RAFFLES INSTITUTION
H2 Mathematics (9758)
2025 Year 5

Additional Practice Questions for Chapter 6B: Applications of Differentiation (Solutions)

1 **9758/2017/01/Q5 (modified)**

A curve has equation $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}x + 7$.

(i) Show that the gradient of the curve is always positive. Hence explain why the equation $\mathrm{f}(x) = 0$ has only one real root and find this root. **[3]**

(ii) Find the x-coordinates of the points where the tangent to the curve is parallel to the line $y = 2x - 3$. **[3]**

[(ii) $x = -0.145$ or $x = 1.15$**]**

1(i)	$\mathrm{f}(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}x + 7$ $\mathrm{f}'(x) = 3x^2 - 3x + \frac{3}{2} = 3(x - \frac{1}{2})^2 + \frac{3}{4}$ Since $3(x - \frac{1}{2})^2 > 0$ for all x. Therefore $\mathrm{f}'(x) > 0$ for all x. $\mathrm{f}(x) \to -\infty$ as $x \to -\infty$ and $\mathrm{f}(x) \to \infty$ as $x \to \infty$. Since gradient of curve is always positive, curve is always increasing and thus will cut x-axis at only one point. So, $\mathrm{f}(x) = 0$ has only one root. From GC, $x = -1.33$
(ii)	$\mathrm{f}'(x) = 3(x - \frac{1}{2})^2 + \frac{3}{4} = 2$ From GC, $x = -0.145$ or $x = 1.15$

2 **IJC Prelim 9740/2007//01/Q9**

The equation of a closed curve is $(x+2y)^2+3(x-y)^2=27$.

(i) Show, by differentiation, that the gradient at the point (x, y) on the curve may be expressed in the form $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y-4x}{7y-x}$.

(ii) Find the equations of the tangents to the curve that are parallel to

(a) the x-axis, **(b)** the y-axis.

[(ii)(a) $y=\pm 2$ **(b)** $x=\pm\sqrt{7}$ **]**

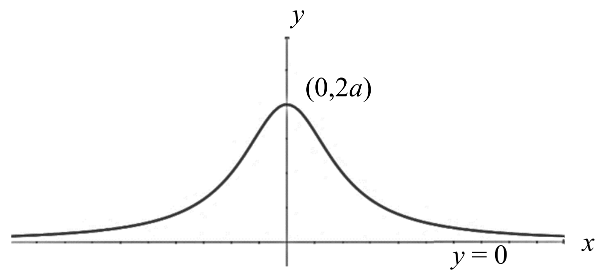
2(i)	$(x+2y)^2+3(x-y)^2=27$ --- (1) Differentiating both sides with respect to x: $2(x+2y)(1+2\dfrac{\mathrm{d}y}{\mathrm{d}x})+6(x-y)(1-\dfrac{\mathrm{d}y}{\mathrm{d}x})=0$ $(x+2y+3x-3y)+(2x+4y-3x+3y)\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ $(4x-y)+(-x+7y)\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y-4x}{7y-x}$ (Shown)
2(ii)	**(a)** To find tangents parallel to x-axis, set $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$, $y-4x=0 \Rightarrow y=4x$ ...(2) Substitute (2) into (1), $(x+8x)^2+3(x-4x)^2=27$ $81x^2+27x^2=27$ $x^2=\dfrac{1}{4} \Rightarrow x=\pm\dfrac{1}{2}$ From (2), equation of tangents: $y=\pm 2$ **(b)** To find tangents parallel to y-axis, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is not defined: $7y-x=0 \Rightarrow x=7y$ ...(3) Substitute (3) into (1), $(9y)^2+3(6y)^2=27$ $189y^2=27$ $y=\pm\dfrac{1}{\sqrt{7}}$ From (3), equation of tangents: $x=\pm\sqrt{7}$

3 **NYJC JC2 CT1 9758/2018/01/Q6 (modified)**

The diagram shows the curve C with parametric equations

$x = -2a \cot t$, $y = 2a \sin^2 t$,

where $0 < t < \pi$ and a is a positive constant.



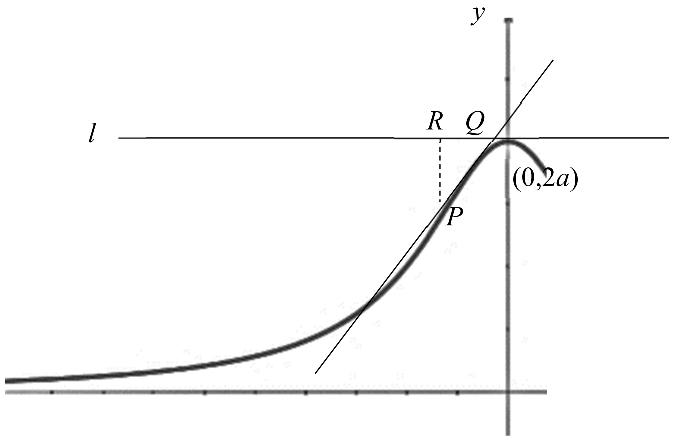
(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\sin^3 t \cos t$. **[1]**

(ii) Hence find, in terms of a, the equation of the tangent l to C which is parallel to the x-axis. **[2]**

(iii) A point P on C has parameter $t = \dfrac{\pi}{3}$. Given that the tangent to C at P meets l at the point Q and the point R is the foot of the perpendicular from P to l, find the exact area of triangle PQR in terms of a. **[5]**

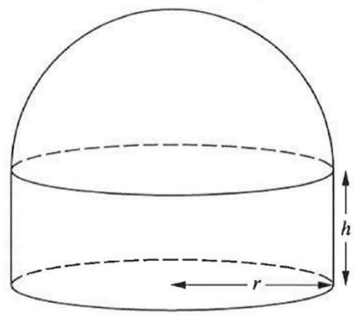
[(ii) $y = 2a$ **(iii)** $\dfrac{a^2}{3\sqrt{3}}$ **]**

(i)	$x = -2a \cot t$ $\quad \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2a \operatorname{cosec}^2 t$ $y = 2a \sin^2 t$ $\quad \dfrac{\mathrm{d}y}{\mathrm{d}t} = 4a \sin t \cos t$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{4a \sin t \cos t}{2a \operatorname{cosec}^2 t} = 2\sin^3 t \cos t$
(ii)	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\sin^3 t \cos t = 0$ $\sin t = 0$ or $\cos t = 0$ $t = 0,\ \pi$ or $t = \dfrac{\pi}{2}$ Since $0 < t < \pi$, $t = \dfrac{\pi}{2}$. Equation of the tangent parallel to x-axis is $y = 2a$.
(iii)	At P, $t = \dfrac{\pi}{3}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\sin^3\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{\pi}{3}\right) = \dfrac{3\sqrt{3}}{8}$ $x = -2a\cot\left(\dfrac{\pi}{3}\right) = \dfrac{-2a}{\sqrt{3}}$ and $y = 2a\sin^2\left(\dfrac{\pi}{3}\right) = \dfrac{3a}{2}$ Equation of tangent at P is $y - \dfrac{3a}{2} = \dfrac{3\sqrt{3}}{8}\left(x + \dfrac{2a}{\sqrt{3}}\right)$ $y = \dfrac{3\sqrt{3}}{8}x + \dfrac{3a}{4} + \dfrac{3a}{2} = \dfrac{3\sqrt{3}}{8}x + \dfrac{9a}{4}$

	At Q, $y=2a$, $2a=\frac{3\sqrt{3}}{8}x+\frac{9a}{4} \Rightarrow x=\frac{8}{3\sqrt{3}}\left(2a-\frac{9a}{4}\right)=-\frac{2a}{3\sqrt{3}}$ $P\equiv\left(-\frac{2a}{\sqrt{3}},\frac{3a}{2}\right)$ $Q\equiv\left(-\frac{2a}{3\sqrt{3}},2a\right)$ $R\equiv\left(-\frac{2a}{\sqrt{3}},2a\right)$  Area of triangle $PQR=\frac{1}{2}PR\times QR$ $=\frac{1}{2}\times\left(\frac{a}{2}\right)\times\left(-\frac{2a}{3\sqrt{3}}+\frac{2a}{\sqrt{3}}\right)$ $=\frac{1}{2}\times\left(\frac{a}{2}\right)\times\left(\frac{4a}{3\sqrt{3}}\right)=\frac{a^2}{3\sqrt{3}}$

4 9740/2012/01/Q10

[It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



A model of a concert hall is made up of three parts.

- The roof is modelled by the curved surface area of a hemisphere of radius r cm.
- The walls are modelled by the curved surface of a cylinder of radius r cm and height h cm.
- The floor is modelled by a circular disc of radius r cm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

(i) It is given that the volume of the model is a fixed value k cm^3, and the external surface area is a minimum. Use differentiation to find the values of r and h in terms of k. Simplify your answers. **[7]**

(ii) It is given instead that the volume of the model is 200 cm^3 and its external surface area is 180 cm^2. Show that there are two possible values of r. Given also that $r < h$, find the value of r and the value of h. **[5]**

[(i) $h = r = \sqrt[3]{\dfrac{3k}{5\pi}}$ **(ii)** $r = 3.04$, $h = 4.88$ **]**

(i)	Considering the volume of the model gives $$k = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + \pi r^2 h \;\Rightarrow\; h = \frac{k}{\pi r^2} - \frac{2}{3}r.$$ The external surface of the model is given by $$S = \frac{1}{2}\left(4\pi r^2\right) + \pi r^2 + 2\pi r h$$ $$= 3\pi r^2 + 2\pi r\left(\frac{k}{\pi r^2} - \frac{2}{3}r\right) = \frac{5}{3}\pi r^2 + \frac{2k}{r}. \qquad \cdots(*)$$ For minimum S, $\frac{\mathrm{d}S}{\mathrm{d}r} = 0$, giving $$\frac{\mathrm{d}S}{\mathrm{d}r} = \frac{10\pi r}{3} - \frac{2k}{r^2} = 0 \;\Rightarrow\; r^3 = \frac{3k}{5\pi}.$$

	Since $h=\dfrac{k}{\pi r^2}-\dfrac{2}{3}r$, we have $$\frac{h}{r}=\frac{k}{\pi}\frac{1}{r^3}-\frac{2}{3}=\frac{k}{\pi}\frac{5\pi}{3k}-\frac{2}{3}=1.$$ Therefore $h=r=\sqrt[3]{\dfrac{3k}{5\pi}}$. $\dfrac{\mathrm{d}^2S}{\mathrm{d}r^2}=\dfrac{10\pi}{3}+\dfrac{4k}{r^3}$ Since $r > 0$ and $k > 0$, $\dfrac{\mathrm{d}^2S}{\mathrm{d}r^2}>0$ for all $r > 0$ $h=r=\sqrt[3]{\dfrac{3k}{5\pi}}$ gives minimum external area.
(ii)	Given that k = 200, from (*) in **(i)**, $$\frac{5}{3}\pi r^2+\frac{2(200)}{r}=180$$ $$\Rightarrow\ 5\pi r^3-540r+1200=0$$ $$\Rightarrow\ \pi r^3-108r+240=0.$$ Using GC, we get $r=3.0372,\ 3.7215$ or -6.7587. Since r is clearly positive, $r=3.04$ (3 s.f.) or $r=3.72$ (3 s.f.) The corresponding values of h are $$h=\frac{200}{\pi(3.0372)^2}-\frac{2}{3}(3.0372)=4.88\text{ (3 s.f.)}$$ or $$h=\frac{200}{\pi(3.7215)^2}-\frac{2}{3}(3.7215)=2.12.\text{ (3 s.f.)}$$ Since $r<h$, we have $r=3.04$ and $h=4.88$ to 3 s.f.

5 **9740/2013/02/Q2**

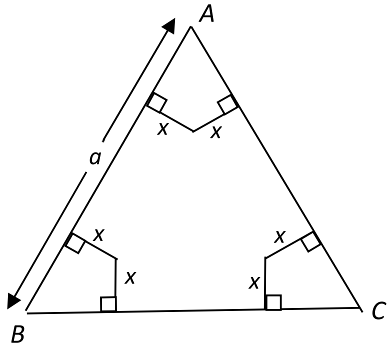


Fig.1

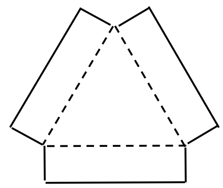


Fig. 2

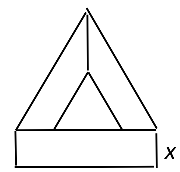


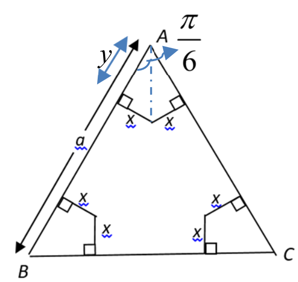
Fig. 3

Fig. 1 shows a piece of card, ABC, in the form of an equilateral triangle of side a. A kite shape is cut from each corner, to give the shape shown in Fig. 2. The remaining card shown in Fig. 2 is folded along the dotted lines, to form the open triangular prism of height x shown in Fig. 3.

(i) Show that the volume V of the prism is given by $V=\frac{1}{4}x\sqrt{3}(a-2x\sqrt{3})^2$. **[3]**

(ii) Use differentiation to find, in terms of a, the maximum value of V, proving that it is a maximum. **[6]**

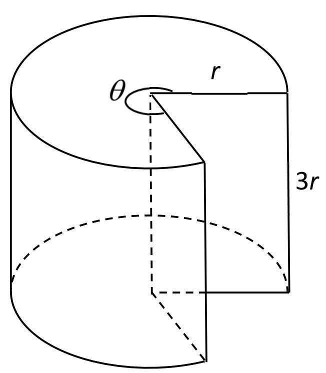
[(ii) $\frac{a^3}{54}$]

<table>
<tr><td>(i)</td><td>Consider one of the kite shapes:


we have $\tan\frac{\pi}{6}=\frac{x}{y} \Rightarrow y=x\sqrt{3}.$
The base of the prism is
an equilateral triangle with side $a-2y=a-2x\sqrt{3}.$
The area of this base, S, is
$$\begin{aligned}S&=\frac{1}{2}\times(\text{base})\times(\text{height})\\&=\frac{1}{2}(a-2x\sqrt{3})\left[(a-2x\sqrt{3})\sin\frac{\pi}{3}\right]\\&=\frac{\sqrt{3}}{4}(a-2x\sqrt{3})^2.\end{aligned}$$
Therefore $V=Sx=\frac{1}{4}x\sqrt{3}(a-2x\sqrt{3})^2.$</td></tr>
<tr><td>(ii)</td><td>We have
$$\frac{\mathrm{d}V}{\mathrm{d}x}=\frac{\sqrt{3}}{4}\left[(a-2x\sqrt{3})^2-4x\sqrt{3}(a-2x\sqrt{3})\right]=\frac{\sqrt{3}}{4}(a-2x\sqrt{3})(a-6x\sqrt{3}).$$</td></tr>
</table>

	At stationary values, $\frac{\mathrm{d}V}{\mathrm{d}x}=0 \Rightarrow \frac{\sqrt{3}}{4}(a-2x\sqrt{3})(a-6x\sqrt{3})=0 \Rightarrow x=\frac{a}{6\sqrt{3}}$ or $x=\frac{a}{2\sqrt{3}}$. We have $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}=\frac{\sqrt{3}}{4}\left[(-2\sqrt{3})(a-6x\sqrt{3})+(-6\sqrt{3})(a-2x\sqrt{3})\right]$ $=3(-2a+6x\sqrt{3})$. At $x=\frac{a}{6\sqrt{3}}$, $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}=-3a<0$, so this gives a maximum value of $V=\frac{1}{4}\left(\frac{a}{6\sqrt{3}}\right)\sqrt{3}\left(a-2\frac{a}{6\sqrt{3}}\sqrt{3}\right)^2=\frac{a^3}{54}\text{units}^3$ At $x=\frac{a}{2\sqrt{3}}$, $\frac{\mathrm{d}^2V}{\mathrm{d}x^2}=3a>0$, so $x=\frac{a}{2\sqrt{3}}$ gives a minimum value of V.

6 NJC Prelim 9740/2014/02/Q1



A solid has height $3r$ cm and a base that is made up of a circular sector with radius r cm. The sector subtends an angle of θ radians, where $0<\theta<2\pi$, at the centre of the circle as shown in the diagram. Given that the total surface area of the container is 10 cm^2, find the exact values of r and θ such that the volume of the container is a maximum. **[8]**

[$r=\frac{\sqrt{5}}{3}$, $\theta=3$ **]**

6

Let V denote the volume of the container.

$$2\left(\frac{1}{2}r^2\theta\right)+2(r)(3r)+r\theta(3r)=10$$

$$4r^2\theta+6r^2=10$$

$$\theta=\frac{10-6r^2}{4r^2}$$

$$=\frac{5-3r^2}{2r^2}$$

$$V=\frac{1}{2}r^2\theta(3r)$$

$$=\frac{3}{2}r^3\left(\frac{5-3r^2}{2r^2}\right)$$

$$=\frac{3}{4}\left(5r-3r^3\right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{3}{4}\left(5-9r^2\right)=\frac{3}{4}\left(\sqrt{5}-3r\right)\left(\sqrt{5}+3r\right)$$

For stationary V, $\frac{\mathrm{d}V}{\mathrm{d}r}=0$

$r=\frac{\sqrt{5}}{3}$ or $r=-\frac{\sqrt{5}}{3}$ (reject since $r>0$)

Corresponding value of $\theta=\dfrac{5-3\left(\frac{5}{9}\right)}{2\left(\frac{5}{9}\right)}=3$

<u>Second Derivative Test</u>

$$\left.\frac{\mathrm{d}^2V}{\mathrm{d}r^2}\right|_{r=\frac{\sqrt{5}}{3}}=\left.\frac{3}{4}(-18r)\right|_{r=\frac{\sqrt{5}}{3}}=-\frac{9\sqrt{5}}{2}<0$$

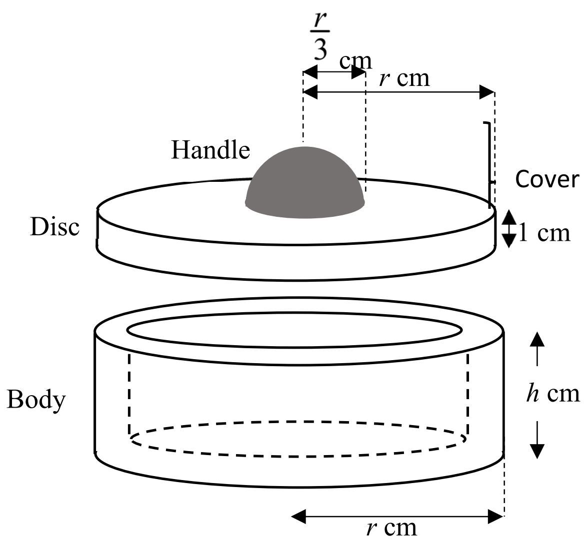
<u>Alternative: First Derivative Test</u>

r	$\left(\frac{\sqrt{5}}{3}\right)^-$	$\frac{\sqrt{5}}{3}$	$\left(\frac{\sqrt{5}}{3}\right)^+$
$\left(\sqrt{5}-3r\right)$	+ve	0	-ve
$\frac{\mathrm{d}V}{\mathrm{d}r}$	+ve	0	-ve

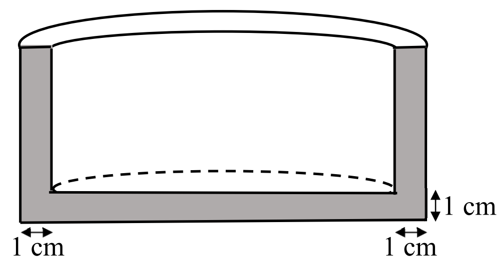
Hence, V is a maximum when $r=\frac{\sqrt{5}}{3}$.

7 SAJC JC2 CT1 9758/2018/01/Q8

[It is given that a sphere of radius r has volume $\frac{4}{3}\pi r^3$.]



Front view of container



Cross-section of body

A container is made up of 2 parts.

- A cylindrical body, opened at one end, of external radius r cm (where $r > 1$), external height h cm and thickness 1 cm throughout.
- A cover consisting of a solid cylindrical disc with radius r cm and height 1 cm and a hemispherical handle with radius one-third of the radius of the disc.

The internal volume of the cylindrical body is 30 cm^3 when full. The material used to make the body and the cover of the container costs \$0.50 per cm^3.

(i) Find an expression for h in terms of r. **[2]**

(ii) Show that the cost, \$ C, to make the container is given by

$$C = \frac{1}{81}\pi r^3 + \frac{15r^2}{(r-1)^2} + \pi r^2 - 15.$$ **[3]**

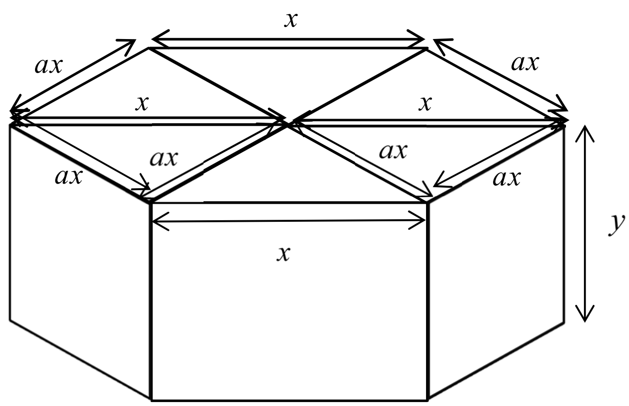
(iii) Given that $r = r_1$ is the value of r which gives the **stationary** value of C, show that r_1 satisfies the equation $\frac{\pi r^2}{27} - \frac{30r}{(r-1)^3} + 2\pi r = 0$. Hence find the values of r and h which give the **minimum** value of C. **[6]**

[(i) $h = \frac{30}{\pi(r-1)^2} + 1$ **(iii)** $r = 2.66$ cm, $h = 4.48$ cm**]**

(i)	Internal volume of the container $= \pi\left(r_m\right)^2 h_m$ $= \pi\left(r-1\right)^2\left(h-1\right)$ $30 = \pi\left(r-1\right)^2\left(h-1\right)$ $\dfrac{30}{\pi\left(r-1\right)^2} = \left(h-1\right) \Rightarrow h = \dfrac{30}{\pi\left(r-1\right)^2}+1$
(ii)	Let C be the cost of material needed to make the container. $C = 0.5$[volume of cover(thin cylinder plus hemispherical handle) + volume of container(cylinder minus interval volume)] $= 0.5\left[\pi r^2\left(1\right)+\dfrac{2}{3}\pi\left(\dfrac{r}{3}\right)^3+\pi r^2 h-30\right]$ $= 0.5\left[\pi r^2+\dfrac{2}{3}\pi\left(\dfrac{r^3}{27}\right)+\pi r^2\left(\dfrac{30}{\pi\left(r-1\right)^2}+1\right)-30\right]$ $= \dfrac{\pi r^3}{81}+\dfrac{15r^2}{\left(r-1\right)^2}+\pi r^2-15$
(iii)	$\dfrac{\mathrm{d}C}{\mathrm{d}r} = \dfrac{\pi r^2}{27}+15\left(\dfrac{\left(r-1\right)^2\left(2r\right)-r^2\left(2\right)\left(r-1\right)}{\left(r-1\right)^4}\right)+2\pi r$ $= \dfrac{\pi r^2}{27}+\dfrac{15\left(2r^2-2r-2r^2\right)}{\left(r-1\right)^3}+2\pi r$ $= \dfrac{\pi r^2}{27}-\dfrac{30r}{\left(r-1\right)^3}+2\pi r$ **<u>For stationary value of *C*</u>**, $\dfrac{\mathrm{d}C}{\mathrm{d}r}=0$, Hence $\dfrac{\pi r^2}{27}-\dfrac{30r}{\left(r-1\right)^3}+2\pi r=0$ ----- (*) Using GC, $r = 0$ (Not application as $r > 1$) or $r = 2.6571 \approx 2.66$ (from GC) Using second derivative test $\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2} = \dfrac{2\pi r}{27}-\dfrac{30\left(r-1\right)^3-90r\left(r-1\right)^2}{\left(r-1\right)^6}+2\pi$ When $r = 2.66$, $\dfrac{\mathrm{d}^2C}{\mathrm{d}r^2} \approx 32.0 > 0$ (**must find and state the value**) Hence $r = 2.66$ gives a minimum C. Hence $h = \dfrac{30}{\pi\left(2.6571-1\right)^2}+1 = 4.4773 = 4.48$ (to 3 sf) Cost is minimum (C is minimum) at $r = 2.66$ cm and $h = 4.48$ cm.

8 **NYJC Prelim 9740/2014/01/Q5**

A company requires a box made of cardboard of negligible thickness to hold 300 cm^3 of powder when full. The top and the base of the box are made up of six identical isosceles triangles. The two identical sides of the isosceles triangle are of length ax cm, where a is a constant and $a > \frac{1}{2}$, and the remaining side is of length x cm. The height of the box is y cm (see diagram).



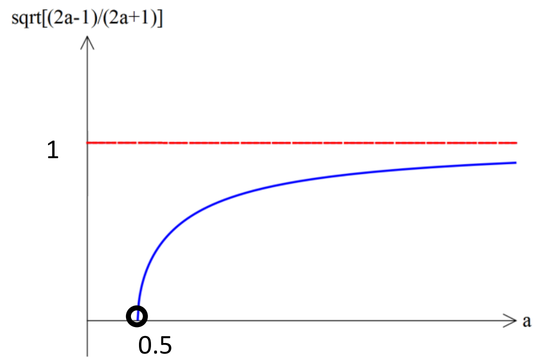
(i) Use differentiation to find, in terms of a, the value of x which gives a minimum surface area of the box. **[7]**

(ii) Show that, in this case, $\frac{y}{x} = 3\sqrt{\frac{2a-1}{2a+1}}$. Hence find the range of $\frac{y}{x}$. **[3]**

[(i) $x = \left(\frac{200}{3(2a-1)}\right)^{\frac{1}{3}}$ **(ii)** $0 < \frac{y}{x} < 3$**]**

(i)	Height of an isosceles $\Delta = \sqrt{(ax)^2 - \left(\frac{x}{2}\right)^2} = \frac{x}{2}\sqrt{4a^2-1}$ Area of one $\Delta = \frac{1}{2}x\left(\frac{x}{2}\sqrt{4a^2-1}\right) = \frac{x^2}{4}\sqrt{4a^2-1}$ Area of base $= 6\cdot\frac{x^2}{4}\sqrt{4a^2-1} = \frac{3x^2}{2}\sqrt{4a^2-1}$ $300 = \left(\frac{3x^2}{2}\sqrt{4a^2-1}\right)y = \left(\frac{3}{2}\sqrt{4a^2-1}\right)x^2y\cdots\cdots(1)$ Surface Area, $S = \left(\frac{3}{2}x^2\sqrt{4a^2-1}\right)2 + 2xy + 4axy$ From (1), $y = \frac{200}{\left(\sqrt{4a^2-1}\right)x^2}$ Therefore

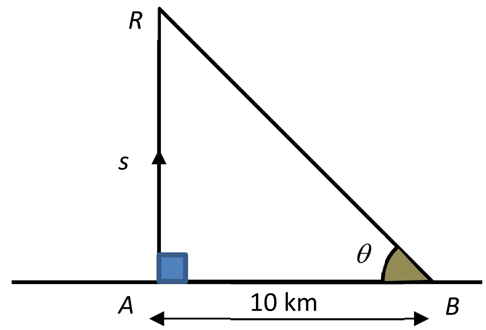
	$S=\left(\frac{3}{2}x^2\sqrt{4a^2-1}\right)2+2x\frac{200}{\left(\sqrt{4a^2-1}\right)x^2}+4ax\frac{200}{\left(\sqrt{4a^2-1}\right)x^2}$ $=\left(3\sqrt{4a^2-1}\right)x^2+\frac{400}{\left(\sqrt{4a^2-1}\right)x}+4a\frac{200}{\left(\sqrt{4a^2-1}\right)x}$ $=\left(3\sqrt{4a^2-1}\right)x^2+\left(\frac{400\left(1+2a\right)}{\sqrt{4a^2-1}}\right)\frac{1}{x}$ $\frac{\mathrm{d}S}{\mathrm{d}x}=\left(6\sqrt{4a^2-1}\right)x-\left(\frac{400(1+2a)}{\sqrt{4a^2-1}}\right)\frac{1}{x^2}$ Let $\frac{\mathrm{d}S}{\mathrm{d}x}=0$ $\left(6\sqrt{4a^2-1}\right)x--\left(\frac{400(1+2a)}{\sqrt{4a^2-1}}\right)\frac{1}{x^2}=0$ $\left(6\sqrt{4a^2-1}\right)x=\left(\frac{400(1+2a)}{\sqrt{4a^2-1}}\right)\frac{1}{x^2}$ $x^3=\frac{200}{3\left(4a^2-1\right)}\left(1+2a\right)$ $x^3=\frac{200}{3\left(2a+1\right)\left(2a-1\right)}\cdot\left(1+2a\right)$ $x^3=\frac{200}{3\left(2a-1\right)}$ $x=\left(\frac{200}{3\left(2a-1\right)}\right)^{\frac{1}{3}}$ $\frac{\mathrm{d}^2S}{\mathrm{d}x^2}=\left(6\sqrt{4a^2-1}\right)+2\left(\frac{400(1+2a)}{\sqrt{4a^2-1}}\right)\frac{1}{x^3}$ Since $6\sqrt{4a^2-1}>0$ and $a>\frac{1}{2}\Rightarrow\frac{800(1+2a)}{\sqrt{4a^2-1}}>0,\ \frac{\mathrm{d}^2S}{\mathrm{d}x^2}>0\ \forall\ x>0$ $\therefore$ Value of x which gives minimum surface area of box is $\left(\frac{200}{3\left(2a-1\right)}\right)^{\frac{1}{3}}$.
(ii)	Recall that $y=\frac{200}{\left(\sqrt{4a^2-1}\right)x^2}$ therefore $\frac{y}{x}=\frac{200}{\left(\sqrt{4a^2-1}\right)x^3}=\frac{200}{\left(\sqrt{4a^2-1}\right)\left(\frac{200}{3\left(2a-1\right)}\right)}$

	$$=\frac{3(2a-1)}{\left(\sqrt{4a^2-1}\right)}=\frac{3(2a-1)}{\sqrt{(2a-1)(2a+1)}}=\frac{3\sqrt{2a-1}}{\sqrt{2a+1}}=3\sqrt{\frac{2a-1}{2a+1}}$$ **Method 1(Graphical):** From graph, for $a>\frac{1}{2}$, $0<\sqrt{\frac{2a-1}{2a+1}}<1$, Therefore $0<3\sqrt{\frac{2a-1}{2a+1}}<3$  **Method 2(Algebraic)** $\frac{3\sqrt{2a-1}}{\sqrt{2a+1}}=3\sqrt{\frac{2a-1}{2a+1}}=3\sqrt{1-\frac{2}{2a+1}}$ Since $a>\frac{1}{2}$, $\Rightarrow\ 2a+1>2>0\quad\Rightarrow 0<\frac{1}{2a+1}<\frac{1}{2}$ $\Rightarrow 0>-\frac{2}{2a+1}>-1$ $\Rightarrow 1>1-\frac{2}{2a+1}>0$ $\Rightarrow 0<\sqrt{1-\frac{2}{2a+1}}<1$ $\Rightarrow 0<3\sqrt{1-\frac{2}{2a+1}}<3$

9 **(a)** A rocket, R, rises vertically from a point A on level ground. It is observed from another point B on the ground where B is 10 km from A. When the angle of elevation has the value $\frac{\pi}{4}$ radians, this angle is increasing at the rate of 0.005 radians per second. Find, in m s^{-1}, the velocity of the rocket at that instant.

(b) **JPJC Promo 9758/2021/Q8(b)**
The height of an upright cone is twice the radius, r, of its circular base. It is known that the volume of the cone is increasing at the rate of 15 cm^3 min^{-1} when the radius is 3 cm. Find the rate of increase of the base area of the cone at this instant. **[4]**

[(a) 100 m s^{-1} **(b)** 5 cm^2 min^{-1}**]**

(a)	Let θ be the angle of elevation. Given, at $\theta=\frac{\pi}{4}$, $\frac{\mathrm{d}\theta}{\mathrm{d}t}=0.005$, find $v=\frac{\mathrm{d}s}{\mathrm{d}t}$ where s is the distance travelled by the rocket. Now, $\tan\theta=\frac{s}{10}$  Diff. w.r.t θ: $\sec^2\theta=\frac{1}{10}\frac{\mathrm{d}s}{\mathrm{d}\theta}\Rightarrow\frac{\mathrm{d}s}{\mathrm{d}\theta}=10\sec^2\theta$ $\therefore\frac{\mathrm{d}s}{\mathrm{dt}}=\frac{\mathrm{d}s}{\mathrm{d}\theta}\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$ $=10\sec^2\theta\ \frac{\mathrm{d}\theta}{\mathrm{d}t}$ At $\theta=\frac{\pi}{4}$, $\frac{\mathrm{d}\theta}{\mathrm{d}t}=0.005$ $\frac{\mathrm{d}s}{\mathrm{d}t}=10\sec^2\left(\frac{\pi}{4}\right)(0.005)$ $=0.05\left(\sqrt{2}\right)^2=0.1$ $\therefore$ The velocity of the rocket at that instant is 0.1 km s^{-1} i.e. 100 m s^{-1} OR Diff. with respect to t, $\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{10}\frac{\mathrm{d}s}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}s}{\mathrm{d}t}=10\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=10\sec^2\left(\frac{\pi}{4}\right)(0.005)=0.1$
(b)	Volume of Cone $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi r^2(2r)=\frac{2}{3}\pi r^3\Rightarrow\frac{\mathrm{d}V}{\mathrm{d}r}=2\pi r^2$ Base area $A=\pi r^2\Rightarrow\frac{\mathrm{d}A}{\mathrm{d}r}=2\pi r$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}=\left(2\pi r^2\right)\frac{\mathrm{d}r}{\mathrm{d}t}\Rightarrow 15=2\pi(3)^2\frac{\mathrm{d}r}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{5}{6\pi}$. $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t}=\left(2\pi r\right)\frac{\mathrm{d}r}{\mathrm{d}t}=\left(2\pi\right)(3)\left(\frac{5}{6\pi}\right)=5\text{ cm}^2\text{ min}^{-1}$. Alternatively, $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}V}\frac{dV}{dt}=\left(2\pi r\right)\left(\frac{1}{2\pi r^2}\right)(15)=5\text{ cm}^2\text{ min}^{-1}$.

10 EJC Promo 9758/2021/Q9(b)

Fig. 2 shows a 2.6 m long pole sliding down along a vertical wall such that the top of the pole is slipping at the constant rate of 0.3 m/s. Determine the speed at which the foot of the pole is moving along the ground when the foot is 1.0 m from the base of the wall. **[4]**

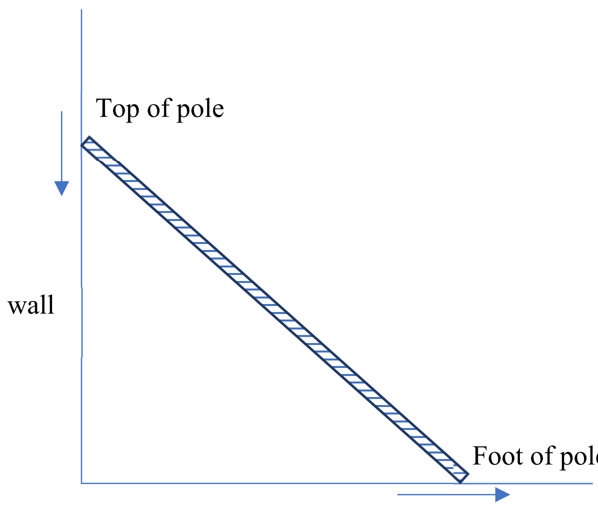
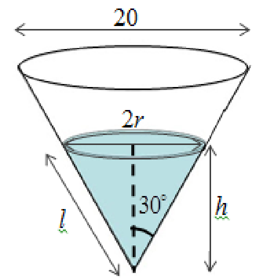


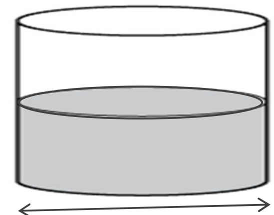
Fig. 2

[0.72 m/s**]**

10	Let y m be the vertical distance between the top of the pole and the ground. Let x m be the horizontal distance between the foot of the pole and the base of the wall. $x^2+y^2=2.6^2$ (Pythagoras' Theorem) When $x=1.0$, $y=2.4$ $\frac{\mathrm{d}y}{\mathrm{d}t}=-0.3$ By implicit differentiation, $2x\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+2y\cdot\frac{\mathrm{d}y}{\mathrm{d}t}=0$ $2(1.0)\frac{\mathrm{d}x}{\mathrm{d}t}+2(2.4)(-0.3)=0$ $\frac{\mathrm{d}x}{\mathrm{d}t}=0.72$ The foot of the pole is moving at a speed of 0.72 m/s.

11 AJC/2014/Prelim/02/Q4





An experiment is conducted using the conical filter which is held with its axis vertical as shown. The filter has a radius of 10cm and semi-vertical angle 30°. Chemical solution flows from the filter into the cylindrical container, with radius 10cm, at a constant rate of 3 cm^3/s. At time t seconds, the amount of solution in the filter has height h cm and radius r cm.

(i) Find the rate of decrease of radius of the solution in the filter when $h = 5$ cm.

(ii) Let S denotes the curved surface area of filter in contact with the solution.

Show that $\dfrac{\mathrm{d}S}{\mathrm{d}t} = -\dfrac{4\sqrt{3}}{r}$ cm²/s.

(iii) When the height of solution in the cylindrical container measures 0.81 cm, volumes of solution left in the filter and the container are the same. Find the rate of change of S at this instant.

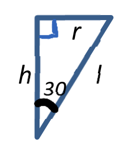
[The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$ and the curved surface area is $\pi r l$ where l is the slant height of the cone.]

[(i) -0.0662 cm/s **(iii)** $-\dfrac{4}{3}$ cm^2/s **]**

(i)

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2\left(\sqrt{3}r\right) = \frac{1}{\sqrt{3}}\pi r^3$$



$$\frac{r}{h} = \tan 30^{o} = \frac{1}{\sqrt{3}}$$

$$h = \sqrt{3}r$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \sqrt{3}\pi r^2$$

Then, $$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}V} \times \frac{\mathrm{d}V}{\mathrm{d}t}$$

$$= \frac{1}{\sqrt{3}\pi r^2}\frac{\mathrm{d}V}{\mathrm{d}t}$$

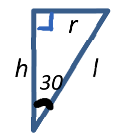
$$= \frac{-3}{\sqrt{3}\pi r^2}$$

$$= \frac{-\sqrt{3}}{\pi r^2}$$

When $h = 5$, $r = \dfrac{5}{\sqrt{3}}$.

So, $$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\sqrt{3}}{\pi\left(\frac{5}{\sqrt{3}}\right)^2} = \frac{-3\sqrt{3}}{25\pi} = -0.0662 \text{ cm/s}$$

$\therefore$ The rate of decrease of radius of the solution is 0.0662 cm/s.

	OR: $V=\frac{1}{\sqrt{3}}\pi r^3$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\sqrt{3}\pi r^2\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t}=\frac{1}{\sqrt{3}\pi r^2}\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)=\ldots$
(ii)	$S=\pi rl$ $=2\pi r^2$ $\frac{\mathrm{d}S}{\mathrm{d}r}=4\pi r$  $\frac{r}{l}=\sin 30^o=\frac{1}{2}$ $l=2r$ Then $\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{\mathrm{d}S}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$ $=4\pi r\times\frac{-\sqrt{3}}{\pi r^2}=\frac{-4\sqrt{3}}{r}$
(iii)	Volume of solution in the cylinder $=\pi r^2 h=\pi(10)^2(0.81)=81\pi$ Volume of solution in the conical filter $=\frac{1}{\sqrt{3}}\pi r^3$ Since volume of solution left in the filter and the container are the same, $81\pi=\frac{1}{\sqrt{3}}\pi r^3$ $\Rightarrow r=\left(81\sqrt{3}\right)^{1/3}=3\sqrt{3}$ $\therefore$ Rate of change of S, $\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{-4\sqrt{3}}{3\sqrt{3}}=-\frac{4}{3}$ cm^2/s

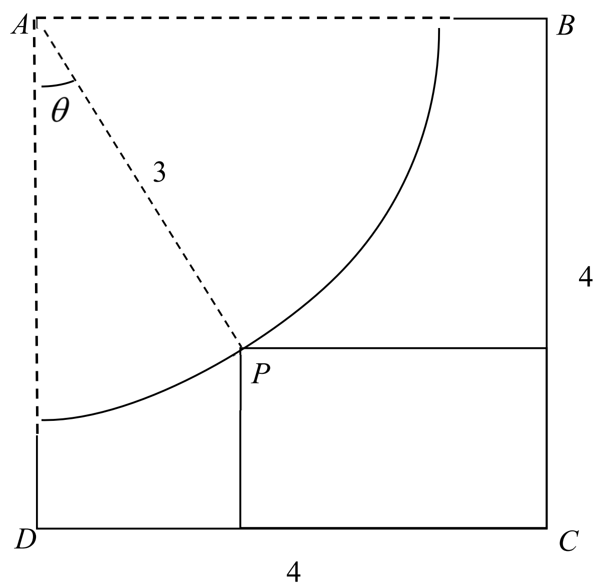
12 NYJC Promos 9740/2013/Q12

(a) The curve C has parametric equations

$$x=\theta^2+4\theta, \qquad y=\frac{2}{\theta}, \qquad \text{for } \theta>0.$$

A point $P(x, y)$ moves on the curve C in such a way that the x-coordinate of P decreases at a constant rate of 4 units per second. Find the rate at which the y-coordinate of P is changing when $x = 4$. **[4]**

(b)



The diagram above shows the floor plan of a storeroom. The floor plan consists of a square $ABCD$ of side 4 units from which a quadrant of a circle with centre A and radius 3 units has been removed. The owner intends to store a rectangular crate with one corner of the base at C, and the opposite corner of the base at P against the curved wall. The base of the crate has area y unit^2 and angle DAP is θ radians, where $0\le\theta\le\frac{\pi}{4}$.

Show that $\frac{dy}{d\theta}=3(\sin\theta-\cos\theta)(4-3\sin\theta-3\cos\theta)$. **[2]**

Hence, find the least possible value of y. **[5]**

[(a) 2.06 units/sec **(b)** 3.50**]**

(a)	$x=\theta^2+4\theta$, $\quad y=\dfrac{2}{\theta}$ $\dfrac{dy}{dt}=\dfrac{dy}{dx}\cdot\dfrac{dx}{dt}$ $=\left(\dfrac{dy}{d\theta}\cdot\dfrac{d\theta}{dx}\right)\cdot\dfrac{dx}{dt}$ $=\dfrac{-2}{\theta^2}\cdot\dfrac{1}{2\theta+4}\cdot(-4)$ $=\dfrac{4}{\theta^2(\theta+2)}$ When $x=4$, $\theta^2+4\theta=4\Rightarrow\theta=0.82843$ (5sf) since $\theta>0$ $\dfrac{dy}{dt}=2.06$ (3sf) units/sec Rate of change of y-coordinate is 2.06 units/sec.
(b)	$y=(4-3\cos\theta)(4-3\sin\theta)$ $\dfrac{dy}{d\theta}=(4-3\sin\theta)(3\sin\theta)+(4-3\cos\theta)(-3\cos\theta)$ $=3\left[4\sin\theta-3\sin^2\theta-4\cos\theta+3\cos^2\theta\right]$ $=3\left[3\left(\cos^2\theta-\sin^2\theta\right)+4\sin\theta-4\cos\theta\right]$ $=3\left[3(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)+4(\sin\theta-\cos\theta)\right]$ $=3(\sin\theta-\cos\theta)(4-3\sin\theta-3\cos\theta)$ (shown) $\dfrac{dy}{d\theta}=0$ $3(\sin\theta-\cos\theta)(4-3\sin\theta-3\cos\theta)=0$ $\sin\theta-\cos\theta=0$ or $4-3\sin\theta-3\cos\theta=0$ $\sin\theta+\cos\theta=\dfrac{4}{3}$ $\tan\theta=1$ $\theta=\dfrac{\pi}{4}$ or Using GC, $\theta=0.44556$ or θ=1.1252 (rej $0\le\theta\le\dfrac{\pi}{4}$) Alternatively, we can apply R-formula and solve $\sqrt{2}\sin\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{4}{3}$ $\dfrac{d^2y}{d\theta^2}=3(\sin\theta-\cos\theta)(3\sin\theta-3\cos\theta)+3(\sin\theta+\cos\theta)(4-3\sin\theta-3\cos\theta)$ When $\theta=\dfrac{\pi}{4}$, $\dfrac{d^2y}{d\theta^2}=12\sqrt{2}-18\approx-1.03<0\Rightarrow y$ is max When $\theta=0.44556$, $\dfrac{d^2y}{d\theta^2}=2>0\Rightarrow y$ is min Min $y=(4-3\cos 0.44556)(4-3\sin 0.44556)=3.50$.