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MACLAURIN SERIES

SYLLABUS

- Standard series expansion of $(1+x)^n$ for any rational n , e^x , $\sin x$, $\cos x$ and $\ln(1+x)$
- Derivation of the first few terms of the Maclaurin series by
 - repeated differentiation, e.g. $\sec x$
 - repeated implicit differentiation, e.g. $y^3 + y^2 + y = x^2 - 2x$
 - using standard series, e.g. $e^x \cos 2x$, $\ln\left(\frac{1+x}{1-x}\right)$
- Range of values of x for which a standard series converges
- Concept of “approximation”
- Small angle approximations: $\sin x \approx x$, $\cos x \approx 1 - \frac{1}{2}x^2$, $\tan x \approx x$

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LECTURE 1

Lesson Outline

- Introduction to power series
 - Maclaurin series: derivation, standard series expansions & some basic examples
-

0 INTRODUCTION: POWER SERIES

Generally, functions can be complicated, thereby making calculations and manipulations with them tough. Thankfully, many functions can be represented as an “infinite polynomial”, called a power series.

This enables many functions to be well approximated by an appropriate polynomial. It is useful since polynomials are easier to evaluate, differentiate and integrate. Power series are applied in many fields, including Physics and Economics, as a way of simplifying computations.

Formally, a **power series** is an infinite series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$

where the coefficients $a_0, a_1, a_2, a_3, \dots, a_n, \dots$ are constants, e.g.
 $1 + 5x + 6x^2 + 3x^4 + 7x^6 + \cdots$

The following geometric series are examples of power series,

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \cdots, & |x| < 1 \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - \cdots, & |x| < 1 \end{aligned}$$

1 MACLAURIN SERIES

1.1 Derivation of Maclaurin Series

Let $f(x)$ be a function of x . Suppose $f(x)$ can be expanded as a power series of x ,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

Problem

To find $a_0, a_1, a_2, a_3, \dots, a_n, \dots$ so that the power series expansion is completely determined.

Solution

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \Rightarrow f(0) = a_0$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots \Rightarrow f'(0) = a_1 \Rightarrow a_1 = f'(0)$$

$$f''(x) = 2a_2 + (2)(3)a_3x + (3)(4)a_4x^2 + \dots \Rightarrow f''(0) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2!}$$

$$f'''(x) = (2)(3)a_3 + (2)(3)(4)a_4x + \dots \Rightarrow f'''(0) = (2)(3)a_3 \Rightarrow a_3 = \frac{f'''(0)}{3!}$$

$$f^{(4)}(x) = (2)(3)(4)a_4 + \dots \Rightarrow f^{(4)}(0) = (2)(3)(4)a_4 \Rightarrow a_4 = \frac{f^{(4)}(0)}{4!}$$

Hence, the **Maclaurin series of $f(x)$** is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(r)}(0)}{r!}x^r + \dots$$

Note

The general formula for the Maclaurin series of $f(x)$ is given in MF27 as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

Example 1

Find the Maclaurin series expansion of the following functions up to and including the term in x^4 .

(a) $\ln(1+x)$ (b) $\cos x$

Solution

(a) Let $f(x) = \ln(1+x)$ $\Rightarrow$

Using Maclaurin series formula above:

$$\begin{aligned}
 f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\
 &= 0 + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-2(3)}{4!}x^4 + \dots \\
 &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots
 \end{aligned}$$

(b) Let $f(x) = \cos x \quad \Rightarrow f(0) = \cos 0 = 1$
 $f'(x) = -\sin x \quad \Rightarrow f'(0) = -\sin 0 = 0$
 $f''(x) = -\cos x \quad \Rightarrow f''(0) = -\cos 0 = -1$
 $f'''(x) = \sin x \quad \Rightarrow f'''(0) = \sin 0 = 0$
 $f^{(4)}(x) = \cos x \quad \Rightarrow f^{(4)}(0) = \cos 0 = 1$

Using Maclaurin series formula above:

$$\begin{aligned}
 f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\
 &= 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \dots \\
 &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots
 \end{aligned}$$

■

The Maclaurin series of a function, when it converges, gives an equivalent form of the function. In practice, the first few terms of a Maclaurin series provide a sufficiently accurate approximation of the function. How does the number of terms in the Maclaurin series affect the level of accuracy of approximation? Does the approximation work for all values of x ?



Learning Experience Activity

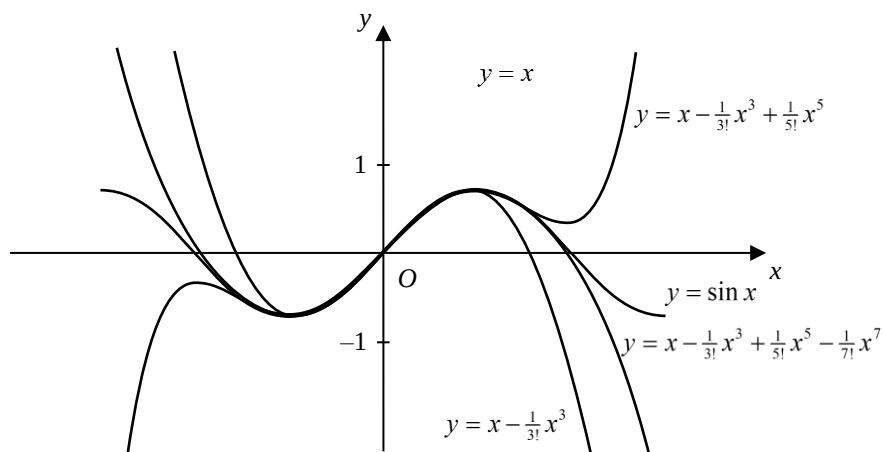
Access the GeoGebra worksheet at <http://ggbtu.be/mez563VJB>

Example

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Using GC, compare the graph of $y = \sin x$ with the graphs of the following equations.

- $y = x$
- $y = x - \frac{x^3}{3!}$
- $y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$
- $y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

**Observations**

1. The graphs of the Maclaurin series (for $\sin x$) approximate the graph of $y = \sin x$. This approximation gets better as the number of terms in the Maclaurin series increases.
2. The graphs are close to that of $y = \sin x$ when x is near 0. Thus, Maclaurin series can be used to approximate a function in the vicinity of $x = 0$.

Learning points

In general, there are two ways to improve the approximation of a function using Maclaurin series:

- As the number of terms in a Maclaurin series increases, the level of accuracy of the approximation increases.
- For a given Maclaurin series of a function, the approximation to the function gets better when the **x value is closer to 0**.

1.2 Some Standard Series Expansions

Note that the following expansions are standard expansions in MF27 under “Maclaurin expansion”:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots \quad \text{for all values of } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \cdots \quad \text{for all values of } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \cdots \quad \text{for all values of } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{r+1} \frac{x^r}{r} + \cdots \quad \text{for } -1 < x \leq 1$$

Using the appropriate standard expansions and substitutions for x , we can write down the following expansions.

$$\begin{aligned} e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots + \frac{(-x)^r}{r!} + \cdots \\ &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^r x^r}{r!} + \cdots \quad \text{for all values of } x \end{aligned}$$

$$\begin{aligned} \ln(1-x^2) &= (-x^2) - \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3} - \frac{(-x^2)^4}{4} + \cdots + (-1)^{r+1} \frac{(-x^2)^r}{r} + \cdots \\ &= -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \cdots - \frac{x^{2r}}{r} - \cdots \end{aligned}$$

$$\text{for } -1 < -x^2 \leq 1 \Rightarrow 0 \leq x^2 < 1 \Rightarrow -1 < x < 1$$

$$\begin{aligned} \ln(2+x) &= \ln\left(2\left(1+\frac{x}{2}\right)\right) = \ln 2 + \ln\left(1+\frac{x}{2}\right) \\ &= \ln 2 + \left(\frac{x}{2}\right) - \frac{\left(\frac{x}{2}\right)^2}{2} + \frac{\left(\frac{x}{2}\right)^3}{3} - \frac{\left(\frac{x}{2}\right)^4}{4} + \cdots \\ &= \ln 2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} - \frac{x^4}{64} + \cdots \end{aligned}$$

$$\text{for } -1 < \frac{x}{2} \leq 1 \Rightarrow -2 < x \leq 2$$

Why not expand from “ $\ln(1+(1+x))$ ”?

Example 2

Using the standard results given in the List of Formulae (MF27), show that for small values of x , $(1+2x)e^{-x} + \ln(1+2x) \approx 1+3x - \frac{7}{2}x^2 + \frac{7}{2}x^3$ and state the range of validity.

Solution

$$(1+2x)e^{-x} + \ln(1+2x)$$

$$= 1 + (-x + 2x) + \left(\frac{x^2}{2} - 2x^2 \right) + \left(-\frac{x^3}{6} + x^3 \right) + 2x - 2x^2 + \frac{8}{3}x^3 + \dots$$

$$\approx 1 + 3x - \frac{7}{2}x^2 + \frac{7}{2}x^3 \text{ (shown)}$$

Range of validity: ■

Example 3

It is given that $y = \sin(\ln(1+x))$.

- (i) Show that $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 0$.
- (ii) Find the Maclaurin series for y , up to and including the terms in x^3 .
- (iii) Verify that the same result is obtained if the standard series expansions for $\ln(1+x)$ and x^3 are used.

Solution

$$(i) \quad y = \sin(\ln(1+x)) \quad \text{---(1)}$$

Rearrange before
differentiating again
– use Product Rule
instead of Quotient
Rule

$$(1+x) \frac{dy}{dx} = \cos(\ln(1+x)) \quad \text{---(2)}$$

Differentiate with respect to x

$$(1+x) \frac{d^2y}{dx^2} + \frac{dy}{dx}(1) = -(y) \cdot \frac{1}{1+x}$$

$$(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 0 \text{ (shown)} \quad \text{---(3)}$$

(ii) Differentiate with respect to x

$$(1+x)^2 \frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} 2(1+x) + (1+x) \frac{d^2 y}{dx^2} + \frac{dy}{dx}(1) + \frac{dy}{dx} = 0$$

$$(1+x)^2 \frac{d^3 y}{dx^3} + (3+3x) \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = 0 \quad \text{---(4)}$$

When $x = 0$,

$$(1) \Rightarrow y = \sin(\ln(1+0)) = \sin 0 = 0$$

$$(2) \Rightarrow (1+0) \frac{dy}{dx} = \cos(\ln(1+0)) = \cos 0 = 1 \quad \Rightarrow \frac{dy}{dx} = 1$$

$$(3) \Rightarrow (1+0)^2 \frac{d^2 y}{dx^2} + (1+0)(1) + 0 = 0 \quad \Rightarrow \frac{d^2 y}{dx^2} = -1$$

$$(4) \Rightarrow (1+0)^2 \frac{d^3 y}{dx^3} + (3+0)(-1) + 2(1) = 0 \Rightarrow \frac{d^3 y}{dx^3} = 1$$

Using Maclaurin series,

$$\begin{aligned} y = f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &= 0 + (1)x + \frac{(-1)}{2!}x^2 + \frac{(1)}{3!}x^3 + \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{6} + \dots \end{aligned}$$

(iii) Using standard series expansions from MF27,

$$y = \sin(\ln(1+x))$$

$$\approx \sin\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)$$

$$\approx \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \frac{1}{3!}\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^3$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{1}{6}(x^3 + \dots)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} + \dots \text{(verified)}$$

Collate all the respective $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, $\frac{d^3 y}{dx^3}$ etc. terms first before further differentiation or calculation!

Use the expansion of $\ln(1+x)$ up to the term in x^3

Use the expansion of $\sin x$ up to the term in x^3 and replace x by $x - \frac{x^2}{2} + \frac{x^3}{3}$

LECTURE 2

Lesson Outline

- Maclaurin series: More examples
 - Binomial series: Partial fractions, formula for binomial series, some examples
-

Self-Practice Example 1

Given that $y = \ln(\sec x)$, show that

(i) $\frac{d^3 y}{dx^3} = 2 \frac{d^2 y}{dx^2} \frac{dy}{dx},$

(ii) the value of $\frac{d^4 y}{dx^4}$ when $x = 0$ is 2.

Write down the Maclaurin series for $\ln(\sec x)$ up to and including the term in x^4 .

By substituting $x = \frac{1}{4}\pi$, show that $\ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536}$. [9233 N2007/I/13]

Answer: (ii) $\frac{1}{2}x^2 + \frac{1}{12}x^4 + \dots$

Solution

(i) $y = \ln(\sec x) \quad \text{---(1)}$

Differentiate with respect to x :

Differentiate with respect to x :

Differentiate with respect to x :

$$\frac{d^3 y}{dx^3} = (2 \sec x)(\sec x \tan x) = 2 \sec^2 x \tan x$$

$$= 2 \frac{d^2 y}{dx^2} \frac{dy}{dx} \text{ (Shown) } \quad \text{---(4)}$$

(ii) Differentiate with respect to x :

$$\frac{d^4 y}{dx^4} = 2 \left(\frac{d^3 y}{dx^3} \frac{dy}{dx} + \left(\frac{d^2 y}{dx^2} \right)^2 \right) \quad \text{---(5)}$$

When $x = 0$,

$$(1) \Rightarrow y = \ln(\sec 0) = \ln 1 = 0$$

$$(2) \Rightarrow \frac{dy}{dx} = \tan 0 = 0$$

$$(3) \Rightarrow \frac{d^2 y}{dx^2} = \sec^2 0 = 1$$

$$(4) \Rightarrow \frac{d^3 y}{dx^3} = 2(1)(0) = 0$$

$$(5) \Rightarrow \frac{d^4 y}{dx^4} = 2((0)(0) + 1^2) = 2 \quad (\text{Shown})$$

The Maclaurin series for $\ln(\sec x)$ is

$$y = 0 + (0)x + \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{2}{4!}x^4 + \dots = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \dots$$

Substitute $x = \frac{\pi}{4}$ into Maclaurin series,

$$\ln\left(\sec \frac{\pi}{4}\right) \approx \frac{1}{2}\left(\frac{\pi}{4}\right)^2 + \frac{1}{12}\left(\frac{\pi}{4}\right)^4$$

$$\ln \sqrt{2} \approx \frac{\pi^2}{32} + \frac{\pi^4}{12(4^4)} \Rightarrow \ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536} \quad (\text{Shown}) \quad \blacksquare$$

Example 4

It is given that $y = \sqrt{1 + e^{\tan x}}$. Show that $2y \frac{dy}{dx} = (y^2 - 1)\sec^2 x$, and hence

show that when $x = 0$, $\frac{d^2 y}{dx^2} = \frac{3\sqrt{2}}{16}$. [5]

(i) Find the first three terms of the Maclaurin series for y , and [1]

(ii) the first two terms of the Maclaurin series for $\frac{dy}{dx}$, [2]

giving your answers in surd form. [MI/PU2/H2/I/7(modified)]

Solution

Square y before
differentiation as it
is tedious to
differentiate the
square root

$$y = \sqrt{1 + e^{\tan x}}$$

$$y^2 = 1 + e^{\tan x} \quad \text{---(1)}$$

Differentiate with respect to x :

Replace $e^{\tan x}$ with
 $y^2 - 1$

$$2y \frac{dy}{dx} = (y^2 - 1) \sec^2 x \text{ (Shown)} \quad \text{---(2)}$$

Differentiate with respect to x :

When $x = 0$,

$$(1) \Rightarrow y = \sqrt{2}$$

$$(2) \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$(3) \Rightarrow 2\sqrt{2} \frac{d^2y}{dx^2} + 2 \left(\frac{1}{2\sqrt{2}} \right)^2 = 0 + 2\sqrt{2} \left(\frac{1}{2\sqrt{2}} \right) \quad (1)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{2}} \left(1 - \frac{2}{(4)(2)} \right) \left(\frac{\sqrt{2}}{\sqrt{2}} \right) = \frac{3\sqrt{2}}{16} \text{ (Shown)}$$

(i) Using Maclaurin series,

$$\begin{aligned} y &= \sqrt{2} + \frac{\sqrt{2}}{4}x + \frac{3\sqrt{2}}{16} \frac{x^2}{2!} + \dots \\ &= \sqrt{2} + \frac{\sqrt{2}}{4}x + \frac{3\sqrt{2}}{32}x^2 + \dots \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \frac{dy}{dx} &\approx \frac{d}{dx} \left(\sqrt{2} + \frac{\sqrt{2}}{4}x + \frac{3\sqrt{2}}{32}x^2 \right) \\ &= \frac{\sqrt{2}}{4} + \frac{3\sqrt{2}}{32}(2x) = \frac{\sqrt{2}}{16}(4 + 3x) \end{aligned}$$

Note

$$\frac{d}{dx}(y^2) \neq 2y, \text{ it should be } \frac{d}{dx}(y^2) = 2y \frac{dy}{dx};$$

$$\text{and } \frac{d}{dx} \left(\left(\frac{dy}{dx} \right)^2 \right) \neq 2 \frac{dy}{dx}, \text{ it should be } \frac{d}{dx} \left(\left(\frac{dy}{dx} \right)^2 \right) = 2 \frac{dy}{dx} \frac{d^2y}{dx^2}.$$

Example 5

It is given that $\frac{dy}{dx} = y + 2x$ and $y = 1$ when $x = 0$. Show that $\frac{d^2y}{dx^2} = 3$ when $x = 0$. Find the first three terms in the Maclaurin series for y .

Solution

$$\frac{dy}{dx} = y + 2x$$

Differentiate with respect to x

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + 2$$

When $x = 0$,

$$y = 1 \text{ (given)}$$

$$\frac{dy}{dx} = 1 + 2(0) = 1$$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy}{dx} + 2 = 1 + 2 = 3 \text{ (Shown)}$$

Using Maclaurin series,

$$\begin{aligned} y &= 1 + \frac{1}{1!}x + \frac{3}{2!}x^2 + \cdots \\ &= 1 + x + \frac{3}{2}x^2 + \cdots \end{aligned}$$

■

Example 6

If $\sin y = \frac{1}{2} \cos x$, prove that $\frac{d^2y}{dx^2} = \tan y \left(\left(\frac{dy}{dx} \right)^2 - 1 \right)$.

If $-\frac{\pi}{2} < y < \frac{\pi}{2}$, obtain the expansion of y in ascending powers of x as far as and including the term in x^2 , and show that the coefficient of x^3 is zero.

Solution

$$\sin y = \frac{1}{2} \cos x$$

Differentiate with respect to x ,

$$\cos y \frac{dy}{dx} = -\frac{1}{2} \sin x \quad \text{---(1)}$$

Differentiate with respect to x ,

$$\cos y \frac{d^2 y}{dx^2} - \sin y \left(\frac{dy}{dx} \right)^2 = -\sin y$$

$$\frac{d^2 y}{dx^2} - \tan y \left(\frac{dy}{dx} \right)^2 = -\tan y$$

$$\frac{d^2 y}{dx^2} = \tan y \left(\left(\frac{dy}{dx} \right)^2 - 1 \right) \text{ (Proven)} \quad \text{--- (2)}$$

When $x = 0$, $\sin y = \frac{1}{2} \cos 0 = \frac{1}{2} \Rightarrow y = \frac{\pi}{6}$

When $x = 0$ and $y = \frac{\pi}{6}$, substitute into (1):

$$\left(\cos \frac{\pi}{6} \right) \frac{dy}{dx} = -\frac{1}{2} \sin 0 \Rightarrow \frac{dy}{dx} = 0$$

Substitute into (2):

$$\frac{d^2 y}{dx^2} = \left(\tan \frac{\pi}{6} \right) (0^2 - 1) \Rightarrow \frac{d^2 y}{dx^2} = -\frac{1}{\sqrt{3}}$$

Using Maclaurin series,

$$\begin{aligned} y &= \frac{\pi}{6} + (0)x + \left(-\frac{1}{\sqrt{3}} \right) \frac{x^2}{2!} + \dots \\ &= \frac{\pi}{6} - \frac{1}{2\sqrt{3}} x^2 + \dots \end{aligned}$$

To show coefficient of x^3 is zero, differentiate (2) with respect to x :

$$\frac{d^3 y}{dx^3} = (\tan y) \left(2 \left(\frac{dy}{dx} \right) \frac{d^2 y}{dx^2} \right) + \left(\left(\frac{dy}{dx} \right)^2 - 1 \right) \sec^2 y \frac{dy}{dx}$$

When $x = 0$,

$$\frac{d^3 y}{dx^3} = \left(\tan \frac{\pi}{6} \right) \left(2(0) \left(-\frac{1}{\sqrt{3}} \right) \right) + \left((0)^2 - 1 \right) \left(\sec^2 \frac{\pi}{6} \right) (0) = 0$$

Hence coefficient of x^3 in the Maclaurin series is $\frac{f^{(3)}(0)}{3!} = 0$. ■

2 BINOMIAL SERIES

2.1 Recap: Partial Fractions

Consider a function, say, $f(x) = \frac{2}{x+3} + \frac{1}{x-2}$. $f(x)$ may be expressed as a single fraction as:

$$f(x) = \frac{2}{x+3} + \frac{1}{x-2} = \frac{2(x-2) + (x+3)}{(x+3)(x-2)} = \frac{3x-1}{(x+3)(x-2)} = \frac{3x-1}{x^2+x-6}.$$

It is often useful to be able to reverse the operation. In other words, if given a function such as $f(x) = \frac{3x-1}{x^2+x-6}$, we would want to be able to express $f(x)$ as a sum of two (or more in some cases) separate fractions. This process is called **expressing $f(x)$ in partial fractions**.

The **general form of the partial fractions** for the various types are as follows and can be found in MF27:

(a) Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

(b) Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

(c) Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$

Refer to Annex A for the procedure to resolve a rational function $\frac{f(x)}{g(x)}$ into partial fractions.

2.2 Binomial Expansion for Positive Integer Indices

The binomial expansion with a positive integer index is as follows and is found in MF27:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Notes

- The expansion has $n+1$ terms.
- $r! = r(r-1)(r-2)\dots(3)(2)(1)$.
- The $(r+1)^{\text{th}}$ term is $\binom{n}{r}a^{n-r}b^r$.
- The sum of powers of a and b in each term is n .

2.3 Binomial Series for Rational Indices that are not Positive Integers

If n is not a positive integer, we can expand $(1+x)^n$ as a series of ascending powers of x as follows:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$$

This formula is found in MF27 under Maclaurin expansion.

This expansion is **valid** for $-1 < x < 1$, that is, the series will **converge** for $-1 < x < 1$ or $|x| < 1$.

Notes

- This is an infinite series.
- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots, \quad -1 < x < 1.$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots, \quad -1 < x < 1.$

To expand $(a+x)^n$, where a is a constant:

If x is **small**, the expansion is in **ascending powers** of x . Therefore,

$$\begin{aligned}(a+x)^n &= \left(a \left(1 + \frac{x}{a} \right) \right)^n = a^n \left(1 + \frac{x}{a} \right)^n \\ &= a \left(1 + n \frac{x}{a} + \frac{n(n-1)}{2!} \left(\frac{x}{a} \right)^2 + \dots \right)\end{aligned}$$

The expansion is valid for $\left| \frac{x}{a} \right| < 1 \Rightarrow |x| < |a|$.

Recall that for $k \in \mathbb{R}^+$

(a) $|x| < a \Leftrightarrow -a < x < a$

(b) $|x| > a \Leftrightarrow x > a \text{ or } x < -a$

If x is **large**, then $\frac{a}{x}$ is **small**, and the expansion is in **descending powers** of x . Therefore,

$$\begin{aligned}(a+x)^n &= \left(x \left(\frac{a}{x} + 1 \right) \right)^n = x^n \left(1 + \frac{a}{x} \right)^n \\ &= x^n \left(1 + n \frac{a}{x} + \frac{n(n-1)}{2!} \left(\frac{a}{x} \right)^2 + \dots \right) \\ &= x^n + nax^{n-1} + \frac{n(n-1)a^2}{2} x^{n-2} + \dots\end{aligned}$$

The expansion is valid for $\left| \frac{a}{x} \right| < 1 \Rightarrow |x| > |a|$.

Recall that for $k \in \mathbb{R}^+$

(a) $|x| < a \Leftrightarrow -a < x < a$

(b) $|x| > a \Leftrightarrow x > a \text{ or } x < -a$

Example 7

Expand each of the following as a series of ascending powers of x up to and including the term in x^3 , stating the range of values of x for which each expansion is valid.

(a) $\frac{1}{\sqrt[3]{8-3x}}$

(b) $\left(\frac{1+2x}{1-x} \right)^2$

Solution

(a) $\frac{1}{\sqrt[3]{8-3x}} = (8-3x)^{-\frac{1}{3}}$

$$= \left(8 \left(1 - \frac{3}{8}x \right) \right)^{-\frac{1}{3}} = 8^{-\frac{1}{3}} \left(1 - \frac{3}{8}x \right)^{-\frac{1}{3}}$$

$$\begin{aligned}
&= \frac{1}{2} \left(1 + \frac{1}{8}x + \frac{1}{32}x^2 + \frac{7}{768}x^3 + \dots \right) \\
&= \frac{1}{2} + \frac{1}{16}x + \frac{1}{64}x^2 + \frac{7}{1536}x^3 + \dots
\end{aligned}$$

The expansion is valid for

(b) $\left(\frac{1+2x}{1-x} \right)^2 =$

$$\begin{aligned}
&= (1+4x+4x^2)(1+2x+3x^2+4x^3+\dots) \\
&= (1+4x+4x^2) + (2x+8x^2+8x^3) + (3x^2+12x^3) + 4x^3 + \dots \\
&= 1+6x+15x^2+24x^3+\dots
\end{aligned}$$

The expansion is valid for $|-x| < 1 \Rightarrow -1 < x < 1$. ■

Self-Practice Example 2

Expand $\frac{\sqrt{1+x}}{1-2x}$ as a series of ascending powers of x up to and including the term in x^3 , and state the range of values of x for which the expansion is valid.

$$\text{Answer: } 1 + \frac{5}{2}x + \frac{39}{8}x^2 + \frac{157}{16}x^3 + \dots; -\frac{1}{2} < x < \frac{1}{2}$$

Solution

$$\frac{\sqrt{1+x}}{1-2x} = (1+x)^{\frac{1}{2}}(1-2x)^{-1}$$

$$\begin{aligned} &= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots\right) \left(1 + 2x + 4x^2 + 8x^3 + \dots\right) \\ &= \left(1 + 2x + 4x^2 + 8x^3\right) + \left(\frac{1}{2}x + x^2 + 2x^3\right) + \left(-\frac{1}{8}x^2 - \frac{1}{4}x^3\right) + \frac{1}{16}x^3 + \dots \\ &= 1 + \frac{5}{2}x + \frac{39}{8}x^2 + \frac{157}{16}x^3 + \dots \end{aligned}$$

The expansion is valid for $|x| < 1$ **and** $|-2x| < 1$

$$\Rightarrow -1 < x < 1 \text{ **and** } -\frac{1}{2} < x < \frac{1}{2}$$

Therefore, the expansion is valid for $-\frac{1}{2} < x < \frac{1}{2}$.

■

LECTURE 3

Lesson Outline

- Binomial series: Further examples & applications
- Small angle approximations

Example 8

Expand $(1+x)^{\frac{1}{4}}$ in ascending powers of x up to and including the term in x^2 . By substituting a suitable value for x , prove that $\sqrt[4]{17} \approx \frac{8317}{4096}$.

Solution

$$(1+x)^{\frac{1}{4}} = 1 + \frac{1}{4}x + \frac{\frac{1}{4}(-\frac{3}{4})}{2!}x^2 + \dots = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \dots$$

Expansion is valid for $|x| < 1 \Rightarrow -1 < x < 1$

In making approximations, the value chosen to be substituted for x must fall within the range of validity of x .

$$\sqrt[4]{17} \approx 2 \left(1 + \frac{1}{4} \left(\frac{1}{16} \right) - \frac{3}{32} \left(\frac{1}{16} \right)^2 \right) = \frac{8317}{4096} \text{ (shown)} \quad \blacksquare$$

Example 9

Given that $x > 1$, write down the first 4 non-zero terms in the expansion of $\sqrt{1+x^2}$ in descending powers of x and hence obtain an appropriate value of $\sqrt{26}$, correct to 4 decimal places.

Solution

$$\begin{aligned} \sqrt{1+x^2} &= \left(x^2 \left(\frac{1}{x^2} + 1 \right) \right)^{\frac{1}{2}} = x \left(1 + \frac{1}{x^2} \right)^{\frac{1}{2}} \\ &= x \left(1 + \frac{1}{2} \left(\frac{1}{x^2} \right) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!} \left(\frac{1}{x^2} \right)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!} \left(\frac{1}{x^2} \right)^3 + \dots \right) \\ &= x + \frac{1}{2x} - \frac{1}{8x^3} + \frac{1}{16x^5} + \dots \end{aligned}$$

Expansion valid for $\left| \frac{1}{x^2} \right| < 1 \Rightarrow x < -1 \text{ or } x > 1$

Hence, $\sqrt{26} = \sqrt{1+5^2} \approx 5 + \frac{1}{2(5)} - \frac{1}{8(5)^3} + \frac{1}{16(5)^5} \approx 5.0990$ (to 4 d.p.) ■

Example 10

Express $f(x) = \frac{2x+4}{(2x-1)(x+3)}$ in partial fractions.

- (a) If x is small, obtain an expression of $f(x)$ in ascending powers of x , giving the first three non-zero terms, and find the coefficient of x^r , where $r = 0, 1, 2, \dots$
- (b) If x is large, obtain an expression of $f(x)$ in descending powers of x , giving the first three non-zero terms, and find the coefficient of x^{-7} .

State in each case the set of values of x for which the expansion is valid.

Solution

$$f(x) = \frac{2x+4}{(2x-1)(x+3)} \equiv \frac{A}{2x-1} + \frac{B}{x+3} \equiv \frac{A(x+3) + B(2x-1)}{(2x-1)(x+3)}$$

By comparing the numerator,

$$2x+4 \equiv A(x+3) + B(2x-1).$$

By substituting $x = -3$,

$$2(-3)+4 = A((-3)+3) + B(2(-3)-1) \Rightarrow B = \frac{2}{7}$$

By substituting $x = \frac{1}{2}$,

$$2\left(\frac{1}{2}\right)+4 = A\left(\left(\frac{1}{2}\right)+3\right) + B\left(2\left(\frac{1}{2}\right)-1\right) \Rightarrow A = \frac{10}{7}$$

Therefore, $f(x) = \frac{10}{7(2x-1)} + \frac{2}{7(x+3)}$.

$$\begin{aligned} \text{(a)} \quad f(x) &= \frac{10}{7(2x-1)} + \frac{2}{7(x+3)} \\ &= \frac{10}{7}(2x-1)^{-1} + \frac{2}{7}(x+3)^{-1} \\ &= \frac{10}{7}(-1)^{-1}(1-2x)^{-1} + \frac{2}{7}(3)^{-1}\left(1+\frac{x}{3}\right)^{-1} \\ &= -\frac{10}{7}\left(1+2x+4x^2+\dots\right) + \frac{2}{21}\left(1-\frac{x}{3}+\frac{x^2}{9}+\dots\right) \\ &= -\frac{4}{3} - \frac{26}{9}x - \frac{154}{27}x^2 + \dots \end{aligned}$$

$$\text{Coefficient of } x^r = -\frac{10}{7}(2^r) + \frac{2}{21}\left(-\frac{1}{3}\right)^r$$

$$\text{Range of validity: } |2x| < 1 \text{ and } \left|\frac{x}{3}\right| < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$$

$$\begin{aligned} \text{(b)} \quad f(x) &= \frac{10}{7(2x-1)} + \frac{2}{7(x+3)} \\ &= \frac{10}{7}(2x-1)^{-1} + \frac{2}{7}(x+3)^{-1} \\ &= \frac{10}{7}(2x)^{-1}\left(1-\frac{1}{2x}\right)^{-1} + \frac{2}{7}(x)^{-1}\left(1+\frac{3}{x}\right)^{-1} \\ &= \frac{10}{14x}\left(1+\frac{1}{2x}+\frac{1}{4x^2}+\dots\right) + \frac{2}{7x}\left(1-\frac{3}{x}+\frac{9}{x^2}+\dots\right) \\ &= \frac{1}{x} - \frac{1}{2x^2} + \frac{11}{4x^3} + \dots \end{aligned}$$

$$\text{Coefficient of } x^{-7} = \frac{10}{14}\left(\frac{1}{2}\right)^6 + \frac{2}{7}(-3)^6 = \frac{13331}{64}$$

$$\text{Range of validity: } \left|\frac{1}{2x}\right| < 1 \text{ and } \left|\frac{3}{x}\right| < 1 \Rightarrow x < -3 \text{ or } x > 3 \quad \blacksquare$$

2.4 Small Angle Approximation

If x is sufficiently small for x^3 and higher powers of x to be neglected and x is measured in radians, then from MF 27, we have

$$\sin x \approx x, \quad \cos x \approx 1 - \frac{1}{2}x^2$$

$$\text{Since } \tan x = \frac{\sin x}{\cos x},$$

$$\tan x \approx \frac{x}{1 - \frac{1}{2}x^2} = x\left(1 - \frac{1}{2}x^2\right)^{-1} = x\left(1 + \frac{1}{2}x^2 + \dots\right) \approx x$$

Therefore, for x is sufficiently small such that x^3 and higher powers of x can be neglected,

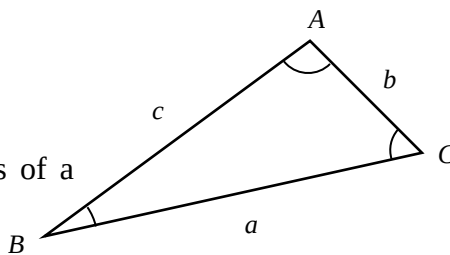
$$\sin x \approx x \qquad \cos x \approx 1 - \frac{1}{2}x^2 \qquad \tan x \approx x$$

'O' LEVEL RECAP

Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine Rule: $a^2 = b^2 + c^2 - 2bc \cos A$

where a, b, c are lengths of the sides of a triangle with opposite angles A, B, C respectively.

**Example 11**

Given that x is small, show that $\frac{1}{1 - \sin 2x} \approx 1 + 2x$. Hence, show that

$$\frac{17 - 16 \cos\left(\frac{x}{2}\right) - 5 \tan x}{1 - \sin 2x} \approx 1 - 3x.$$

[H2 Mathematics Vol 1]

Solution

$$\frac{1}{1 - \sin 2x} \approx \frac{1}{1 - 2x}$$

Using $\sin x \approx x$

Using binomial series to expand $(1 - 2x)^{-1}$ up to and including term in x

$$\approx 1 + 2x \text{ (shown)}$$

$$\frac{17 - 16 \cos\left(\frac{x}{2}\right) - 5 \tan x}{1 - \sin 2x}$$

$$= (1 - 5x + 2x^2) \cdot (1 + 2x)$$

$$\approx 1 + 2x - 5x + \dots$$

$$\approx 1 - 3x \text{ (shown)}$$

■

Example 12

In triangle ABC , angle BAC is $\frac{\pi}{6}$ radians, angle ABC is $\frac{\pi}{3} + x$ radians.

Given that x is sufficiently small, show using the sine rule that

$$\frac{BC}{AC} \approx \frac{1}{\sqrt{3}\left(1 - \frac{x^2}{2}\right) + x} \approx a + bx + cx^2,$$

where a , b and c are constants to be determined.

[4]

[2012 Prelim SAJC/P1/Q1]

Solution

Using sine rule,

$$\frac{BC}{\sin \frac{\pi}{6}} = \frac{AC}{\sin\left(\frac{\pi}{3} + x\right)}$$

$\sin\left(\frac{1}{3}\pi + x\right)$ is not
approximately
 $\frac{1}{3}\pi + x$

$$\frac{BC}{AC} = \frac{\sin \frac{\pi}{6}}{\sin \frac{\pi}{3} \cos x + \cos \frac{\pi}{3} \sin x}$$

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x}$$

$$\approx \frac{1}{\sqrt{3}\left(1 - \frac{x^2}{2}\right) + x}$$

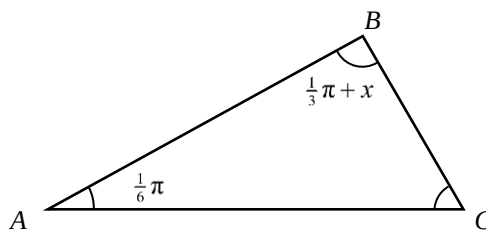
$$= \frac{1}{\sqrt{3}} \left(1 + \frac{x}{\sqrt{3}} - \frac{x^2}{2}\right)^{-1}$$

$$= \frac{1}{\sqrt{3}} \left(1 - \left(\frac{x}{\sqrt{3}} - \frac{x^2}{2}\right) + \left(\frac{x}{\sqrt{3}} - \frac{x^2}{2}\right)^2 + \dots\right)$$

$$\approx \frac{1}{\sqrt{3}} \left(1 - \frac{x}{\sqrt{3}} + \frac{5}{6}x^2\right)$$

$$= \frac{1}{\sqrt{3}} - \frac{1}{3}x + \frac{5}{6\sqrt{3}}x^2 \quad \text{or} \quad \frac{\sqrt{3}}{3} - \frac{1}{3}x + \frac{5\sqrt{3}}{18}x^2 \quad (\text{shown})$$

Hence, $a = \frac{\sqrt{3}}{3}$, $b = -\frac{1}{3}$, $c = \frac{5\sqrt{3}}{18}$. ■



Example 13

Given that $y = e^{\tan^{-1} x}$, where $\tan^{-1} x$ denotes the principal value, show that

$$(1+x^2) \frac{dy}{dx} = y.$$

By repeated differentiation of this result, obtain the Maclaurin's expansion of $e^{\tan^{-1} x}$ up to and including the term in x^3 . [5]

Hence, obtain an expansion for $\frac{e^{\tan^{-1} x}}{\sqrt{4-x^2}}$ up to and including the term in x^3 . [3]

[2011 Prelim NJC/II/Q1]

Solution

$$y = e^{\tan^{-1} x} \Rightarrow \ln y = \tan^{-1} x$$

Differentiating with respect to x ,

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$(1+x^2) \frac{dy}{dx} = y \quad \text{(shown)} \quad \text{---(1)}$$

Differentiating (1) with respect to x ,

$$(1+x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = \frac{dy}{dx}$$

$$(1+x^2) \frac{d^2 y}{dx^2} + (2x-1) \frac{dy}{dx} = 0 \quad \text{---(2)}$$

Differentiating (2) with respect to x ,

$$(1+x^2) \frac{d^3 y}{dx^3} + 2x \frac{d^2 y}{dx^2} + (2x-1) \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = 0$$

$$(1+x^2) \frac{d^3 y}{dx^3} + (4x-1) \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = 0 \quad \text{---(3)}$$

When $x = 0$, $y = e^{\tan^{-1} 0} = e^0 = 1$

$$(1) \Rightarrow \frac{dy}{dx} = 1$$

$$(2) \Rightarrow \frac{d^2 y}{dx^2} + (-1)(1) = 0 \Rightarrow \frac{d^2 y}{dx^2} = 1$$

$$(3) \Rightarrow \frac{d^3 y}{dx^3} + (-1)(1) + 2(1) = 0 \Rightarrow \frac{d^3 y}{dx^3} = -1$$

Therefore, $e^{\tan^{-1} x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots$

Rewrite $\frac{1}{\sqrt{4-x^2}}$ as
 $(4-x^2)^{-\frac{1}{2}}$, factorise
 out 4 and apply
 binomial expansion
 formula.

$$\begin{aligned}
 \frac{e^{\tan^{-1}x}}{\sqrt{4-x^2}} &= \left(e^{\tan^{-1}x}\right)(4-x^2)^{-\frac{1}{2}} \\
 &= 4^{-\frac{1}{2}} \left(e^{\tan^{-1}x}\right) \left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}} \\
 &= \frac{1}{2} \left(e^{\tan^{-1}x}\right) \left[1 + \left(-\frac{1}{2}\right) \left(-\frac{x^2}{4}\right) + \dots\right] \\
 &= \frac{1}{2} \left(1+x+\frac{1}{2}x^2-\frac{1}{6}x^3+\dots\right) \left(1+\frac{1}{8}x^2+\dots\right) \\
 &= \frac{1}{2} \left(1+\frac{1}{8}x^2+x+\frac{1}{8}x^3+\frac{1}{2}x^2-\frac{1}{6}x^3+\dots\right) \\
 &= \frac{1}{2} \left(1+x+\frac{5}{8}x^2-\frac{1}{24}x^3+\dots\right) \\
 &= \frac{1}{2} + \frac{1}{2}x + \frac{5}{16}x^2 - \frac{1}{48}x^3 + \dots \quad (\text{Ans})
 \end{aligned}$$

■

ANNEX A: PARTIAL FRACTIONS

Procedure for resolving a rational function $\frac{f(x)}{g(x)}$ into partial fractions

- Convert all improper rational functions into proper ones. $\frac{f(x)}{g(x)}$ is an **improper** rational function if the degree of the polynomial $f(x)$ (numerator) is **higher** than or **equal** to that of the polynomial $g(x)$ (denominator). e.g. $\frac{2x^2}{x^2 + x - 2} = 2 + \frac{4 - 2x}{(x-1)(x+2)}$
- Suppose $\frac{f(x)}{g(x)}$ is a proper rational function. Then, **factorise the denominator** $g(x)$ completely if it hasn't already been done.
- The **general form of the partial fractions** for the various types are as follows:

- (a) Non-repeated linear factors:

$$\frac{px + q}{(ax + b)(cx + d)} = \frac{A}{(ax + b)} + \frac{B}{(cx + d)}$$

- (b) Repeated linear factors:

$$\frac{px^2 + qx + r}{(ax + b)(cx + d)^2} = \frac{A}{(ax + b)} + \frac{B}{(cx + d)} + \frac{C}{(cx + d)^2}$$

- (c) Non-repeated quadratic factor:

$$\frac{px^2 + qx + r}{(ax + b)(x^2 + c^2)} = \frac{A}{(ax + b)} + \frac{Bx + C}{(x^2 + c^2)}$$

- Equate the numerators on both sides** to get an identity to solve for the unknown constants A , B and C .

Method 1: Compare coefficients of like terms on both sides of this equation.

Method 2: **Substitute suitable values of x** into this identity.

Method 3: Cover up rule.

Notes

- An **identity** is a statement which is true for **all** values of the unknown variables in the statement. E.g.

$$(x + y)^2 \equiv x^2 + 2xy + y^2 \text{ is true for all values of } x \text{ and } y.$$

- An **equation** is a statement which is true only for **some** values of the unknown variables in the statement. E.g.

$$x^2 - 4 = 0 \text{ is true when } x = 2 \text{ or } x = -2$$

ANNEX B: PRACTICE QUESTIONS**Practice Questions on Maclaurin Series**

- 1** Find, by repeated differentiation, the Maclaurin series, up to the term in x^3 , of
(i) $\sin 2x$ **(ii)** e^{-3x}
 Verify that your expansions are correct using the standard series provided in MF27.
- 2** Given that $y = \tan x$, show that $\frac{d^2y}{dx^2} = 2y \frac{dy}{dx}$. Hence, find the Maclaurin series for y , up to and including the term in x^3 .
- 3** Using the standard series provided in MF27, obtain the Maclaurin series of the following functions:
(i) $(2+x)^{-2}$ up to the term in x^3 ,
(ii) $\cos(3x^2)$ up to the term in x^8 ,
(iii) $\ln(2-2x)$ up to the term in x^3 .

Answers

- 1** **(i)** $2x - \frac{4}{3}x^3$ **(ii)** $1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3$
- 2** $x + \frac{1}{3}x^3$
- 3** **(i)** $\frac{1}{4}\left(1 - x + \frac{3}{4}x^2 - \frac{1}{2}x^3\right)$ **(ii)** $1 - \frac{9}{2}x^4 + \frac{27}{8}x^8$ **(iii)** $\ln 2 - x - \frac{1}{2}x^2 - \frac{1}{3}x^3$