Anglo-Chinese School

(Independent)

FINAL EXAMINATION 2022

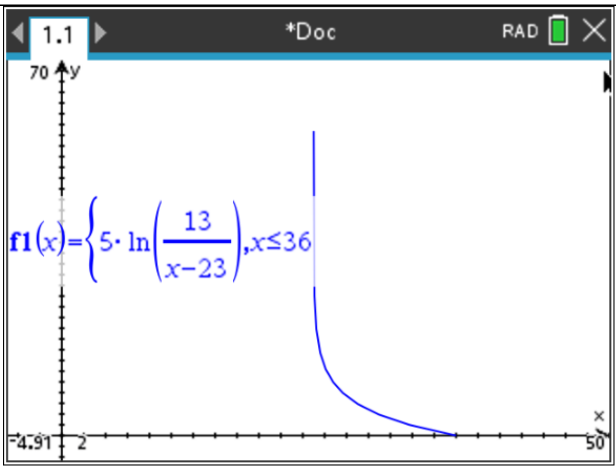
YEAR 5 IB DIPLOMA PROGRAMME

MATHEMATICS HIGHER LEVEL PAPER 2

ANALYSIS AND APPROACHES

SECTION A

Qn	Solution	Marks
1*.	In an infinite geometric sequence, $u_1 = a, u_2 = a^2 - 2$.	[6 marks]
1(a)	Given that $a = \frac{9}{5}$ show that the sum to infinity of the sequence exists.	Students found the sum to infinity, when what they are required to do is to find the common ratio r and show that -1<r<1. Common error to conclude that r < 1 for sum to infinity to exist.
	When $a = \frac{9}{5}, r = \frac{a^2-2}{a} = \frac{31}{45}$ Since $-1 < \frac{31}{45} < 1$, the sum to infinity exists	
1(b)	Find the least value of n such that $S_\infty - S_n < \frac{1}{100}$.	There is mostly error in computing least value of n and handling of equations involving inequalities.
	$S_\infty - S_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{ar^n}{1-r}$ $\frac{\frac{9}{5}\left(\frac{31}{45}\right)^n}{1-\frac{31}{45}} < \frac{1}{100}$ By GDC, $n > 17.1$ The least value of n is 18	

Qn	Solution	Marks
2*.	The time T hours taken for a machine to cool down to temperature x°C after it has been turned off can be modelled by $T = 5ln\left(\frac{13}{x-k}\right)$, where $T \geq 0$ and k is a positive constant. The machine has an initial temperature of 36°C when it is turned off.	[4 marks]
2(a)	Show that $k = 23$.	Done correctly by most students
	$0 = 5ln\left(\frac{13}{36-k}\right)$ $k = 23$ (Shown)	
2(b)	Sketch the graph of the machine's temperature for $T \geq 0$, clearly stating the coordinates of any axial intercept and the equation of any asymptote.	Sketching x against T which leads to wrong graph. Note that the equation given has T as the y-axis variable by convention from the way it is expressed.
	 x-intercept, x = 36 Vertical asymptote, x = 23	

Qn	Solution	Marks
3*.	On 1st January 2022, Sam invested \$600 in an account that pays a nominal annual interest rate of 1.2%, compounded monthly. The amount of money in Sam's account at the end of each year follows a geometric sequence with common ratio, r.	[6 marks]
3(a)	Find the value of r, giving your answer correct to 4 significant figures.	Generally well done
	$r = \left(1 + \frac{0.012}{12}\right)^{12}$ $= 1.012$	
3(b)	Sam makes no further deposits or withdrawals from the account. Find the year in which the amount of money in Sam's account becomes more than \$1000.	There are several ways of doing it, by

Qn	Solution	Marks
	$600 \times 1.012^n > 1000$ $n > \frac{\ln\left(\frac{1000}{600}\right)}{\ln(1.012)} = 42.8$ Year 2064	letting the general term of a GP to be greater than 1000. Common error is to confuse the term with the sum of a GP.
OR	GDC Financial Solver I(%) = 1.2 PV = -600 FV = 1000 PpY = 12 CpY = 12 N = 511. 08 months (number of times investment is compounded) = 42.59 years Year = 2064	

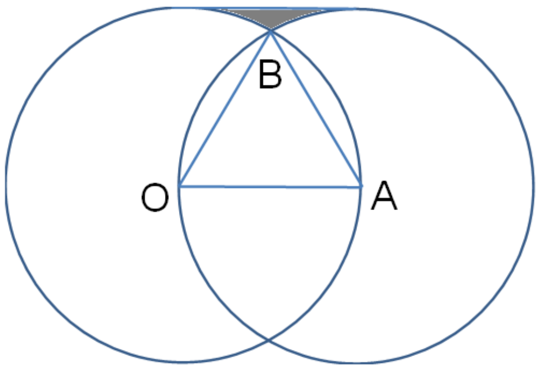
Qn	Solution	Marks
4*.	The functions f and g are defined by $f(x) = \frac{3-2x}{x+1}$, $x \in R, x \neq -1$ and $g(x) = x - a$, where $x \in R, a \in R$.	[5 marks]
4(a)	Find the range of f.	Generally well done. Common answer is to state the range as simply f(x) not equal to -2
	From GDC or otherwise, range of f $=]-\infty, -2[\ \cup\ [-2, \infty[$	
4(b)	Given that $(f \circ g)(x)$ has an asymptote at $x = -4$ for all $x \in R$, determine the value of a.	Generally well done.
	$f \circ g(x) = \frac{3-2(x-a)}{(x-a)+1}$ [composite function] $= \frac{3+2a-2x}{x-a+1}$ $x - a + 1 = 0$ [equate denominator = 0] $x = a - 1 = -4$ $a = -3$	

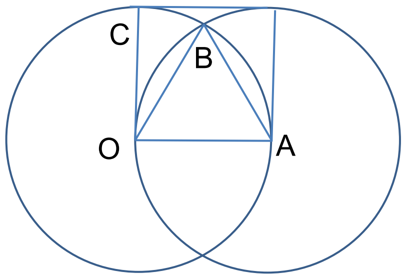
Qn	Solution	Marks
5.	Consider the quartic equation $z^4 - 3z^3 + az^2 + bz + c = 0$, $z \in C$. Two of the roots of this equation are 2 and $1 + 2i$. Find the possible values of a, b and c where $a, b, c \in R$.	[7 marks]

	Another root is $1-2i$. [since all real coefficients] $(z-1+2i)(z-1-2i)=z^2-2z+5$ [find quadratic root] Let $d \in R$ be the last unknown root $$z^4-3z^3+az^2+bz+c=(z-2)(z^2-2z+5)(z-d)$$ [M1: equate coefficients] Equating coefficient of z^3, $-3=-d-2-2, \quad d=-1$ Equating constant term, $c=10d=10$ Equating coefficient of z^2 $a=2d+5+2d+4=\ 5$ Equating coefficient of z $b=-10-5d-4d=-1$	Considerable effort in solving this part using factor/remainder theorem and leading to wrong answers because of careless mistakes.
OR	Another root is $1-2i$. [since all real coefficients] Let $d \in R$ be the last unknown root Sum of roots $=-\frac{-3}{1}$ [sum/ product of roots] $2+1+2i+1-2i+d=3$ $d=-1$ Product of roots $=(-1)^3c$ $2(1+2i)(1-2i)(-1)=-c$ $c=10$ Hence, $$z^4-3z^3+az^2+bz+10=(z-2)(z^2-2z+5)(z+1)$$ [M1: equate coefficients] Equating coefficient of z^2 $a=-2-2+4+5=\ 5$ Equating coefficient of z $b=-10+5+4=-1$	

Qn	Solution	Marks
6.	Find the roots of the equation $(w-2i)^3=8i, w \in C$. Give your answers in Cartesian form.	[7 marks]

Qn	Solution	Marks
	$(w-2i)^3 = 8i$ $(w-2i)^3 = 8\left(e^{\frac{\pi}{2}+2k\pi}\right), k \in Z$ $w-2i = 8^{\frac{1}{3}}\left(e^{\frac{\pi}{6}+\frac{2k\pi}{3}}\right)$ [simplify, $2k\pi$] $w-2i = \sqrt{3}+i, -\sqrt{3}+i, -2i$ [Answers in cartesian form] $w = \sqrt{3}+3i, -\sqrt{3}+3i, 0$	Possible to solve this using GDC using Cpolyroots as the answer required is not specified to be exact. Those who use analytic method and got it wrong failed to consider 2kpi to find other possible solutions.

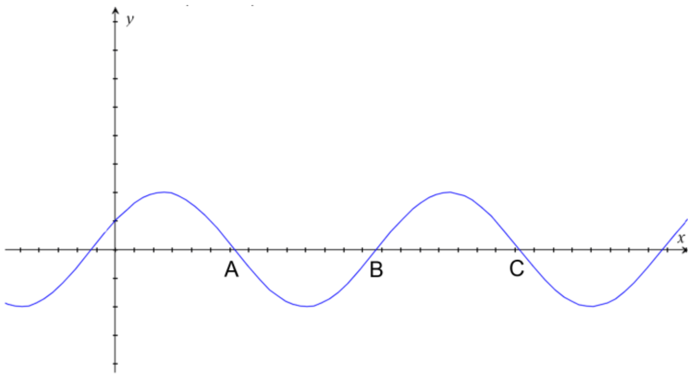
Qn	Solution	Marks
7.	The following diagram shows two circles of radius 1 unit, with centres at O and A respectively. The circles intersects at B.  	[7 marks]
7(a)	Find the value of angle AOB, giving your answers in radians.	Generally well done. Some forget to express answer in radians.
	OB = OA = AB = 1 since they are the radius of the circle. Angle AOB = $\frac{\pi}{3}$ (since OAB is an equilateral triangle) [correct angle]	
7(b)	Find the area of the shaded region. Give your answer correct to 3 significant figures.	

<table>
<tr><th>Qn</th><th>Solution</th><th>Marks</th></tr>
<tr><td></td><td>


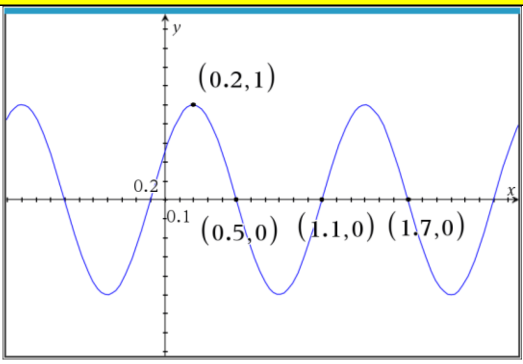
Area of shaded region = area of square – 2(area of sector OCB) – area of equilateral triangle
Angle COB = $\frac{\pi}{6}$
Area of square = 1 (sides of length = radius)
Area of equilateral triangle = $\frac{1}{2}\sin(BOA) = \frac{1}{2}\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}$
Area of sector OCB = $\frac{1}{2} \times 1^2 \times \frac{\pi}{6} = \frac{\pi}{12}$

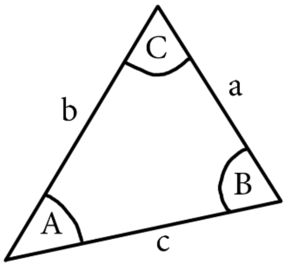
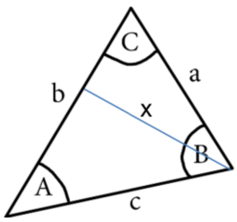
Area of shaded region = area of square – 2(area of sector OCB) – area of equilateral triangle
= $1 - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\, units^2$
= 0.0434 $units^2$
</td><td>Students who added the lines to form the square were able to solve the question easily.

Note: to use radians for angles in area of sector/ segment or arc length formulas.</td></tr>
</table>

<table>
<tr><th>Qn</th><th>Solution</th><th>Marks</th></tr>
<tr><td>8.</td><td>
Consider the graph of the function $f(x) = \cos(px - q), x \in R$ where p and q are positive constants.
The graph of f intersects the x-axis at point A, point B and point C. This is shown in the following diagram. The coordinates of B are $(1.1, 0)$ and the coordinates of C are $(1.7, 0)$.


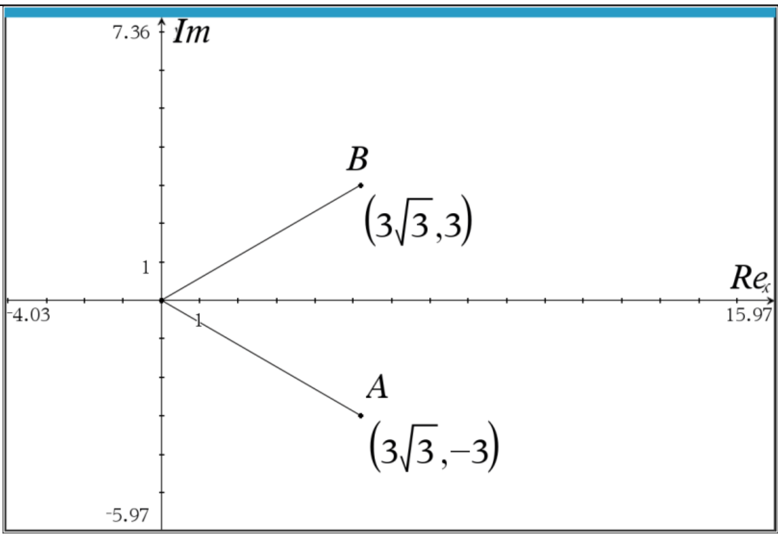
</td><td>[5 marks]</td></tr>
<tr><td>8(a)</td><td>Find the coordinates of A.</td><td></td></tr>
</table>

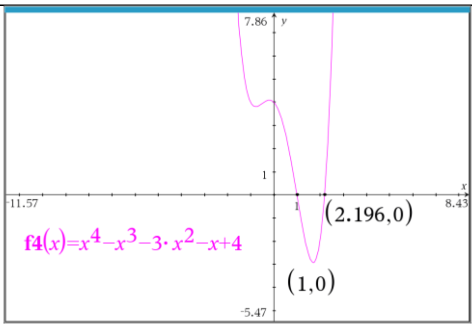
Qn	Solution	Marks
	 $A(0.5, 0)$	Generally well done.
8(b)	Find the value of p and the value of q.	
	Period = $2(1.7 - 1.1) = 1.2$ $\frac{2\pi}{p} = 1.2$ $p = \frac{\pi}{0.6} = \frac{5\pi}{3}$	Most could not recall formula: $\frac{2\pi}{p} = period$
8(c)	Find the smallest positive value of q.	
	$cosx$ undergoes a translation of 0.2 rad in the positive x-dir $q = \frac{5\pi}{3} \times 0.2 = \frac{\pi}{3}$ [scale the translation by p] **OR** At $(0.2, 1)$ $f(0.2) = \cos\left(\frac{5\pi}{3}(0.2) - q\right) = 1$ $\frac{5\pi}{3}(0.2) - q = cos^{-1}(1) = 0$ $q = \frac{5\pi}{3} \times 0.2 = \frac{\pi}{3}$ **OR** At $(0.5, 0)$ $f(0.5) = \cos\left(\frac{5\pi}{3}(0.5) - q\right) = 0$ $\frac{5\pi}{3}(0.5) - q = cos^{-1}(0) = \frac{\pi}{2}$ $q = \frac{5\pi}{3} \times 0.5 - \frac{\pi}{2} = \frac{\pi}{3}$	Most were not able to solve this question. Common mistake was to take the translated distance as the value of q.

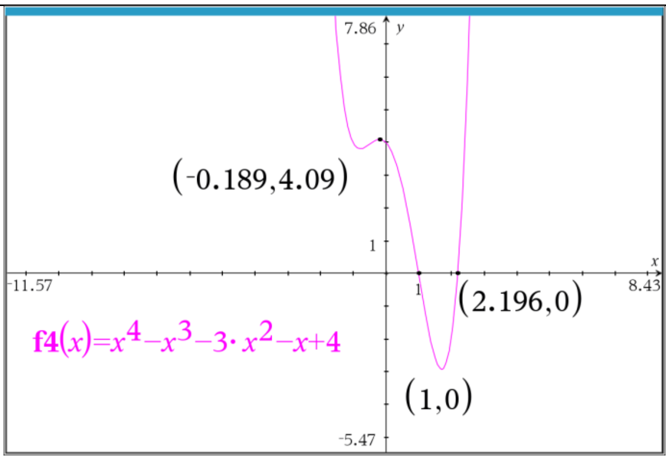
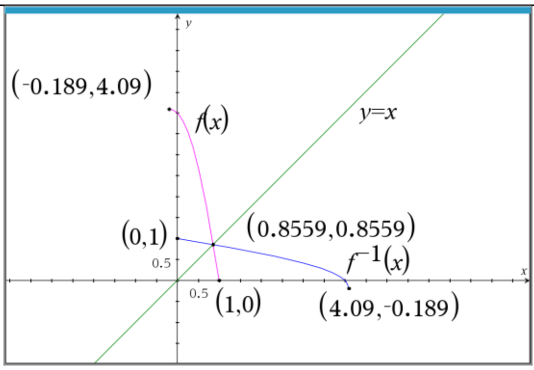
Qn	Solution	Marks
9.	A triangle is shown in the following diagram with interior angles A, B and C.  Show that $\frac{a}{c}sin2C + \frac{c}{a}sin2A = 2sin(\pi - B)$.	[8 marks]
9a.	Show that $\frac{a}{c} = \frac{sinA}{sinC}$.	
	By Sine rule, $\frac{a}{sinA} = \frac{c}{sinC}$ $\frac{a}{c}sinC = sinA$ $\frac{a}{c} = \frac{sinA}{sinC}$ (Shown) **OR**  Let the line perpendicular to b and passing through B be x. $sinA = \frac{opp}{hpy} = \frac{x}{c}$ $\quad x = c \times sinA$ $sinC = \frac{opp}{hpy} = \frac{x}{a}$ $\quad x = a \times sinC$ $\frac{a}{c} = \frac{sinA}{sinC}$	Well done by many.
9b.	Hence show that $\frac{a}{c}sin2C + \frac{c}{a}sin2A = 2sin(\pi - B)$.	
	$\frac{c}{a}sinA = sinC$ and $\frac{a}{c}sinC = sinA$ LHS $= \frac{a}{c}sin2C + \frac{c}{a}sin2A$ $= \frac{a}{c}2sinC \times cosC + \frac{c}{a}2sinA \times cosA$ [double angle formula] $= 2(sinAcosC + sinCcosA)$ [Simplify, Sub sinA and sinC] $= 2(\sin(A + C))$ [addition formula] $= 2(\sin(\pi - B))$ [sum of angles in a triangle] = RHS	Generally well done. Some student make careless mistakes when simplifying after substitution.

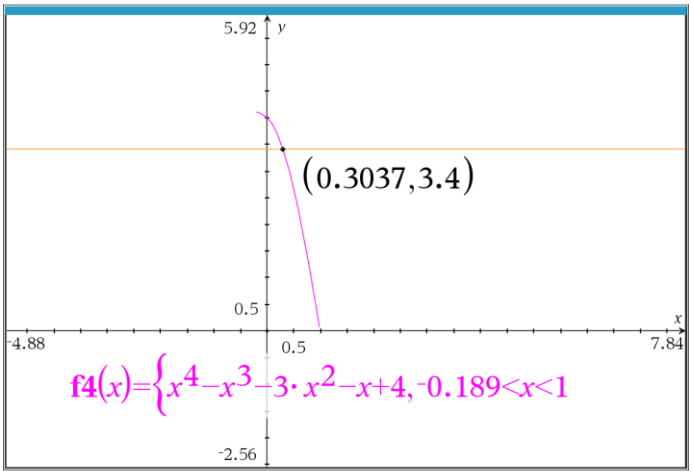
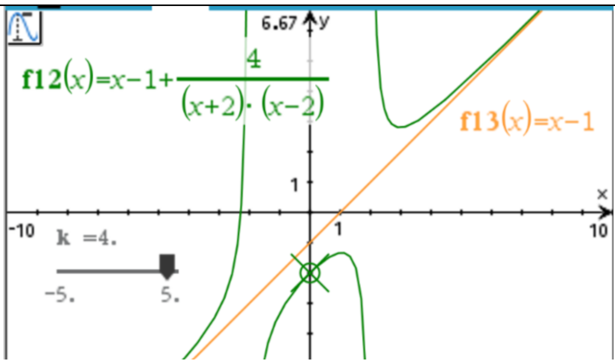
Section B

Qn	Solution	Marks
10	The complex number z is given by $z = a - 3i$, where a is a real number.	[16 marks]
10(a)	Find the possible values of a if $\frac{z^2}{z^*}$ is purely imaginary.	Some students did not rationalize the denominator and were unable to identify the correct real and imaginary parts of $\frac{z^2}{z^*}$.
	$\frac{z^2}{z^*} = \frac{(a-3i)^2}{a+3i} \times \frac{a-3i}{a-3i}$ [rationalize denominator] $= \frac{a(a^2-9)-18a}{a^2+9} - i\frac{3(a^2-9)+6a^2}{a^2+9}$ [Expand terms] if $\frac{z^2}{z^*}$ is purely imaginary, Re(z) = 0. $\frac{a(a^2-9)-18a}{a^2+9} = 0$ [Equate Re(z) = 0] $a(a^2-27) = 0$ [Equate numerator = 0] $a = 0, 3\sqrt{3}\ or -3\sqrt{3}$	
	For the rest of the question, it is further given that $a = 3\sqrt{3}$.	
10(bi)	Find the smallest integer value of n such that $\lvert z^n \rvert \geq 1000$.	Generally well done.
	$a = 3\sqrt{3}$ $\lvert 3\sqrt{3} - 3i \rvert^n \geq 1000$ $\left(\sqrt{(3\sqrt{3})^2 + 3^2}\right)^n \geq 1000$ $(6)^n \geq 1000$ $n \geq \frac{ln1000}{ln6} = 3.86$ Smallest $n = 4$	
10(bii)	On a single Argand diagram, plot and label the points A and B representing the complex numbers z and z^* respectively.	Generally well done. Points must be labeled A and B (as per question).
	Convert $3\sqrt{3} - 3i$ into A$(3\sqrt{3}, -3)$ $3\sqrt{3} + 3i$ into B$(3\sqrt{3}, 3)$	

	 [B is the reflection of A in the x-axis]	
10(biii)	The complex numbers z and z^* satisfies the equation $z^* = wz$. Using the Argand diagram in 10bii), find w in the form $re^{i\alpha}$ where $-\pi < \alpha \leq \pi$.	A number of students were unable to identify $\arg(z) = -\frac{\pi}{6}$ on the argand diagram. $\arg(z)$ is the angle between the positive x-axis and the line from the origin to the point representing z.
	$\arg(z) = tan^{-1}\left(\frac{-3}{3\sqrt{3}}\right) = tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ From the Argand diagram, z^* is an anticlockwise rotation of z by $2\arg(z)$ about the origin. Angle of rotation is $= 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$ $w = e^{i\frac{\pi}{3}}$	

	Solution	**Marks**
11.	The function f is defined as $f(x) = x^4 - x^3 - 3x^2 - x + 4$, $x \in R$.	[16 marks]
11(a)	Find the solutions of $f(x) > 0$. Give your answer correct to 3 significant figures.	Generally well done.
		

	x < 1 or x > 2.20	
11(b)	The domain of f is now restricted to $\in [a, 1]$.	
11b(i)	Find the smallest value of a for which f has an inverse. Give your answer correct to 3 significant figures.	Generally well done.
	 f is a one-to-one function if it has an inverse. Hence restrict f to the closest turning point. a = -0.189 [a is the maximum point]	
(ii)	For this value of a, sketch the curves $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes, clearly stating the coordinates of the end points of each curve and the coordinates of any point(s) of intersection of the curves.	A few students sketched $y = \frac{1}{f(x)}$ instead. Note that $f^{-1}(x) \neq \frac{1}{f(x)}$.
	 Intersection of $y = f(x)$ and $y = f^{-1}(x)$ is also the intersection of $y = f(x)$ and y. Intersection at (0.856, 0.856). End points for f : (-0.189, 4.09) and (1,0) End points for g : (4.09, -0.189) and (0, 1)	
(iii)	Find the value of $f^{-1}(3.4)$. Give your answer correct to 3 significant figures.	

<table>
<tr><td></td><td>Solving for $f(x) = 3.4, x = 0.304$

</td><td>Not well attempted. Not many used the graph in (ii) to solve.</td></tr>
<tr><td>11(c)</td><td>A function h has an oblique asymptote at $y = x - 1$, vertical asymptotes at $x = -2$ and $x = 2$, and passes through the point with coordinates (0,-2).
Find the equation of h in the form $a(x) + \frac{k}{b(x)}$, where $k \in R$ and $a(x), b(x)$ are functions to be determined.</td><td>Generally well done.</td></tr>
<tr><td></td><td>

$h(x) = x - 1 + \frac{k}{(x+2)(x-2)}$ [oblique and vertical asympts]

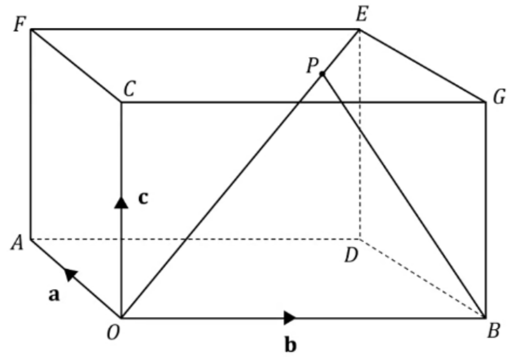
Sub coordinates (0,-2),

$0 = 2 - 1 + \frac{k}{(2+2)(2-2)}$ [Solve for k]

$k = 4$ [A1]

$h(x) = x - 1 + \frac{4}{(x+2)(x-2)}, x \neq 2, x \neq -2$</td><td></td></tr>
</table>

Qn	Solution	Marks

12	The following diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are shown on the diagram, where $\boldsymbol{a} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$. P is a point on the diagonal OE that divides OE in the ratio $3:1$.  **Diagram not drawn to scale.**	[23 marks]
12(a)	Find $\overrightarrow{OP}$ and $\overrightarrow{BP}$.	Generally well done. Some made careless mistakes in their calculations.
	$\overrightarrow{OP} = \frac{3}{3+1}\overrightarrow{OE}$ $= \frac{3}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) = \frac{3}{4}\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$ $\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = \frac{3}{4}\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ $= \begin{pmatrix} -1 \\ \frac{9}{4} \\ \frac{15}{4} \end{pmatrix}$	
12(b)	Write down a vector equation of the line that passes through B and P.	Generally well done.
	$L: \boldsymbol{r} = \boldsymbol{b} + k\overrightarrow{BP} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + k\begin{pmatrix} -1 \\ \frac{9}{4} \\ \frac{15}{4} \end{pmatrix}, k \in R$ OR $L: \boldsymbol{r} = \boldsymbol{b} + k\overrightarrow{BP} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + k\begin{pmatrix} -4 \\ 9 \\ 15 \end{pmatrix}, k \in R$	

OR	$L: \boldsymbol{r} = \overrightarrow{OP} + k\overrightarrow{BP} = \frac{3}{4}\begin{pmatrix}4\\3\\5\end{pmatrix} + k\begin{pmatrix}-1\\\frac{9}{4}\\\frac{15}{4}\end{pmatrix}, k \in R$ OR $L: \boldsymbol{r} = \boldsymbol{b} + k\overrightarrow{BP} = \frac{3}{4}\begin{pmatrix}4\\3\\5\end{pmatrix} + k\begin{pmatrix}-4\\9\\15\end{pmatrix}, k \in R$	
12©	Show that if the line segment BP is extended, it will intersect FE.	Most were able to show that the two lines intersect but not able to explain how the intersection point lie on the line segment FE.
	Let $L2$ be the vector equation of the line that passes through F and E. $L2: \boldsymbol{r} = \overrightarrow{OF} + h\overrightarrow{FE} = \begin{pmatrix}0\\3\\5\end{pmatrix} + h\begin{pmatrix}4\\0\\0\end{pmatrix}, h \in R$ $\begin{pmatrix}0\\3\\5\end{pmatrix} + h\begin{pmatrix}4\\0\\0\end{pmatrix} = \begin{pmatrix}4\\0\\0\end{pmatrix} + k\begin{pmatrix}-4\\9\\15\end{pmatrix}$ [Equate lines L and $L2$] $\begin{pmatrix}4h\\3\\5\end{pmatrix} = \begin{pmatrix}4-4k\\9k\\15k\end{pmatrix}$ $k = \frac{1}{3}$ $h = 1 - k = \frac{2}{3}$ The values of k and h are consistent, if the lines with segment BP and FE intersects. Since $0 < h < 1$ the line with segment BP will intersect line segment FE.	
12(d)	Hence show that FE is divided in the ratio $2:1$ by the point of intersection.	Some did not define the point of intersection clearly. Defining the point as X allows it to be referenced and used for calculation of vector/ length.
	Let the point of intersection be X. $\overrightarrow{OX} = \begin{pmatrix}0\\3\\5\end{pmatrix} + h\begin{pmatrix}4\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{8}{3}\\3\\5\end{pmatrix}$ [Using $L2$] $\overrightarrow{FX} = \begin{pmatrix}\frac{8}{3}\\3\\5\end{pmatrix} - \begin{pmatrix}0\\3\\5\end{pmatrix} = \begin{pmatrix}\frac{8}{3}\\0\\0\end{pmatrix}$ $\overrightarrow{XE} = \begin{pmatrix}4\\3\\5\end{pmatrix} - \begin{pmatrix}\frac{8}{3}\\3\\5\end{pmatrix} = \begin{pmatrix}\frac{4}{3}\\0\\0\end{pmatrix}$	

	Hence, $\overrightarrow{FX} = 2\overrightarrow{XE}$, FE is divided in the ratio $2:1$ by the point of intersection. **OR** $\frac{\lvert\overrightarrow{XE}\rvert}{\lvert\overrightarrow{FX}\rvert} = \frac{\left\lvert\begin{pmatrix}\frac{4}{3}\\0\\0\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}\frac{8}{3}\\0\\0\end{pmatrix}\right\rvert} = \frac{\frac{4}{3}}{\frac{8}{3}} = \frac{1}{2}$ $\lvert\overrightarrow{FX}\rvert : \lvert\overrightarrow{XE}\rvert = 2:1$	
12e)	Show that the area of the triangle OBP is $\frac{3}{8}\lvert\boldsymbol{b} \times (\boldsymbol{a} + \boldsymbol{c})\rvert$. Hence find the value of the area of triangle OBP, giving your answer correct to 3 significant figures.	Most omitted $\boldsymbol{b} \times \boldsymbol{b} = \boldsymbol{0}$ in their solution. Solving for the area using the coordinates was well done.
	Area of the triangle $OBP = \frac{1}{2}\left\lvert\overrightarrow{OB} \times \overrightarrow{OP}\right\rvert$ $= \frac{1}{2}\left\lvert\boldsymbol{b} \times \frac{3}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\right\rvert$ [Area formula] $= \frac{3}{8}\lvert\boldsymbol{b} \times (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})\rvert$ $= \frac{3}{8}\lvert\boldsymbol{b} \times \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{c}\rvert$ $[\boldsymbol{b} \times \boldsymbol{b} = \boldsymbol{0}]$ $= \frac{3}{8}\lvert\boldsymbol{b} \times (\boldsymbol{a} + \boldsymbol{c})\rvert$ [Shown] $= \frac{3}{8}\left\lvert\begin{pmatrix}4\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\3\\5\end{pmatrix}\right\rvert$ [Cross Product] $= \frac{3}{8}\left\lvert\begin{pmatrix}0\\-20\\12\end{pmatrix}\right\rvert$ $= \frac{3\sqrt{10^2+6^2}}{4} = 8.75$ units^2	