac{4}{3}\mathbf{b}$

(b) $\overrightarrow{PN} = \overrightarrow{PT} + \overrightarrow{TN}$

$= 4\mathbf{b} + \frac{4}{3}(\mathbf{a} - \mathbf{b})$

$= \frac{4}{3}\mathbf{a} + \frac{8}{3}\mathbf{b}$

$= \frac{4}{3}(\mathbf{a} + 2\mathbf{b})$

(c) $\overrightarrow{PM} = \overrightarrow{PS} + \overrightarrow{SM} = 2\mathbf{b} + \mathbf{a}$

(d)(i) $\overrightarrow{PN} = \frac{4}{3}(\mathbf{a} + 2\mathbf{b}) = \frac{4}{3}\overrightarrow{PM}$

$\Rightarrow PN = \frac{4}{3}PM$

$\frac{PM}{PN} = \frac{3}{4}$

(ii) $$\frac{\text{Area } \Delta MTP}{\text{Area } \Delta MTN} = \frac{\frac{1}{2} \times h \times PM}{\frac{1}{2} \times h \times MN}$$

$$= \frac{PM}{MN}$$

$$= \frac{3}{1}$$

$$= 3$$

5 Introduction to Modulus

In certain contexts, we often have to consider the numerical value rather than the actual value of a quantity. For instance, when we consider all positive integers which *differ* from 2 by at most 3, this includes positive integers 3, 4 and 5 which are larger than 2, 2 itself and integers -1, 0 and 1 which are smaller than 2. We are not exactly concerned about whether the resulting integers are larger or smaller than 2, but rather the focus is on the numerical value of the difference between the two numbers.

5.1 Basic definition and Properties

The absolute value or the modulus of *x*, is denoted by $|x|$, is defined by

$$\boxed{|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}}$$

Intuitively, the absolute value or modulus of a real number x is the numerical value (or magnitude) of *x* without regard to its sign. For instance, $|3| = |-3| = 3$.

Some useful properties:

1) $|x| \geq 0$

2) $|x - y| = 0 \quad \Leftrightarrow \quad x = y$

3) $|xy| = |x||y|$

4) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ where $y \neq 0$

5) $|x^2| = |x|^2 = x^2$

6) For $a > 0$, $|x| = a \quad \Leftrightarrow \quad x = \pm a$

7) $|x| = \sqrt{x^2}$

Eg $\sqrt{4^2} = \sqrt{(-4)^2} = 4$

Example 25

x and *y* are two real numbers such that $x \in [-5, 3]$ and $y \in [1, 6]$.

Find the largest and least possible values of

(a) $|x - y|$, **(b)** $|x + y|$, **(c)** $|xy|$, **(d)** $|x^2 y|$.

Solution

(a)	Largest value of $|x - y|$ occurs when $x = -5$ and $y = 6$, ie largest value is 11. Least value of $|x - y|$ occurs when $x = y$, ie least value is 0.	**(b)**	Largest value of $|x + y|$ occurs when $x = 3$ and $y = 6$, ie largest value is 9. Least value of $|x + y|$ occurs when $x = -y$, ie least value is 0.

(c)	$\lvert xy\rvert=\lvert x\rvert\lvert y\rvert$ Largest value of $\lvert xy\rvert$ occurs when $\lvert x\rvert$ and $\lvert y\rvert$ are largest, ie $x=-5$ and $y=6$. Largest value is 30. Least value of $\lvert xy\rvert$ occurs when $\lvert x\rvert=0$ ie least value is 0.	**(d)**	$x\in[-5,3]\Rightarrow x^2\in\left[0,5^2\right]$ and $\lvert x^2y\rvert=x^2\lvert y\rvert$ Largest value of $\lvert x^2y\rvert$ occurs when $\lvert x\rvert$ and $\lvert y\rvert$ are largest, ie $x=-5$ and $y=6$. Largest value is 150. Least value of $\lvert x^2y\rvert$ occurs when $\lvert x\rvert=0$ ie least value is 0.

5.2 Modulus of a Function

In general, for any function f(x), $\lvert \mathrm{f}(x)\rvert=\begin{cases}\mathrm{f}(x), & \mathrm{f}(x)\ge 0,\\ -\mathrm{f}(x), & \mathrm{f}(x)<0.\end{cases}$

Example 26

Let $\mathrm{f}(x)=x-2$ and $\mathrm{g}(x)=\lvert \mathrm{f}(x)\rvert$.

Evaluate $\mathrm{f}(x)$ and $\mathrm{g}(x)$ for **(a)** x=2, **(b)** x= 3, **(c)** $x=1$.

State the geometrical relationship between the points

(i) $(2,\mathrm{f}(2))$ and $(2,\mathrm{g}(2))$, **(ii)** $(3,\mathrm{f}(3))$ and $(3,\mathrm{g}(3))$, **(iii)** $(1,\mathrm{f}(1))$ and $(1,\mathrm{g}(1))$.

Solution

x	$\mathrm{f}(x)=x-2$	$\mathrm{g}(x)=\lvert \mathrm{f}(x)\rvert$	$(x,\mathrm{f}(x))$	$(x,\mathrm{g}(x))$
2	0	0	(2,0)	(2,0)
3	1	1	(3,1)	(3,1)
1	−1	1	$(1,-1)$	(1,1)

(i) $(2,\mathrm{f}(2))$ and $(2,\mathrm{g}(2))$ represent the same point.

(ii) Similarly, $(3,\mathrm{f}(3))$ and $(3,\mathrm{g}(3))$ represent the same point.

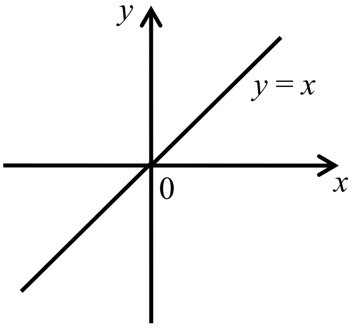
(iii) $(1,\mathrm{f}(1))$ and $(1,\mathrm{g}(1))$ are reflections of each other about the x-axis.

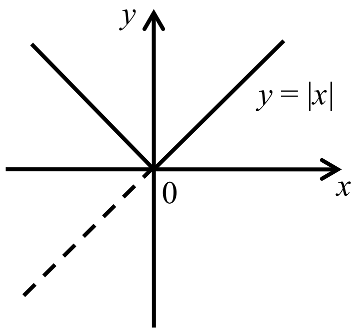
From the above example, we can derived the following result on graphs of modulus functions.

5.3 Graphs of Modulus Functions

To sketch the graph of $y = |\mathrm{f}(x)|$, we retain the portion of the graph of $y = \mathrm{f}(x)$ that is "above or on the x-axis", and reflect about the x-axis the remaining portion of the graph of $y = \mathrm{f}(x)$ that is "below the x-axis".

For example, the graphs of $y = x$ and $y = |x|$ are shown below.





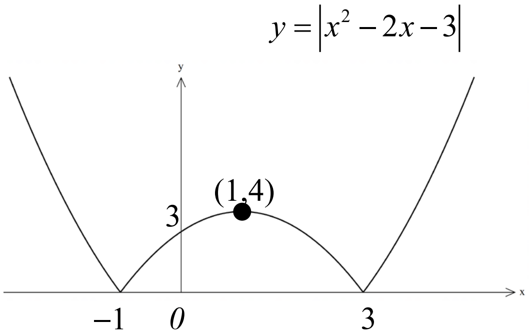
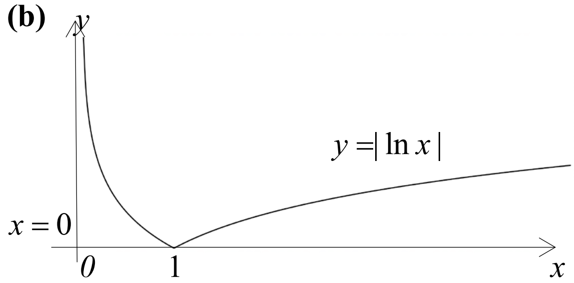
Example 27

Sketch the graphs of

(a) $y = |x^2 - 2x - 3|$ **(b)** $y = |\ln x|$.

indicating in each case the axial intercepts and equations of asymptotes.

Solution

<table>
<tr>
<td>(a)</td>
<td>
</td>
<td>(b) 
</td>
</tr>
</table>

Example 28

Solve the following equations

(a) $|x^2-2x-3|=5$ **(b)** $|x^2-2x-3|=-1$ **(c)** $|\ln x|=1$

Solution

(a) To solve the equation, we must first simplify it by "getting rid" of the modulus sign

Method 1:

$|x^2-2x-3|=5$

$x^2-2x-3=\pm 5$

$x^2-2x-3=5$ or $x^2-2x-3=-5$

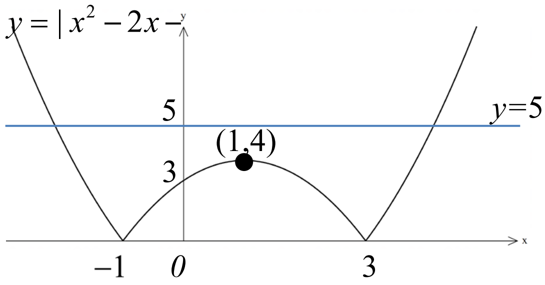
$x^2-2x-8=0$ or $x^2-2x+2=0$

$(x-4)(x+2)=0$ or $(x-1)^2+1=0$

Since $\left[(x-1)^2+1\right]>0$ for all $x\in\mathbb{R}$, $x=-2$ or 4

We are applying the result:
For k>0, $|\mathrm{f}(x)|=k \Leftrightarrow \mathrm{f}(x)=\pm k$

Method 2: Using the graph of $y=|x^2-2x-3|$ in Example 27a



By drawing the line y=5, it can be observed that y=5 intersects the graph of $y=|x^2-2x-3|$ when $x^2-2x-3>0$.

Solve $|x^2-2x-3|=5$ is equivalent to solving $x^2-2x-3=5$.

From Method 1, $x=-2$ or 4

Note: You can use your graphing calculator to solve $|x^2-2x-3|=5$ by plotting the graphs of the 2 functions and finding the x co-ordinates of the points of intersection. These techniques will be covered in the chapter on functions.

(b) $|x^2-2x-3|=-1$

This equation has no real solution since $|x^2-2x-3|\geq 0$ for all real x and -1<0.

Note that the result "$|x|=k$ is equivalent to $x=\pm k$" cannot be applied here since k has to be a positive integer.

(c) $|\ln x|=1$

$\ln x=\pm 1$ (Note: This conclusion can also be derived by using the graph sketched in Example 27b and observing that $y=1$ intersects the graph of $y=|\ln x|$ at both its branches)

$x=\mathrm{e}^{-1}$ or e