Anglo-Chinese School

(Independent)

FINAL EXAMINATION 2017

YEAR 5 IB DIPLOMA PROGRAMME

MATHEMATICS

HIGHER LEVEL

Paper 2 Solutions with Marker's Comments

Qn	Solution	Marker's Comments
	Section A	
1)		[6 marks]
	$2^{x+3y}=\frac{1}{e}$ $\log_2\left(3x+y\right)=e$	
	$x+3y=-\log_2 e$ $3x+y=2^e$	
	Solving with GDC (Solving with Linear Equations Function and Log Function) or alternative $\mathrm{linSolve}\left(\begin{cases} x+3\cdot y=-\log_2(e) \\ 3\cdot x+y=2^{e} \end{cases},\{x,y\}\right)$	Most students used graphical method to solve.
	$\{2.64817,-1.36362\}$ Final answer $x=2.65, y=-1.36$	

Qn	Solution	Marker's Comments
2)		[7 marks]
	At the end of year 1 The total in bank account $S = 1000 \times 1.012 - p$	
	At the end of second year $S_2 = (1000 \times 1.012 - p)1.012 - p$ $= 1000 \times 1.012^2 - p(1 + 1.012)$	
	At the end of seven years $S_7 = 1000 \times 1.0012^7 - p\sum_{r=1}^{7} 1.012^{r-1}$	Many students were unable to find the general expression for S_7
	Using nsolve on GDC (or plotting graphs) $\text{nSolve}\left(1000\cdot(1.012)^7 - x\cdot \sum_{r=1}^{7}\left((1.012)^{r-1}\right)=439.52, x\right)$ 89.2319 $p = \$89.20$	
3)		[6 marks]
(a)	$z_2 = z_3 - z_1 = 4 + (4 + \sqrt{3})i - 3 - 4i$ $= 1 + i\sqrt{3}$	
(b)	$\lvert z_2 \rvert = \lvert 1 + i\sqrt{3} \rvert = 2$ $Arg(z_2) = 60°$	
	Observe that z_2^2 has argument $120°$ (giving bearing of $330°$) and a modulus of 4, giving a distance of 4 km from A	Very few students were able to make this observation.
	Hence the complex number representing B $z_B = z_1 + z_2^2 = 3 + 4i + (1 + i\sqrt{3})^2$ $= 3 + 4i + 1 + i2\sqrt{3} - 3$ $= 1 + (4 + 2\sqrt{3})i$ Coordinates of B is $(1, 4 + 2\sqrt{3})$	A majority of those who found the coordinates of B correctly did not use complex numbers to obtain the answer.
4)		[5 marks]
	Consider first number odd, no of ways $3 \times {}^4P_2 \times 3 = 108$	
	Consider first number even, no of ways $2 \times {}^4P_2 \times 2 = 48$	
	Total number of ways = $108 + 48 = 156$	Most students were able to do this question.

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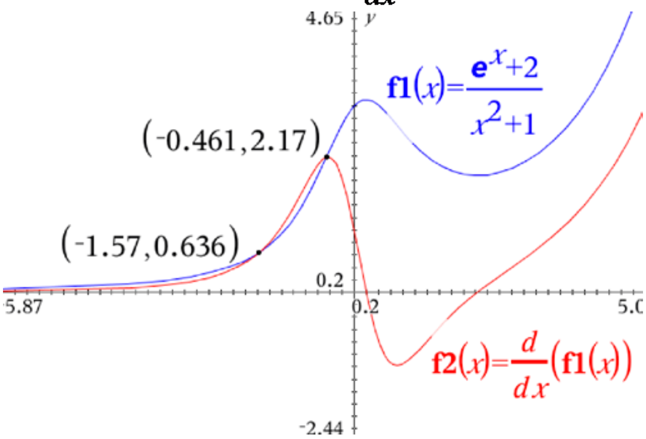
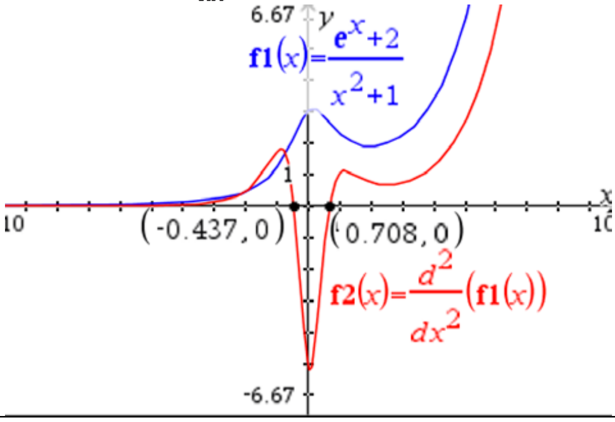
	Alternative method:	
	No. of 4-digit numbers ending with 0 = $^5P_3 \times 1 = 60$ No. of 4-digit numbers ending with 2 or 4 = $4 \times {}^4P_2 \times 2 = 96$ Total number of ways = 60 +96 = 156	
5)		[5 marks]
	For an even function $f(x) = f(-x)$	
	$f(-x) = \ln\left(\left(-x\right)^2 + 3\right) + \left(-x-1\right)^2 + a\left(-x\right)$ $= \ln\left(x^2+3\right) + \left(x+1\right)^2 - ax$	Generally well done. But a minority of students do not know the definition of an even function.
	$f(-x) = f(x)$	
	Equating , $\ln\left(x^2+3\right) + \left(x+1\right)^2 - ax = \ln\left(x^2+3\right) + \left(x-1\right)^2 + ax$ $2ax = \left(x+1\right)^2 - \left(x-1\right)^2$ $a = \dfrac{\left(x+1-x+1\right)\left(x+1+x-1\right)}{2x} = \dfrac{2(2x)}{2x} = 2$	
6)		[7 marks]
(a)	Use GDC to plot $f(x)$ and $\dfrac{df(x)}{dx}$ 	Many attempt to solve the problem manually, without using the GDC which can solve this problem graphically which is more efficient.
	$x \le -1.57$ or $x \ge -0.461$	

Qn	Solution	Marker's Comments
(b)	Plot graph of $\dfrac{d^2 f(x)}{dx^2}$ 	
	The graph is concave downwards where $\dfrac{d^2 f}{dx^2} < 0$ $-0.437 < x < 0.708$	Many students confuse point of inflexion with stationary points, wrongly finding the range by setting $f'(x) < 0$
7)		[7 marks]
(a)	$\mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$	
	A possible unit vector $\hat{\mathbf{n}} = \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$	Students did not follow instruction and did not find the unit vector and the plane equation in the requested form.
	Equation of plane π_1 $\mathbf{r} \cdot \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \sqrt{3}$	
	$\mathbf{r} \cdot \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \sqrt{3}$	
(b)	Possible equations of plane π_2 $\mathbf{r} \cdot \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 3\sqrt{3} + \sqrt{3} = 4\sqrt{3}$ or $\mathbf{r} \cdot \dfrac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \sqrt{3} - 3\sqrt{3} = -2\sqrt{3}$	Students not heeding the hence requirement and taking a much longer method to solve the problem.

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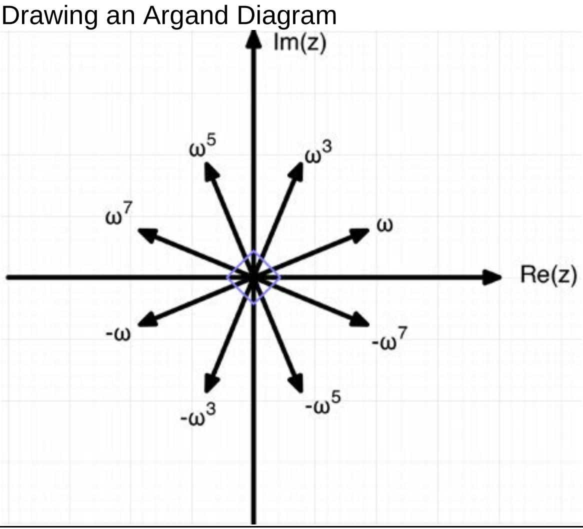
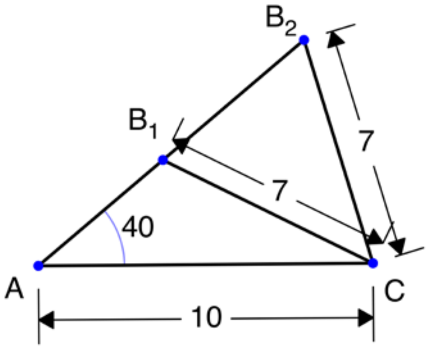
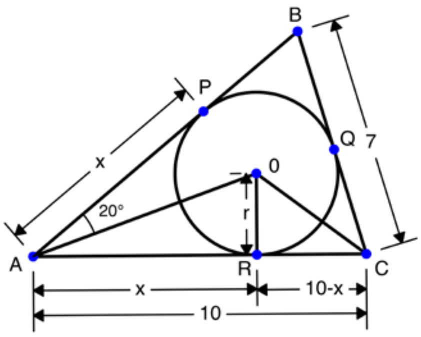
	Alternatively, $x-y+z=12$ or $x-y+z=-6$	
8)		[7 marks]
(a)	Perimeter of inner hexagon $=r\times 6=6r$	Many fail to see that the polygon is a hexagon and each triangle is equilateral and that means they can infer that the perimeter is 6r, and there is no need for lengthy proof.
	Perimeter of outer hexagon $=r\tan 30^\circ\times 2\times 6=\dfrac{12r}{\sqrt{3}}=4\sqrt{3}r$	Many gave very unnecessary long proof for this.
(b)	Perimeter of enclosed circle = $2\pi r$	
	$6r\le 2\pi r\le 4\sqrt{3}r$ Note: The objective is to find an estimate for π and the above represents the bound for the lower and upper bound for the perimeter of the circle	
	Hence $3\le\pi\le 2\sqrt{3}$ Note that the above simply mean that an estimate of π is between the given range.	
	Section B	
9)		[18 marks]
(a)	Using ti Nspire,	
	$\text{cPolyRoots}\left(z^3-3\cdot z^2+(3-i)\cdot z-2\cdot(1-i),z\right)$ $\{-i,1+i,2\}$	Many gave very unnecessary long working for this, and failed to use cpolyroots function on their Ti Nspire.
	The zeros are $z=-i$ or $z=1+i$ or $z=2$	
(b)(i)	$\sin\theta+i\cos\theta=i(\cos\theta-i\sin\theta)=iz^*$	
(ii)	$(\sin\theta+i\cos\theta)^3=\sin^3\theta+3\sin^2\theta(i\cos\theta)+3\sin\theta(i\cos\theta)^2+(i\cos\theta)^3$	Students use De Moivre's theorem on $(\sin\theta+i\cos\theta)^3$ Failure to make use of b(i) to find $\sin 3\theta$
	$(iz^*)^3=-i(\cos(-\theta)+i\sin(-\theta))^3=i(-\cos 3\theta+i\sin 3\theta)$	
	$=-\sin 3\theta-i\cos 3\theta$	This step was often left out
	Comparing coefficients,	
	$-\sin 3\theta=\sin^3\theta-3\sin\theta\cos^2\theta$	

Qn	Solution	Marker's Comments
	$=\sin^3\theta-3\sin\theta\left(1-\sin^2\theta\right)$	
	$\sin 3\theta=3\sin\theta-4\sin^3\theta$	
(c)(i)	$f(z)=0$ $z^8=-1=e^{i(\pi+2k\pi)}$	Students choose $-\pi$ instead of π which is a principal angle
	$z=e^{i\left(\frac{\pi}{8}+\frac{2k\pi}{8}\right)}$	
	Considering conjugate roots, or otherwise	
	$z=e^{\pm i\frac{\pi}{8}}$	
	$z=e^{\pm i\frac{3\pi}{8}}$	
	$z=e^{\pm i\frac{5\pi}{8}}$	
	$z=e^{\pm i\frac{5\pi}{8}}$	
(ii)	Drawing an Argand Diagram 	Many students drew the Argand Diagram wrongly as a series of points or simply a closed polygon.
(iii)		
	Noting that $\left(\omega,-\omega^7\right)$,$\left(\omega^3,-\omega^5\right)$, $\left(\omega^5,-\omega^3\right)$ and $\left(\omega^7,-\omega\right)$ are conjugate pairs and the sum of all distinct roots equal to zero (see Argand diagram)	Many variety of proofs including trigonometric identity types, which are less preferred given the "hence" nature of the question. Students should have observed from the Argand diagram that all 8 roots sum up to zero and they form four conjugate pairs.
	$\left(\omega-\omega^7\right)+\left(\omega^3-\omega^5\right)+\left(\omega^5-\omega^3\right)+\left(\omega^7-\omega^1\right)=0$	
	Hence using sum of conjugates, $2\cos\frac{\pi}{8}+2\cos\frac{3\pi}{8}+2\cos\frac{5\pi}{8}+2\cos\frac{7\pi}{8}=0$	
	$\cos\frac{\pi}{8}+\cos\frac{3\pi}{8}+\cos\frac{5\pi}{8}+\cos\frac{7\pi}{8}=0$	

Qn	Solution	Marker's Comments
10)		[17 marks]
(a)		
(i)	Sketch the possible triangles 	Students failed to conceive that the facts given about triangle ABC describe TWO possible triangles ABC, one larger than the other in terms of its area. This idea of two possible triangles is not at odds with the question, which many students thought to be in error because they can only find one. The problem here is a case of the ambiguous triangle that one may get employing sine rule.
	Using Sine Rule, $\frac{\sin \measuredangle ABC}{10} = \frac{\sin 40°}{7}$	
	Using Ti Nspire, $\sin \measuredangle ABC = 66.6742$ or 113.326	
	The two possible lengths of AB given by $AB = \frac{7 \times \sin(180 - 40 - 66.6742)}{\sin 40} = 10.4322 \approx 10.4$ m or $AB = \frac{7 \times \sin(180 - 40 - 113.326)}{\sin 40} = 4.88873 \approx 4.89$ m	
	Area of smaller triangle ABC = $\frac{1}{2} \times 10 \times 4.8883 \times \sin(40°) = 15.7121$ ≈ 15.7 m^2	
(ii)	Ratio of areas of smaller to larger triangle = $\frac{4.88873}{10.4322}$ 0.468619 Ratio $= 0.469$ (3sf)	The ratio can be easily found by considering the ratio of lengths found in part (i). There is no need to evaluate area of larger triangle.
(b)		
(i)		

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	$\dfrac{r}{x}=\tan 20°$ $x=\dfrac{r}{\tan 20°}$	Many students wrongly assume the following a) CQ=QB=3.5 b) Triangle ABC is isosceles with AB=AC=10 c) Points A, O and Q are collinear
	$\measuredangle ACB=180°-40°-66.6742°=73.3258°$	
	$\dfrac{r}{10-x}=\tan\left(\dfrac{73.3258°}{2}\right)$	
	Substituting $\dfrac{r}{10-\dfrac{r}{\tan 20°}}=\tan\dfrac{73.3258}{2}$	
	Using nsolve, $\text{nSolve}\left(\dfrac{x}{10-\dfrac{x}{\tan(20)}}=\tan\left(\dfrac{73.3258}{2}\right),x\right)$ 2.44445 $r=2.44$ m	
	Quicker Alternative Solution $\dfrac{1}{2}\times r\left(10.4322+10+7\right)=\dfrac{1}{2}\times 10.4322\times 10\times\sin 40$ $r\approx 2.44$	
(ii)	Length of AR is $\dfrac{2.44442923014}{\tan\left(20°\right)}=6.716014$ m	
	Area of bounded region $=2\times\dfrac{1}{2}\times 6.716014\times 2.44445-\dfrac{1}{2}\times 2.44445^2\times\left(\dfrac{140\times\pi}{180}\right)$ $=9.116711$ ≈ 9.12 Area = 9.12 m^2	Students who work on this manually tend to round off the values of r and AP early leading to inaccurate value of area.
11)		[15 marks]
(a)		
(i)	$\overrightarrow{OB}=\mathbf{a}+\mathbf{c}$ and $\overrightarrow{AC}=\mathbf{c}-\mathbf{a}$	
	$\overrightarrow{OB}\bullet\overrightarrow{AC}=\left(\mathbf{a}+\mathbf{c}\right)\bullet\left(\mathbf{c}-\mathbf{a}\right)$ $=\mathbf{a}\bullet\mathbf{c}-\lvert\mathbf{a}\rvert^2+\lvert\mathbf{c}\rvert^2-\mathbf{c}\bullet\mathbf{a}$ $=\mathbf{a}\bullet\mathbf{c}-1+1-\mathbf{a}\bullet\mathbf{c}$ $=0$	

Qn	Solution	Marker's Comments
	Hence OB and AC are perpendicular	
(ii)	ΔAOC is isosceles and OB is therefore the perpendicular bisector of AC, thus by similar triangles, OB bisects $\measuredangle AOC$	
(b)		
(i)	Consider any two position vectors on the line, say $\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}$	
	$\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 4$ and $\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = 8$ Line is in π_1	
	$\begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 4$ and $\begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = 8$ Line is in π_2	
	Hence the line is the line of intersection of both planes.	
(ii)	Vector normal to planes π_1 and π_2 is given by $\mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$	
	Equation of required plane perpendicular to planes π_1 and π_2 and passing through A(2,4,0) $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = 6$ $\mathbf{r} \cdot \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = 6$	
(iii)	From (a) we know that the difference of the unit vectors $\mathbf{a} + \mathbf{c}$ bisects the acute angle $\measuredangle AOC$	

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	Note also that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = 0$ hence π_1 and π_2 are perpendicular	
	Therefore the direction vector of line is given by $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 1 \\ 3/5 \end{pmatrix}$ or $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -4/5 \\ 4/5 \\ -3/5 \end{pmatrix}$	
	Vector equation of line is therefore given by $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 4/5 \\ 1 \\ 3/5 \end{pmatrix}$ or $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} -4/5 \\ 4/5 \\ -3/5 \end{pmatrix}$	